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NINE-POINT DIFFERENCE SOLUTIONS FOR POISSON'S EQUATION

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Abstract—It is shown that "stencils" exist for the sixth order solution of Poisson's equation by use of a nine-point difference approximation.

1. INTRODUCTION

Suppose we wish to approximate the solution of

$$\nabla^2 u(x, y) = f(x, y) \tag{1.1}$$

inside some region, given values around the boundary. A standard approach is to introduce a "stencil", say

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \quad u(x, y).$$

The significance of this is that if a coefficient in the stencil is m units above the horizontal center line and n units to the right of the vertical center line (m and/or n may be negative), one forms the product of the coefficient with $u(x + nh, y + mk)$; the entire stencil denotes the sum of these products. Thus the stencil shown above denotes

$$u(x + h, y) + u(x - h, y) + u(x, y + k) + u(x, y - k) - 4u(x, y).$$

To approximate the $u(x, y)$ which solves (1.1) it is traditional to choose $k = h$. Also, to save space, we write

$$\Delta_5 u(x, y) = \begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \; u(x, y). \tag{1.2}$$

This is called the five-point stencil.

If $u(x, y)$ is reasonably smooth, we have

$$\Delta_5 u(x, y) \cong h^2 \nabla^2 u(x, y) \tag{1.3}$$

to order h^4. So, if h is reasonably small,

$$\Delta_5 u(x, y) \cong h^2 f(x, y) \tag{1.4}$$

is a good approximation to (1.1). Let us choose a suitable (x_0, y_0) and define

$$u_{m,n} = u(x_0 + mh, y_0 + nh). \tag{1.5}$$

Then to approximation h^4, we conclude

$$\Delta_5 u_{m,n} \cong h^2 f(x_0 + mh, y_0 + nh). \tag{1.6}$$

The equations (1.6) are a set of linear equations, which have a unique exact solution, $\bar{u}_{m,n}$. If we solve these linear equations, we will get quantities $\bar{u}_{m,n}$ such that $\bar{u}_{m,n}$ differs from $u_{m,n}$ by order h^2 (assuming smooth boundary conditions).

This is the basis for several schemes for computing numerical approximations for $u(x, y)$ at the "grid points" $(x_0 + mh, y_0 + nh)$. However, since the accuracy is only to order h^2, it is not possible to get very high accuracy. Even moderate accuracy requires solving a very large number of simultaneous equations.

If $f(x, y) = 0$, one can get higher order approximations by use of a nine-point stencil, and so improve the situation. The nine-point stencil is given by

$$\Delta_9 u(x, y) = \begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array}\; u(x, y). \tag{1.7}$$

In general, one has

$$\Delta_9 u(x, y) \cong 6h^2 \nabla^2 u(x, y) \tag{1.8}$$

only to order h^4. So if one solves

$$\Delta_9 \bar{u}_{m,n} = 6h^2 f(x_0 + mh, y_0 + nh), \tag{1.9}$$

one will still usually only get an approximation to order h^2. However, if one solves

$$\Delta_9 \bar{u}_{m,n} = 0, \tag{1.10}$$

one will get an approximation to order h^6 for the solution of

$$\nabla^2 u(x, y) = 0, \tag{1.11}$$

provided one has smooth enough boundary conditions.

The object of the present paper is to present methods for solving (1.1) by a nine-point stencil to order h^6 even when $f(x, y)$ is not identically zero. We should warn that these methods will fail unless the boundary conditions and $f(x, y)$ are smooth. In particular, $f(x, y)$ should have bounded derivatives up to order six for the methods of this paper to succeed.

For completeness, we repeat certain material from Rosser[1].

2. A FOURTH ORDER METHOD

In Section 1, we contemplated dividing our region into squares. For some types of regions, it would be convenient to divide the region into rectangles. It is widely believed that difference methods cannot be constructed to give approximations of order greater than two unless the region is divided into squares. This is not so. We will explain a method that gives approximations of order four if rectangles of sides h and k are used.

Let us temporarily set

$$u_{m,n} = u(x_0 + mh, y_0 + nk). \tag{2.1}$$

That is, we use rectangles whose corners are the grid points $(x_0 + mh, y_0 + nk)$.

Define

$$A = \frac{12h^2k^2}{h^2 + k^2} \tag{2.2}$$

$$b = \frac{10k^2 - 2h^2}{h^2 + k^2} \tag{2.3}$$

$$c = \frac{10h^2 - 2k^2}{h^2 + k^2}. \tag{2.4}$$

We have

$$(b + 2)h^2 = A = (c + 2)k^2, \tag{2.5}$$

$$b + c = 8. \tag{2.6}$$

Define a modified nine-point stencil

$$\Delta_9^* u(x, y) = \begin{array}{|c|c|c|} \hline 1 & c & 1 \\ \hline b & -20 & b \\ \hline 1 & c & 1 \\ \hline \end{array} \; u(x, y); \tag{2.7}$$

here motion of one unit in the y-direction in the stencil is supposed to induce a change of k in y, as in our original definition.

If $u(x, y)$ is smooth, we have to order $h^6 + k^6$

$$\Delta_9^* u(x, y) \cong A\nabla^2 u(x, y) + \frac{h^2 A}{12} u_{xxxx}(x, y) + h^2 k^2 u_{xxyy}(x, y) + \frac{k^2 A}{12} u_{yyyy}(x, y). \tag{2.8}$$

By (1.1) we have

$$u_{xxxx}(x, y) + u_{xxyy}(x, y) = f_{xx}(x, y)$$
$$u_{xxyy}(x, y) + u_{yyyy}(x, y) = f_{yy}(x, y).$$

If we multiply the first of these by $h^2 A/12$ and the second by $k^2 A/12$ and add, we see by (2.2) that we can write (2.8) as

$$\Delta_9^* u(x, y) \cong Af(x, y) + \frac{h^2 A}{12} f_{xx}(x, y) + \frac{k^2 A}{12} f_{yy}(x, y). \tag{2.9}$$

Observe that

$$h^2 f_{xx}(x, y) \cong f(x + h, y) + f(x - h, y) - 2f(x, y)$$
$$k^2 f_{yy}(x, y) \cong f(x, y + k) + f(x, y - k) - 2f(x, y).$$

Thus we conclude finally that to order $h^6 + k^6$

$$\Delta_9^* u(x, y) \cong \frac{h^2 k^2}{h^2 + k^2} \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 8 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \; f(x, y). \tag{2.10}$$

So, if we solve

$$\Delta_9^* \bar{u}_{m,n} = \frac{h^2 k^2}{h^2 + k^2} \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 8 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} \; f(x_0 + mh, y_0 + nk), \tag{2.11}$$

we will get $\bar{u}_{m,n}$ that differ from $u_{m,n}$ by the order $h^4 + k^4$.

This gives a method of order four for rectangular grid elements.

3. A SIXTH ORDER METHOD

If there is a sixth order method that permits the use of rectangular grid elements, we have no knowledge of it. So we return to square grid elements, adopting again the notations of Section 1.

If (1.1) holds and $u(x, y)$ is smooth enough, then to order h^8

$$\Delta_9 u(x, y) \cong 6h^2 f(x, y) + \frac{h^4}{2}\nabla^2 f(x, y) + \frac{h^6}{60}\nabla^4 f(x, y) + \frac{h^6}{30} f_{xxyy}(x, y). \tag{3.1}$$

This does not agree exactly with equation (20.57) on p. 194 of Forsythe and Wasow[2]. However, they claim that their (20.57) is copied from another reference, but they made a mistake in copying. Our (3.1) agrees with the formula from which (20.57) was supposed to be copied.

We note that

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & -8 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}\; f(x, y) \cong 3h^2\nabla^2 f(x, y) + \frac{h^4}{4}\nabla^4 f(x, y) + \frac{h^4}{2} f_{xxyy}(x, y) \tag{3.2}$$

to order h^6, if $f(x, y)$ is sufficiently smooth. So we may replace (3.1) by

$$\Delta_9 u(x, y) \cong \frac{h^2}{15}\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 82 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}\; f(x, y) + \frac{3h^4}{10}\nabla^2 f(x, y). \tag{3.3}$$

This formula appears in Collatz[3] as one of the stencils in Table VI on p. 543.

There are occasional circumstances in which this would be quite adequate; for instance if $f(x, y)$ is a harmonic function. However, in general we must do better.

For computational purposes, when we replace the $u_{m,n}$ by $\bar{u}_{m,n}$ and attempt to solve, it does not matter what is on the right side of our equations, as long as we maintain the same form on the left side. So there is no reason why we must restrict the right side of (3.3) to a nine-point stencil.

As we will have a solution to order h^6, we need not take h particularly small. Useful results have been obtained with $h = L/6$, where L is a basic dimension of the entire figure. With $h = L/10$ accurate results should result, and with $h = L/20$ quite high accuracy should result. For a square region, this requires at most 441 grid points, including the boundary. It would not overload the memory to compute $f(x, y)$ in advance at every grid point, store these values, and call such as are needed for each application of (3.3) or its replacement. In fact, it would be an efficient scheme of computation. So we use a larger stencil on the right of (3.3).

We note that

$$h^2 f_{xx}(x, y) \cong -\frac{1}{12}f(x+2h, y) + \frac{4}{3}f(x+h, y) - \frac{5}{2}f(x, y) + \frac{4}{3}f(x-h, y) - \frac{1}{12}f(x-2h, y) \tag{3.4}$$

to order h^6. By using this and the corresponding relation for $f_{yy}(x, y)$ in (3.3), we obtain

$$\Delta_9 u(x, y) \cong \frac{h^2}{120}\begin{array}{|c|c|c|c|c|}\hline 0 & 0 & -3 & 0 & 0 \\ \hline 0 & 8 & 56 & 8 & 0 \\ \hline -3 & 56 & 476 & 56 & -3 \\ \hline 0 & 8 & 56 & 8 & 0 \\ \hline 0 & 0 & -3 & 0 & 0 \\ \hline\end{array}\; f(x, y). \tag{3.5}$$

This will serve very satisfactorily except for points adjacent to the boundary. For these, values of $f(x, y)$ at points outside the region would be required. If such are available, there would be no difficulty. However, they might not be available.

There is of course the off-center difference approximation

$$h^2 f_{xx}(x, y) \cong \frac{5}{6} f(x-h, y) - \frac{5}{4} f(x, y) - \frac{1}{3} f(x+h, y) + \frac{7}{6} f(x+2h, y) - \frac{1}{2} f(x+3h, y) + \frac{1}{12} f(x+4h, y), \quad (3.6)$$

correct to order h^6. Using it, and the corresponding relation for y, we could derive

$$\Delta_9 u(x, y) \cong \frac{h^2}{120} \begin{array}{|c|c|c|c|c|c|} \hline 0 & 3 & 0 & 0 & 0 & 0 \\ \hline 0 & -18 & 0 & 0 & 0 & 0 \\ \hline 0 & 42 & 0 & 0 & 0 & 0 \\ \hline 8 & -4 & 8 & 0 & 0 & 0 \\ \hline 38 & 566 & -4 & 42 & -18 & 3 \\ \hline 8 & 38 & 8 & 0 & 0 & 0 \\ \hline \end{array} f(x, y), \quad (3.7)$$

which is valid to order h^8; on the right of (3.7) the "origin" of the stencil is the square which contains 566.

This certainly brings a method of order h^6 within reach. One wonders if there could be a better stencil than that on the right of (3.7). Perhaps there is not, but we will investigate what is available.

4. A GENERAL APPROACH

In order to take care of grid points that are one unit away from each of two edges, it follows by (3.1) that we require constants $a_{m,n}$ such that

$$\sum_{m=-1}^{S} \sum_{n=-1}^{S} a_{m,n} f(x+mh, y+nh) \cong 6f(x, y) + \frac{h^2}{2} \nabla^2 f(x, y) + \frac{h^4}{60} \nabla^2 f(x, y) + \frac{h^4}{30} f_{xxyy}(x, y) \quad (4.1)$$

to within terms of order h^6. It follows by (3.7) that such constants exist for $S = 4$. We shall show that they do not exist for $S < 4$. For $S = 4$, there are many sets of $a_{m,n}$, and we shall derive the general form.

Because the right side of (4.1) is invariant under interchange of x and y, if $a_{m,n}$ satisfy (4.1), then so would $a^*_{m,n}$, where we take

$$a^*_{m,n} = a_{n,m}.$$

Then so would

$$a^{**}_{m,n} = \frac{1}{2}(a_{m,n} + a^*_{m,n}).$$

So we lose no generality in assuming

$$a_{m,n} = a_{n,m}. \quad (4.2)$$

If $f(x, y)$ is smooth enough to have a double Taylor series out to order h^6, then

$$\sum_{m=-1}^{S} \sum_{n=-1}^{S} a_{m,n} f(x+mh, y+nh) \cong \sum_{r=0}^{5} h^r \sum_{s=0}^{r} \frac{K_{rs}}{s!(r-s)!} D_x^{r-s} D_y^s f(x, y) \quad (4.3)$$

to within terms of order h^6, where D_x and D_y are partial derivatives with respect to x and y respectively, and

$$K_{rs} = \sum_{m=-1}^{S} \sum_{n=-1}^{S} m^{r-s} n^s a_{m,n}. \quad (4.4)$$

Because of (4.2), we have

$$K_{rs} = K_{sr}. \quad (4.5)$$

Define

$$A_m = \sum_{n=-1}^{S} a_{m,n}. \tag{4.6}$$

By (4.4)

$$K_{r0} = \sum_{m=-1}^{S} m^r A_m. \tag{4.7}$$

By (4.3), if we are to satisfy (4.1), we must have

$$K_{00} = 6 \tag{4.8}$$

$$K_{10} = 0 \tag{4.9}$$

$$K_{20} = 1 \tag{4.10}$$

$$K_{30} = 0 \tag{4.11}$$

$$K_{40} = \frac{2}{5} \tag{4.12}$$

$$K_{50} = 0. \tag{4.13}$$

By (4.7), this is a set of six simultaneous linear equations for the A_m. If $S < 4$, they have no solution. So we take $S = 4$, for which we observed earlier that there is a solution, and proceed. The equations (4.8) through (4.13) have the unique solution

$$A_{-1} = \frac{9}{20},\quad A_0 = \frac{209}{40},\quad A_1 = \frac{1}{10},\quad A_2 = \frac{7}{20},\quad A_3 = -\frac{3}{20},\quad A_4 = \frac{1}{40}.$$

Analogously, if we write

$$B_m = \sum_{n=-1}^{S} n a_{m,n}, \tag{4.14}$$

then to satisfy (4.1) we must have

$$\sum_{m=-1}^{S} m^{r-1} B_m = K_{r1} = 0 \qquad (1 \le r \le 5). \tag{4.15}$$

This set of 5 equations has the one fold multiplicity of solutions

$$B_{-1} = -\frac{1}{5}B_0,\quad B_1 = -2B_0,\quad B_2 = 2B_0,\quad B_3 = -B_0,\quad B_4 = \frac{1}{5}B_0.$$

We write also

$$C_m = \sum_{n=-1}^{S} n^2 a_{m,n}. \tag{4.16}$$

To satisfy (4.1) we must have

$$\sum_{m=-1}^{S} C_m = K_{22} = 1 \tag{4.17}$$

$$\sum_{m=-1}^{S} mC_m = K_{32} = 0 \tag{4.18}$$

$$\sum_{m=-1}^{S} m^2 C_m = K_{42} = \frac{4}{15} \tag{4.19}$$

$$\sum_{m=-1}^{S} m^3 C_m = K_{52} = 0. \tag{4.20}$$

This set of 4 equations has the two fold multiplicity of solutions

$$C_{-1} = \frac{19}{60} - \frac{1}{4}C_0 + \frac{1}{4}C_4,\quad C_1 = \frac{37}{30} - \frac{3}{2}C_0 - \frac{5}{2}C_4,\quad C_2 = -\frac{11}{15} + C_0 + 5C_4,\quad C_3 = \frac{11}{60} - \frac{1}{4}C_0 - \frac{15}{4}C_4.$$

To satisfy (4.1) it is sufficient as well as necessary to satisfy equations (4.8)–(4.13), (4.15), and (4.17)–(4.20). This we have accomplished, and with three parameters, B_0, C_0, and C_4, at our disposal.

Given values of the A_m, B_m, and C_m, we have yet to determine the $a_{m,n}$. By (4.2) and (4.6), we have

$$a_{m,0} = A_m - a_{m,-1} - \sum_{n=1}^{4} a_{m,n}. \tag{4.21}$$

Except for $m = 0$, this expresses $a_{m,0}$ (and hence $a_{0,m}$) in terms of $a_{r,s}$ with both $r \neq 0$ and $s \neq 0$. Using (4.2) with (4.21) gives

$$a_{0,0} = A_0 - \left\{A_{-1} - a_{-1,-1} - \sum_{r=1}^{4} a_{-1,r}\right\} - \sum_{n=1}^{4}\left\{A_n - a_{n,-1} - \sum_{r=1}^{4} a_{n,r}\right\}. \tag{4.22}$$

So also $a_{0,0}$ is expressed in terms of $a_{r,s}$ with both $r \neq 0$ and $s \neq 0$.

If we add and subtract (4.14) and (4.16) we will get

$$a_{-1,m} = \frac{1}{2}\left\{C_m - B_m - \sum_{n=2}^{4} n(n-1)a_{m,n}\right\} \tag{4.23}$$

$$a_{1,m} = \frac{1}{2}\left\{C_m + B_m - \sum_{n=2}^{4} n(n+1)a_{m,n}\right\}. \tag{4.24}$$

Except for $m = -1, 0$, and 1, (4.23) expresses $a_{-1,m}$ (and hence $a_{m,-1}$) in terms of $a_{r,s}$ with $r \geq 2$ and $s \geq 2$. If we take $m = -1$ in (4.23), and make another use of (4.23), we get

$$a_{-1,-1} = \frac{1}{2}\left\{C_{-1} - B_{-1} - \frac{1}{2}\sum_{n=2}^{4} n(n-1)\left[C_n - B_n - \sum_{r=2}^{4} r(r-1)a_{n,r}\right]\right\}. \tag{4.25}$$

If we take $m = 1$ in (4.23), and make a use of (4.24), we get

$$a_{-1,1} = \frac{1}{2}\left\{C_1 - B_1 - \frac{1}{2}\sum_{n=2}^{4} n(n-1)\left[C_n + B_n - \sum_{r=2}^{4} r(r+1)a_{n,r}\right]\right\}. \tag{4.26}$$

So, except for $m = 0$, we have $a_{-1,m}$ (and hence $a_{m,-1}$) in terms of $a_{r,s}$ with $r \geq 2$ and $s \geq 2$. Using these in (4.21) gives also $a_{-1,0}$ in terms of $a_{r,s}$ with $r \geq 2$ and $s \geq 2$.

Except for $m = 0$ and 1, (4.24) or (4.26) expresses $a_{1,m}$ in terms of $a_{r,s}$ with $r \geq 2$, and $s \geq 2$. If we take $m = 1$ in (4.24), and make another use of (4.24), we get

$$a_{1,1} = \frac{1}{2}\left\{C_1 + B_1 - \frac{1}{2}\sum_{n=2}^{4} n(n+1)\left[C_n + B_n - \sum_{r=2}^{4} r(r+1)a_{n,r}\right]\right\}. \tag{4.27}$$

If we use this, (4.26), and (4.24) in (4.21) we get also $a_{1,0}$ in terms of $a_{r,s}$ with $r \geq 2$ and $s \geq 2$.

In view of (4.2), there are only six distinct parameters $a_{r,s}$ with $r \geq 2$ and $s \geq 2$. There remain yet unused three of the eighteen original equations. It would be expected that they would give three more conditions among the $a_{r,s}$, but surprisingly they turn out to be dependent on the other fifteen. This is due to the particular relations that subsist among the A_m, B_m, and C_m, and would not be the case with general A_m, B_m, and C_m.

Thus consider (4.23) for $m = 0$, of which we have not yet made any use. If we substitute from (4.23) into (4.21), we will get

$$a_{-1,0} = A_{-1} - \frac{1}{2}\left\{C_{-1} - B_{-1} - \sum_{n=2}^{4} n(n-1)a_{-1,n} + \sum_{n=1}^{4}(C_n - B_n) - \sum_{n=1}^{4}\sum_{r=2}^{4} r(r-1)a_{n,r}\right\}. \tag{4.28}$$

Making use of $a_{n,r} = a_{r,n}$ lets us write the final term as

$$\sum_{n=2}^{4} n(n-1)\sum_{r=1}^{4} a_{n,r}.$$

By (4.15) and (4.17), we have

$$C_{-1} - B_1 + \sum_{n=1}^{4}(C_n - B_n) = 1 - (C_0 - B_0).$$

Also, from the given values of the A_m, we have

$$A_{-1} = \frac{1}{2}\left\{1 - \sum_{n=2}^{4} n(n-1)A_n\right\}.$$

Putting these into (4.28) gives

$$a_{-1,0} = \frac{1}{2}\left\{C_0 - B_0 - \sum_{n=2}^{4} n(n-1)\left[A_n - a_{-1,n} - \sum_{r=1}^{4} a_{n,r}\right]\right\}.$$

Use of (4.21) converts this into (4.23) with $m = 0$.

Consider next (4.24) for $m = 0$, of which we have not yet made any use. If we substitute from (4.24) into (4.21), we will get

$$a_{1,0} = A_1 - \frac{1}{2}\left\{C_{-1} + B_{-1} - \sum_{n=2}^{4} n(n+1)a_{-1,n} + \sum_{n=1}^{4}(C_n + B_n) - \sum_{n=1}^{4}\sum_{r=2}^{4} r(r+1)a_{n,r}\right\}. \tag{4.29}$$

As before, we write the last term as

$$\sum_{n=2}^{4} n(n+1)\sum_{r=1}^{4} a_{n,r},$$

and we have

$$C_{-1} + B_{-1} + \sum_{n=1}^{4}(C_n + B_n) = 1 - (C_0 + B_0),$$

and

$$A_1 = \frac{1}{2}\left\{1 - \sum_{n=2}^{4} n(n+1)A_n\right\}.$$

Substituting these into (4.29), and using (4.21), gives (4.24) with $m = 0$.

Consider finally (4.24) with $m = -1$. Refer back to (4.26). We have

$$\frac{1}{4}\sum_{n=2}^{4} n(n-1)\sum_{r=2}^{4} r(r+1)a_{n,r} = \frac{1}{4}\sum_{n=2}^{4} n(n+1)\sum_{r=2}^{4} r(r-1)a_{n,r}.$$

Also, use of (4.15) and (4.18) gives

$$\frac{1}{2}\left\{C_1 - B_1 - \frac{1}{2}\sum_{n=2}^{4} n(n-1)(C_n + B_n)\right\} = \frac{1}{2}\left\{C_{-1} + B_{-1} - \frac{1}{2}\sum_{n=2}^{4} n(n+1)(C_n - B_n)\right\}.$$

Substituting these into (4.26) and using (4.23) gives (4.24) with $m = -1$.

Thus we can choose $a_{r,s}$ with $r \geq 2$ and $s \geq 2$ at will, subject to $a_{r,s} = a_{s,r}$. Then we can substitute gradually back, and recover all the $a_{r,s}$. Recalling that we have also the free parameters B_0, C_0 and C_4, we see that there is a nine-fold multiplicity of solutions. One would have thought it possible to choose $a_{4,n} = 0$ for all n, thus reducing to the case $S = 3$. However, as $A_4 = 1/40$, this is precluded by (4.6).

We observe that if the principal grid point is at a distance h from the left edge, but further than that from the top or bottom, then one can use the off center difference approximation in the x-direction only. We use the methods given above to see if one can get a stencil which does not extend as far as six grid points in the x-direction. It turns out that one cannot, but we will present the analysis anyhow, since it shows how to generate all possible stencils.

Without causing confusion, we can use the same letters as before, but with slightly altered denotations.

So for our $a_{m,n}$ we will now have $-2 \leq n \leq 2$, $-1 \leq m \leq S$. In place of (4.2), we will have

$$a_{m,n} = a_{m,-n}. \tag{4.30}$$

All summations on n should be from -2 to $+2$. Specifically, this change should be made in (4.3), (4.4), (4.6), (4.14), and (4.16). Delete (4.5).

As before, we see that we must have $S \geq 4$. Taking $S = 4$, we get the same values of A_m as before.

By (4.30) and (4.14), we have $B_m = 0$ for all m. Thus (4.15) is trivially satisfied.

We get the same determination as before for the C_m.

Finally, we write

$$D_m = \sum_{n=-2}^{2} n^4 a_{m,n}. \tag{4.31}$$

We must have

$$D_{-1} + D_0 + D_1 + D_2 + D_3 + D_4 = \frac{2}{5} \tag{4.32}$$

$$-D_{-1} + D_1 + 2D_2 + 3D_3 + 4D_4 = 0. \tag{4.33}$$

Given the A_m, B_m, C_m, and D_m, there is no question how to determine the $a_{m,n}$. We have immediately

$$a_{m,-2} = a_{m,2} = \frac{D_m - C_m}{24},$$

$$a_{m,-1} = a_{m,1} = \frac{4C_m - D_m}{6},$$

$$a_{m,0} = A_m - 2a_{m,1} - 2a_{m,2}.$$

Thus we can easily determine sets of $a_{m,n}$. There are 18 distinct $a_{m,n}$. As they do not depend on the B_m, it appears that we have a six fold multiplicity of solutions. It is surprising that this does not permit the choice $a_{4,0} = a_{4,1} = a_{4,2}$, which would let us reduce S to 3.

5. REGIONS OF UNUSUAL SHAPE

We have been using squares for our grid. There are cases where this is really impractical. For example, suppose our region is a rectangle of sides 1 and $\sqrt{2}$. For rectangles of intractable proportions, a way of handling the matter easily is provided in Rosser[1]. Beyond that, we have not pushed our investigations.

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REFERENCES

1. J. Barkley Rosser, *Finite-difference solution of Poisson's equation in rectangles of arbitrary shape*, Technical Report TR/27. Brunel University (1974) and MRC Technical Summary Report # 1404 (February 1974).
2. G. E. Forsythe and W. R. Wasow, *Finite-difference methods for partial differential equations*, Wiley, New York (1960).
3. L. Collatz, *The Numerical Treatment of Differential Equations*. Springer-Verlag, Berlin (1966).