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Iterative Methods for a Class of Complementarity Problems

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In this paper, we propose and study an algorithm for a new class of complementarity problems of finding $u \in \mathbf{R}^n$ such that $u \geqslant 0$, $Tu + A(u) \geqslant 0$; $(u, Tu + A(u)) = 0$, where T is a continuous mapping and A is a nonlinear transformation from $\mathbf{R}^n$ into itself. It is proved that the approximate solution obtained from the iterative scheme converges to the exact solution. Several special cases are also discussed.

1. INTRODUCTION

Variational inquality technique is being applied to study many unrelated free boundary value problems arising in various branches of mathematical and engineering sciences in a unified and general framework. The theory of variational inequalities has been developed not only to study the fundamental facts on the qualitative behaviour of solutions (regarding existence, uniqueness, and regularity) to important classes of nonlinear boundary value problems, but also to provide highly efficient new numerical methods for solving free boundary value problems. In a variational inequality formulation, the location of the free boundary (contact area) becomes an intrinsic part of the solution and no special devices are needed to locate it. Variational inequalities have been generalized and extended to study a wide class of linear and nonlinear problems arising in mathematical and engineering sciences. A useful and important generalization of the variational inequality problems is the mildly nonlinear variational inequality introduced and considered by Noor [1, 2] for studying mildly nonlinear partial differential equations, when solutions are required to satisfy some extra constraint conditions.

Equally important is the concept of complementarity theory, a relatively new area of operations research, which has received much attention during the last twenty years. It is fairly well known that both the linear and non-

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linear programs can be characterized by a class of complementarity problems. The linear complementarity problem was introduced and studied by Lemke [3] in 1964, but it was Cottle and Dantzig [4], who formally defined the linear complementarity problem and called it the fundamental problem. The complementarity problem has also been generalized and extended to the study of a large class of problems occurring in fluid flow through porous media, contact problems in elasticity, economics and transportation equilibrium, control optimization, and lubrication problems, see Baiocchi and Capelo [5], Oden and Kikuchi [6], Oden and Carey [7], Crank [8], and the references therein.

The relationship between a variational inequality problem and a complementarity problem has been noted implicitly by Lions [9] and Mancino and Stampacchia [10]. However, it was Karamardian [11, 12], who showed that if the set involved in a variational inequality problem and complementarity problem is a convex cone, then both problems are equivalent. Such a relationship is preserved in both the quasi complementarity problem and the quasi variational inequality problem as proved by Pang [13] and Noor [14]. This equivalence has been used by many authors including Ahn [15] and Noor [16, 14] in suggesting new and unified iterative algorithms for solving complementarity problems and the various generalizations.

Motivated and inspired by the recent research work going on in these fields, relative to the mildly nonlinear variational inequalities, we consider and study a new class of complementarity problems, known as mildly (strongly) nonlinear complementarity problems. It is proved that, if the set involved in both problems, is a convex cone, then the mildly nonlinear variational inequality problem and the mildly nonlinear complementarity problem are equivalent. Using this equivalence, we suggest and analyze a new unified and general algorithm for computing the approximate solution of the mildly nonlinear complementarity problem. This algorithm may be viewed as an extension of the algorithm of Mangasarian [17]. It is shown that convergence properties of Mangasarian's algorithm discussed in [13–15, 18] are carried over to this new proposed algorithm using the technique of variational inequalities. Our results are an extension and improvement of the results of Noor [16], Pang [19], Ahn [15, 18], and Mangasarian [17]. For related work on mildly nonlinear variational inequalities, see Chan and Glowinski [20] and Noor [1, 2, 21].

In Section 2, after reviewing some basic notations and results, we introduce the mildly nonlinear complementarity problem and discuss several special cases. Algorithms and convergence results are considered and discussed in Sections 3 and 4. In Section 5, we consider a simple example to illustrate the applications of the present results developed in Sections 3 and 4.

2. Preliminaries and Formulations

We denote the inner product and norm on $\mathbf{R}^n$ by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. Let K be a closed convex set in $\mathbf{R}^n$. For a given continuous mapping T from $\mathbf{R}^n$ into itself, we consider the problem of finding $u \in K^n$ such that

$$(Tu + A(u), v - u) \geqslant 0, \qquad \text{for all} \quad v \in K, \tag{2.1}$$

where A is a nonlinear transformation from $\mathbf{R}^n$ into itself. Inequalities of the type (2.1) are known as mildly (strongly) nonlinear variational inequalities, which were introduced and considered by Noor [1, 2, 21] in the theory of constrained mildly (strongly) nonlinear partial differential equations.

If the nonlinear transformation $A(u) \equiv 0$ (or $A(u)$ is independent of the solution u; that is, $A(u) \equiv f(\text{say})$), then (2.1) is equivalent to finding $u \in K$ such that

$$(Tu, v - u) \geqslant 0, \qquad \text{for all} \quad v \in K. \tag{2.2}$$

Problems of type (2.2) are know as variational inequality problems, originally introduced and considered by Lions and Stampacchia [22], which have been studied extensively, see Lions [9], Glowinski, Lions, and Tremolieres [23], Oden and Kikuchi [6], Baiocchi and Capelo [5], Crank [8], and the references therein. Clearly inequalities of type (2.1) are more general and include inequalities (2.2) as a special case.

If $K = \mathbf{R}^n$, then problem (2.1) is equivalent to finding $u \in \mathbf{R}^n$ such that

$$(Tu + A(u), v) = 0, \qquad \text{for all} \quad v \in \mathbf{R}^n, \tag{2.3}$$

which are known as the weak formulations of mildly (strongly) nonlinear boundary value problems, see Noor [2, 24], Douglas [25], Ciarlet, Schultz, and Varga [26] and Noor and Whiteman [27].

We now consider the complementarity problem. For a given continuous mapping T from $\mathbf{R}^n$ into itself, the complementarity problem is to find u such that

$$u \geqslant 0, \qquad Tu \geqslant 0, \qquad (u, Tu) = 0. \tag{2.4}$$

It is well known that if the mapping T is nonlinear, then (2.4) is called the nonlinear complementarity problems, which have been extensively studied by Cottle [28] and Karamardian [12].

If the mapping T is a linear affine transformation of the type

$T: u \to Mu + q$, $M \in R^{n \times n}$, $q \in \mathbf{R}^n$, then (2.4) is equivalent to finding u such that

$$u \geqslant 0, \qquad Mu + q \geqslant 0, \qquad (u, Mu + q) = 0, \tag{2.5}$$

which is said to be the linear complementarity problem.

The linear complementarity problem (2.5) was originally introduced by Lemke [3] and Cottle and Dantzig [4], and then studied by Mangasarian [17], Ahn [18, 15], Aganagic [29], and Pang [13, 19, 30] by using iterative methods.

If $K^* = \{u \in \mathbf{R}^n, (u, v) \geqslant 0, \text{ for all } v \in K\}$, is the polar of the convex cone in $\mathbf{R}^n$, then we consider the generalized complementarity problem of finding $u \in K$ such that

$$Tu \in K^* \qquad \text{and} \qquad (u, Tu) = 0. \tag{2.6}$$

This natural generalization of the complementarity problem (2.4) is due to Karamardian [11, 12] and Habetler and Price [31]. We note that the complementarity problem (2.4) is a special case of the generalized complementarity problem (2.6), if K is equal to the non-negative orthant $\mathbf{R}^n = \{u \in \mathbf{R}^n : u \geqslant 0\}$, for all u.

Related to the mildly (strongly) nonlinear variational inequality problem (2.1), we now consider and introduce a new class of complementarity problem as follows:

For a given continuous mapping T from $\mathbf{R}^n$ into itself, we consider the problem of finding u such that

$$u \geqslant 0, \qquad Tu + A(u) \geqslant 0, \qquad (u, Tu + A(u)) = 0, \tag{2.7}$$

where A is a nonlinear transformation from $\mathbf{R}^n$ into itself. If T is a nonlinear mapping, then problem (2.7) is called the strongly nonlinear complementarity problem. If T is an affine transformation of the form $T: u \to Mu + q$, $M \in R^{n \times n}$, $q \in \mathbf{R}^n$, then problem (2.7) is known as the mildly nonlinear complementarity problem; that is, of finding u such that

$$u \geqslant 0, \qquad Mu + q + A(u) \geqslant 0, \qquad (u, Mu + q + A(u)) = 0. \tag{2.8}$$

For the applications and mathematical formulations of the problems (2.6) and (2.8), see Noor [32], where one can also find the basic and general theory of such complementarity problems.

In a similar way, we can introduce the generalized mildly (strongly) nonlinear complementarity problem of finding $u \in K$ such that

$$(Tu + A(u)) \in K^* \qquad \text{and} \qquad (u, Tu + A(u)) = 0, \tag{2.9}$$

Obviously problems (2.8) and (2.7) are special cases of the problem (2.9).

Furthermore, if the nonlinear transformation A is identically zero or is independent of the solution u, that is $A(u) \equiv 0$, $(A(u) \equiv q(\text{say}))$, then problems (2.7)–(2.9) reduce to the problems (2.4)–(2.6), respectively, studied previously. Thus it is clear that problem (2.9) is the most general and unifying one, which is the main motivation of this paper.

3. Iterative Methods

It is fairly well known that numerical methods for solving complementarity problems can be divided into two major categories namely direct and indirect (iterative) methods. Direct methods are those based on the process of pivoting, which are mainly due to Lemke [3] and Cottle and Dantzig [4]. The practicality of the direct methods is restricted mainly due to the problem size limitation in computer implementations. Also it has been shown by Mangasarian [33] that there are examples of linear complementarity problems, which cannot be solved by Lemke's method or the principal pivoting method of Cottle and Dantzig [4]. These facts and reasons have stimulated much investigation of alternative approaches for solving complementarity problems, see [15, 17, 34–36].

In this paper, we are only concerned with the iterative methods. Iterative methods are those which produce a sequence of iteratives (trial solutions) and converge to the exact solution. Iterative methods have emerged in the last decade as a powerful technique for solving many large scale complementarity and variational inequalities problems arising from applications, see [5, 13, 28, 35, 37]. Most of these iterative methods are based on the extensions of their counterparts for solving square systems of linear equations. There have been several papers of Mangasarian [17], Ahn [15, 18], Pang [13], and Noor [14], which provide unified frameworks for the study of the convergence of iterative methods for the complementarity problems. Cheng [38], Lin and Cryer [35], and Aganagic [29] have applied iterative methods of gradient-projection, alternating direction implicit, and Newton's type for solving large structured linear complementarity problems arising in the study of free boundary value problems. The situation related to mildly nonlinear complementarity problems (2.7)–(2.9) is very different and much less developed, because the area of mildly nonlinear complementarity theory and its application is much more complex. In this paper, we use the variational inequality technique to propose and analyze general and unified algorithms for mildly (strongly) nonlinear complementarity problems. To do so, we need the following results, the first one is a generalization of Karamardian [11], Noor [14], and Cottle [28].

LEMMA 3.1. *If K is the positive cone in $\mathbf{R}^n$, then $u \in K$ is a solution of the mildly nonlinear variational inequality* (2.1) *if and only if $u \in K$ solve the complementarity problem* (2.9).

Proof. Its proof is similar to that of Lemma 3.1 of Karamardian [11].

LEMMA 3.2 [39]. *If K is a convex set in $\mathbf{R}^n$, then $u \in K$ satisfies* (2.1) *if and only if $u \in K$ satisfies the relation*

$$u = \lambda P_k[u - \rho(Tu + A(u))] + (1 - \lambda) u, \tag{3.1}$$

for some constant $\rho > 0$ and $0 < \lambda \leqslant 1$. Here P_k is a projection of $\mathbf{R}^n$ into K.

Lemma 3.1) implies the equivalence of variational inequality (2.1) and complementarity problem (2.9), whereas Lemma 3.2 together with Lemma 3.1 shows that the complementarity problem (2.9) can be transformed to a fixed point problem of solving

$$u = F(u),$$

where

$$F(u) = (1 - \lambda) u + \lambda P_k[u - \rho(Tu + A(u))].$$

Based on these observations, we now propose the following general and unified algorithms for the complementarity problem (2.9).

ALGORITHM 3.1. For any given $u_0 \in K$, compute

$$u_{n+1} = (1 - \lambda) u_n + \lambda P_k[u_n - \rho(Tu_n + A(u_n))], \qquad n = 0, 1, 2, ..., \tag{3.2}$$

where $0 < \lambda \leqslant 1$ and $\rho > 0$ is a constant.

If T is an affine transformation of the type $T: u \to Mu + q$, $M \in R^{n \times n}$, $q \in \mathbf{R}^n$, then Algorithm 3.1 becomes:

ALGORITHM 3.2. For given $u_0 \in K$, compute

$$u_{n+1} = (1 - \lambda) u_n + \lambda P_k[u_n - \rho E_n\{Mu_n + q + L_n(u_{n+1} - u_n) + A(u_n)\}],$$
$$n = 0, 1, \tag{3.3}$$

where $0 < \lambda \leqslant 1$, $\rho > 0$ is a constant, $\{E_n\}$ and $\{L_n\}$ are bounded sequences of matrices in $R^{n \times n}$. This problem is compatible with the algorithm of Mangasarian [17].

For the Algorithm 3.2 to be practical, L_n may not be strictly lower or upper triangular matrix, because the iterate u_{n+1} may be obtained by solving a variational inequality subproblem, as pointed out in Pang [13]. Here the original data M, remains intact throughout iteration, allowing an algorithm of type (3.2) to be efficient both for large scale and specially structured problems.

For simplicity, we consider the case $E_n = E$ and $L_n = L$. We here consider the following version of Algorithm 3.2.

ALGORITHM 3.3. For given $u_0 \in K$, compute

$$u_{n+1} = (1-\lambda)\, u_n + \lambda P_k[u_n - \rho E\{Mu_n + q + L(u_{n+1} - u_n) + A(u_n)\}],\qquad n = 0, 1, 2, \ldots, \tag{3.4}$$

where $\rho > 0$ is a constant, $0 < \lambda \leqslant 1$ is a relaxation parameter used after the projection, and L is either a strictly lower or upper triangular matrix.

It is clear that each iteration of Algorithms 3.1, 3.2, and 3.3 is itself equivalent to mildly nonlinear variational inequality problem (2.1) as implied by Lemma 3.1 and Lemma 3.2.

Special Cases. 1. If $K = \mathbf{R}^n$, then Algorithm 3.1 is equivalent to computing $u_0 \in \mathbf{R}^n$ such that

$$u_{n+1} = u_n - \lambda\rho\{Tu_n + A(u_n)\},\qquad n = 0, 1, 2, \ldots, \tag{3.5}$$

where $0 < \lambda \leqslant 1$ and $\rho > 0$ is a constant. This algorithm appears to be a new one for solving mildly nonlinear elliptic partial differential equations.

2. If the nonlinear transformation A is identically equal to zero (or a constant); that is, $A(u) \equiv 0$, then Algorithms 3.1, 3.2, and 3.3 reduce to the algorithms of Noor [16], Mangasarian [17], and Ahn [15], respectively.

ALGORITHM 3.4 [16]. For given $u_0 \in K$, compute

$$u_{n+1} = (1-\lambda)\, u_n + \lambda P_k[u_n - \rho T(u_n)],\qquad n = 0, 1, 2, \ldots,$$

where $0 < \lambda \leqslant 1$ and $\rho < 0$ is a constant.

ALGORITHM 3.5 [17]. For given $u_0 \in K$, compute

$$u_{n+1} = (1-\lambda)\, u_n + \lambda P_k[u_n - \rho E_n\{Mu_n + q + L_n(u_{n+1} - u_n)\}],\qquad n = 0, 1, 2, \ldots,$$

where $0 < \lambda \leqslant 1$ and $\rho > 0$ is a constant.

ALGORITHM 3.6 [15]. For given $u_0 \in K$, compute

$$u_{n+1} = (1-\lambda)\, u_n + \lambda P_k[u_n - \rho E\{Mu_n + q + L(u_{n+1} - u_n)\}],$$
$$u = 0, 1, 2, ...,$$

where $0 < \lambda \leqslant 1$ and $\rho > 0$ is a constant.

Ahn [15] has established the convergence criteria of Algorithm 3.6 for both the symmetric and non-symmetric matrix M by using the variational inequality approach, where Noor [16] has studied the convergence properties of Algorithm 3.4. For the case $\lambda = 1$, the convergence criteria has been investigated and considered by Ahn [18] and Pang [13] for Algorithms 3.4 and 3.5, respectively.

3. If the nonlinear transformation A is identically equal to zero, and $K = \mathbf{R}^n_+$, then Algorithms 3.2 and 3.3 are the same as considered and analyzed by Mangasarian [17] and Ahn [18], respectively.

ALGORITHM 3.7. [17]. For given $u_0 \geqslant 0$, compute

$$u_{n+1} = (1-\lambda)\, u_n + \lambda[u_n - \rho E_n\{Mu_n + q + L_n(u_{n+1} - u_n)\}]_+,$$
$$n = 0, 1, 2, ...,$$

for $0 < \lambda \leqslant 1$ and $\rho > 0$ is a constant.

ALGORITHM 3.8 [18]. For given $u_0 \geqslant 0$, compute

$$u_{n+1} = \lambda[u_n - \rho E\{Mu_n + q + L(u_{n+1} - u_n)\}]_+ + (1-\lambda)\, u_n,$$
$$n = 0, 1, 2, ...,$$

where $\rho > 0$ is a constant.

Concerning the convergence of Algorithms 3.7 and 3.8, Mangasarian [17] established a general convergence result under the crucial assumption that M is a symmetric matrix. However, it was Ahn [15, 18], who proved that, if the matrix M is nonsymmetric, then the sequences generated by Algorithms 3.7 and 3.8 converges to the unique solution of the linear complementarity problem (2.5).

4. Algorithm 3.3 reduces to the Jacobi method, if we take $K = \mathbf{R}^n$, $\lambda = 1$, $L = 0$, $\rho = 1$, $E = D^{-1}$ and $A(u) \equiv 0$, where D is the diagonal matrix of M.

In brief, Algorithms 3.1, 3.2, and 3.3 proposed in this paper are more general and include several previously known algorithms as special cases, which are mainly due to Cryer [13], Aganagic [34], Ahn [15, 18], Noor [16], Cottle and Goheen [37], and Pang [13].

4. Convergence Analysis

In this section, we consider the convergence properties of the suggested Algorithms 3.1 and 3.3. These properties are also compared with those of the related algorithms. We here only consider the special case, when $K=[o, b]$ is a closed convex set in $\mathbf{R}^n$. We rely on the projection operator P_k, which is defined as

$$P_k(u)=\arg\min_{v \in K} \|v-u\|.$$

If $K=\mathbf{R}^n_+$, then $(P_k(u))_i=\max\{o, u_i\}$, $i=1, 2, ..., n$. In our case, we have

$$(P_k(u))_i=(P_{[o,b]}(u))_i=\min\{\max(o, u_i), b_i\}, \qquad i=1, 2, ..., n.$$

For notational purpose, $P_{[o,b]}$ will be denoted as P_k. The operator P_k has the following properties.

Lemma 4.1 [15]. *For any u and v in $\mathbf{R}^n$,*

(i) $u \leqslant v$ *implies* $P_k(u) \leqslant P_k(v)$

(ii) $P_k(u)-P_k(v) \leqslant P_k(u-v)$

(iii) $P_k(u+v) \leqslant P_k(u)+P_k(v)$

(iv) $P_k(u)+P_k(-u) \leqslant |u|$; *with equality, if and only if* $-b \leqslant u \leqslant b$.

In additiion, we also need the following concepts. A real matrix $M \in \mathbf{R}^{n \times n}$ is said to be Z-matrix (a P-matrix), if it has non-positive off-diagonal entries (positive principal minors). A square matrix with non-positive off-diagonal elements and with a non-negative inverse is called an M-matrix. It can be shown that a matrix which is both a Z-matrix and P-matrix is an M-matrix, see [13]. *Given $M \in \mathbf{R}^{n \times n}$, we define its comparison matrix*

$$M_c=(C_{ij}) \text{ by } C_{ii}=|M_{ii}| \quad \text{and} \quad C_{ij}=-|M_{ij}|, \qquad i \neq j,\ i, j=1, 2, ..., n.$$

We now state and prove the main results of this section by modifying the technique of Ahn [15] *and Noor* [32].

Theorem 4.1. *Suppose that there exists a non-negative matrix $N \in \mathbf{R}^{n \times n}$ such that*

$$|A(u)-A(v)| \leqslant N|u-v|, \qquad \text{for all} \quad u, v. \tag{4.1}$$

If $\{u_{n+1}\}$ and $\{u_n\}$ are the sequences generated by Algorithm 3.3, then

$$|u_{n+1}-u_n| \leqslant (I-\lambda\rho E|L|)|^{-1}\,[\lambda\rho EN+|I-\lambda\rho E(M-L)|]\,|\,|u_n-u_{n-1}| \tag{4.2}$$

and

$$|u_{n+1} - u| \leqslant (I - \lambda\rho E|L|)^{-1} [\lambda\rho EN + |I - \lambda\rho E(M - L)|] \, |u_n - u| \quad (4.3)$$

for each n and u is the solution of mildly nonlinear complementarity problem (2.8). *Here L is either a strictly upper or lower triangular matrix.*

Proof. From Algorithm 3.3, we have

$$\begin{aligned} u_{n+1} - u_n = {} & (1 - \lambda)(u_n - u_{n-1}) \\ & + \lambda P_k[u_n - \rho E\{Mu_n + q + L(u_{n+1} - u_n) + A(u_n)\}] \\ & - \lambda P_k[u_n - \rho E\{Mu_{n-1} + q + L(u_n - u_{n-1}) + A(u_{n-1})\}], \\ = {} & (1 - \lambda)(u_n - u_{n-1}) + \lambda P_k[(u_n - u_{n-1}) - \rho E(M - L)(u_n - u_{n-1}) \\ & - \rho EL(u_{n+1} - u_n) - \rho E(A(u_n) - A(u_{n-1}))], \end{aligned}$$

Again applying Lemma 4.1 and the fact $P_k^2 = P_k$, we obtain by Lemma 4.1,

$$\begin{aligned} & P_k[(u_{n+1} - u_n) - (1 - \lambda)(u_n - u_{n-1})] \\ & \quad \leqslant \lambda P_k[(u_n - u_{n-1}) - \rho E(M - L) \\ & \qquad \times (u_n - u_{n+1}) - \rho EL(u_{n+1} - u_n) - \rho E(A(u_n) - A(u_{n-1}))]. \end{aligned} \quad (4.4)$$

In a similar way, we have

$$\begin{aligned} & P_k[-\{(u_{n+1} - u_n) - (1 - \lambda)(u_n - u_{n-1})\}] \\ & \quad \leqslant \lambda P_k[-(u_n - u_{n-1}) + \rho E(M - L)(u_n - u_{n-1}) \\ & \qquad + \rho EL(u_{n+1} - u_n) + \rho E(A(u_n) - A(u_{n-1}))]. \end{aligned} \quad (4.5)$$

Adding the inequalities (4.4) and (4.5) and using Lemma 4.1, we have

$$\begin{aligned} & |u_{n+1} - u_n - (1 - \lambda)(u_n - u_{n-1})| \\ & \quad \leqslant |I - \rho E(M - L)| \, |u_n - u_{n-1}| \\ & \qquad + \lambda\rho E|L| \, |u_{n+1} - u_n| + \lambda\rho E|A(u_n) - A(u_{n-1})| \end{aligned}$$

from which we obtain

$$\begin{aligned} & (I - \lambda\rho E|L|) \, |u_{n+1} - u_n| \\ & \quad \leqslant |I - \lambda\rho E(M - L)| \, |u_n - u_{n-1}| + \lambda\rho E \, |A(u_n) - A(u_{n-1})| \\ & \quad = [\lambda\rho EN + |I - \lambda\rho E(M - L)|] \, |u_n - u_{n-1}|, \text{ by (4.1)}. \end{aligned}$$

Since L is either a strictly upper or lower triangular matrix, so the matrix

$(I-\lambda\rho E|L|)$ is invertible for $0<\lambda\leqslant 1$ and its inverse is non-negative, which implies that $(I-\lambda\rho E|L|)$ is an M-matrix. Thus, we have

$$|u_{n+1}-u_n|\leqslant (I-\lambda\rho E|L|)^{-1}\,[\lambda\rho EN+|I-\lambda\rho E(M-L)|]\,|u_n-u_{n-1}|,$$

which is the required result (4.2). In a similar way, we can obtain (4.3).

Theorem 4.1 enables us to establish a sufficient condition for the convergence of the sequence $\{u_{n+1}\}$ generated by Algorithm 3.3 to be bounded and hence have an accumulation point, which is the solution of the mildly nonlinear complementarity problem (2.8), and this is the main motivation of our next result.

THEOREM 4.2 *Assume that*

$$\sigma(G)<1,$$

where

$$G=(I-\lambda\rho E|L|)^{-1}\,[\lambda\rho EN+|I-\lambda\rho E(M-L)|], \tag{4.6}$$

with σ denoting the spectral radius. Then for any initial vector u, the sequence $\{u_{n+1}\}$ generated by Algorithm 3.3 *converges to a solution of* (2.8).

The proof is not given, because its derivation from Theorem 4.1 is identical to the proof of Theorem 4.1 of Ahn [18].

Remark 4.1. The results obtained in this paper are more general than the ones given in Ahn [15] and Pang [13]. In fact, for $A(u)\equiv 0$, the non-negative matrix N becomes the zero matrix and consequently our results are exactly the same as proved by Ahn [15] and Mangasarian [17] for the nonsymmetric and symmetric matrix M, respectively. For $\lambda=1$, and $A(u)\equiv 0$, our results reduce to the result of Ahn [18] and Pang [13] for the nonsymmetric linear complementarity problem (2.5). Thus we conclude that the algorithms of Mangasarian type can be extended to study the mildly nonlinear complementarity problem (2.8). Furthermore, it is clear from Theorem 4.2, that condition $\sigma(G)<1$ provides the existence and uniqueness result for the complementarity problem (2.8). Note that our results hold for both the symmetric and nonsymmetric matrix M.

It is evident that the convergence analysis of Algorithm 3.3 holds only for $K=[o,b]$. The question arises can this restriction be relaxed. The answer to this is partly true. Indeed, this is true for the strongly nonlinear complementarity problem (2.6) as shown below. For the mildly nonlinear complementarity problem (2.8), the question still remains open.

It can be shown using the technique of Pang [13] that condition (4.1) for the nonlinear transformation A is equivalent to the fact that transformation A is Lipshitz continuous; that is, there exists a constant $\gamma > 0$ such that

$$\|A(u) - A(v)\| \leqslant \gamma \|u - v\|, \qquad \text{for all } u, v. \tag{4.7}$$

We also need the following concepts.

DEFINITION 4.1. A mapping $T: \mathbf{R}^n \to \mathbf{R}^n$ is said to be

(i) *strongly monotone*. If there exists a constant $\alpha > 0$ such that

$$(Tu - Tv, u - v) \geqslant \alpha \|u - v\|^2, \qquad \text{for all} \quad u, v \in \mathbf{R}^n.$$

(ii) *Lipschitz continuous*. If there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leqslant \beta \|u - v\|, \qquad \text{for all} \quad u, v \in \mathbf{R}^n.$$

In particular, it follows that $\alpha \leqslant \beta$.

In the next theorem, we study the conditions under which the approximate solution obtained from Algorithm 3.1 converges to the exact solution u of the strongly nonlinear complementarity problem (2.7). This result also shows that the convergence of the approximate solution to the exact solution depends upon the relaxation parameter λ used after projection, like the mildly nonlinear complementarity problem (2.8). At the same time, we prove that the convergence analysis for the strongly nonlinear complementarity problem (2.7) holds for any general closed convex set K in $\mathbf{R}^n$.

THEOREM 4.3. *Let the continuous mapping T from $\mathbf{R}^n$ into itself be strongly monotone and Lipschitz continuous. If the nonlinear transformation A is Lipschitz continuous with Lipschitz constant γ such that $\gamma < \alpha$, where α is the strongly monotonic constant, then*

$$u_{n+1} \to u \qquad \text{strongly in } \mathbf{R}^n,$$

for $0 < \rho < 2(\alpha - \gamma)/(\beta^2 - \gamma^2)$, $\gamma\rho < 1$, *and* $\lambda \leqslant 1/(1 - \gamma\rho + (2\alpha\rho - \beta^2\rho^2 - 1)^{1/2})$, *where β is the Lipschitz constant of T, u and u_{n+1} are the solutions of* (2.7) *and* (3.2), *respectively.*

Proof. By Lemmas 3.1 and 3.2, we know that the solution u satisfying (2.7) is also a solution of (3.1) and conversely. Thus from (3.1) and (3.2), we obtain

$$\begin{aligned}\|u_{n+1}-u\| &= \|(1-\lambda)\,u_n+\lambda P_k[u_n-\rho(Tu_n+A(u_n))]\\ &\quad -(1-\lambda)\,u-\lambda P_k[u-\rho(Tu+A(u))]\|\\ &\leqslant \|(1-\lambda)(u_n-u)\|+\lambda\|(u_n-u)\\ &\quad -\rho(Tu_n-Tu)-\rho(A(u_n)-A(u))\|\end{aligned}$$

by the nonexpansiveness of the operator P_k, see [5, 30]:

$$\begin{aligned}\|u_{n+1}-u\| &\leqslant (1-\lambda)\|u_n-u\|+\lambda\|(u_n-u)-\rho(Tu_n-Tu)\|\\ &\quad +\lambda\rho\|A(u_n)-A(u)\|\\ &\leqslant ((1-\lambda)+\lambda\rho\gamma)\|u_n-u\|\\ &\quad +\lambda\|(u_n-u)-\rho(Tu_n-T_u)\|, \qquad \text{by (4.7)}\end{aligned}$$

Now by the strongly monotonic and Lipschitz continuity of T, we have

$$\|(u_n-u)-\rho(Tu_n-Tu)\|^2\leqslant(1-2\alpha\rho+\beta^2\rho^2)\ \|u_n-u\|^2.$$

Thus using the above inequality, we obtain

$$\|u_{n+1}-u\|=\theta\|u_n-u\|,$$

where $\theta=((1-\lambda)+\lambda\gamma\rho+\lambda\sqrt{1-2\alpha\rho+\beta^2\rho^2})<1$ for $0<\rho<2(\alpha-\gamma)/(\beta^2-\gamma^2)$, $\rho\gamma<1$, and $\lambda\leqslant 1/(1-\rho\gamma-(1-2\alpha\rho+\beta^2\rho^2)^{1/2})$.

Since $\theta<1$, so the fixed point problem (3.1) has a unique solution u and consequently the Picard iterates u_{n+1} converge to the solution u of (2.7) by the Banach–Picard theorem [38], which is the required result.

5. Applications

A large number of moving and free boundary problems can be formulated as (MNCP) mildly nonlinear complementarity problem (2.8). For simplicity, we consider the MNCP of finding u such that

$$\begin{aligned}-\Delta u(\mathbf{x})+f(\mathbf{x},u(\mathbf{x}))+h(\mathbf{x}) &\geqslant 0 && \text{in } D\\ u(\mathbf{x})&\geqslant 0, && \text{in } D\\ u(\mathbf{x})[-\Delta u(\mathbf{x})+f(\mathbf{x},u(\mathbf{x})))+h(\mathbf{x})]&=0, && \text{in } D\\ u(\mathbf{x})&=g(\mathbf{x}) && \text{on } S,\end{aligned} \tag{5.1}$$

where D is a domain in R^2 with boundary S, Δ is the Laplacian operator, f is a given nonlinear function of $\mathbf{x}$ and $u(\mathbf{x})$, h and g are also given functions.

The problem (5.1) is a generalization of the linear complementarity problem (LCP) of type

$$\begin{aligned} -\Delta u(\mathbf{x})+f(\mathbf{x}) &\geqslant 0 && \text{in } D \\ u(\mathbf{x}) &\geqslant 0 && \text{in } D \\ u(\mathbf{x})[-\Delta u(x)+f(\mathbf{x})] &= 0 && \text{in } D \\ u(\mathbf{x}) &= g(\mathbf{x}) && \text{on } S, \end{aligned} \tag{5.2}$$

for which f is a given function only of space variables, have been studied extensively by many authors including Lin and Cryer [35], Crank [8], Baiocchi and Capelo [5], Cottle [28], and Noor [16], using the variational inequality and complementarity techniques. Well-known examples of free boundary problems, which may be written in the form (5.1) and (5.2) include fluid flow through porous media, journal bearing lubrication problems, contact problems in elasticity.

Problems of the type (5.1) arise in gas dynamics, solid state physics, chemical diffusion, etc., where $f(u) \equiv f(\mathbf{x}, u(x))$ is of the form e^{-u}, $e^{u}-1$, u^{n}, $n \geqslant 2$, see Noor [32] for further details.

When problem (5.1) is approximated using finite differences (finite element), one obtains finite-dimensional MNCP, which may be written in the matrix form

$$\begin{aligned} Mu+q+A(u) &\geqslant 0 \\ u &\geqslant 0 \\ (u, Mu+q+A(u)) &= 0, \end{aligned} \tag{5.3}$$

where M is an $n \times n$ matrix, u is the n-vector, and $A(u)+q$ is obtained from $f(\mathbf{x}, u(\mathbf{x}))+h(\mathbf{x})$ by evaluating it at the grid points, see [42].

In a similar way, one obtains the finite-dimensional LCP, which can be written in the matrix form as

$$\begin{aligned} Mu+q &\geqslant 0 \\ u &\geqslant 0 \\ (u, Mu+q) &= 0. \end{aligned} \tag{5.4}$$

There is an extensive literature on the finite-dimensional LCP. In particular, if the matrix M is symmetric and positive definite, as in the case of the Laplacian operator Δ, then the problem (5.3) are equivalent to:

(i) Find a column $n-$ vector u which minimizes

$$I[u]=\tfrac{1}{2}(Mu, u)+(b, u)+F(u) \tag{5.5}$$

subject to the constraints $u \geqslant 0$ and $(A(u), v) = (F'(u), v)$, where $F'(u)$ is the Fréchet differential of $F(u) \equiv \int_D \int_0^u f(\eta)\, d\eta\, dD$, see Noor and Whiteman [27] and Noor [1].

(ii) There exists a unique solution to the nonlinear programming problem $I[u]$, defined by (5.5) and hence to the MNLC (5.3).

In order to apply the results of Sections 3 and 4, it is sufficient to verify the assumptions of Theorems 4.1 and 4.2. In fact, $A(u)$ is a diagonal nonlinearity and satisfies condition (4.1), see Kannan and Ray [43]. Further the matrix M is symmetric and positive definite, so Algorithm 3.3 can be used to compute the approximate solution of MNCP (5.3). In brief, all the results above can be applied to our model problem (5.1). For the applications of LCP (5.4), see Cottle [28], Lin and Cryer [35], and the references therein.

6. Conclusion

In this paper, we have considered and studied a new class of complementarity problems, which includes the previously known ones as special cases. It is shown that the iterative algorithms of Mangasarian type for solving linear complementarity problem can be extended to process the mildly nonlinear complementarity problems. Algorithm 3.3 introduced and analyzed in this paper may be viewed as an extension of the algorithm of Mangasarian. Most of the convergence properties of Mangasarian's algorithm studied earlier are carried over to this new algorithm. Development and improvement of an implementable algorithm of this class of complementarity problem deserve further research efforts.

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