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Bijective Recurrences for Motzkin Paths

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Consider the lattice paths on $\mathbb{Z}^2$ with steps $(1,1)$, $(1,-1)$, and $(1,0)$. For $n \geq 2$, let E_n denote the set of such paths running from $(0,0)$ to $(n,0)$ and remaining strictly above the x-axis except initially and terminally. The cardinalities, $f_n = |E_n|$ for $n \geq 2$, are the Motzkin numbers, $1, 1, 2, 4, 9, 21, 51, 127, \ldots$. We define a bijection yielding the recurrence $(n+1)f_{n+1} = (2n-1)f_n + 3(n-2)f_{n-1}$, for $n \geq 3$. A modification of the bijection proves that the sum of the areas under the paths of E_n, denoted by A_n, satisfies $A_{n+1} = 2A_n + 3A_{n-1}$, for $n \geq 3$. A second modification yields a recurrence for a second moment for the paths of E_n which agrees with Euler's recurrence for the central trinomial numbers.

Key Words: lattice paths; Motzkin numbers; recurrences.

1. THE PATHS AND THEIR MOMENTS

Consider the lattice paths on $\mathbb{Z}^2$ whose permitted steps are the *up diagonal step* $(1,1)$ denoted by U, the *down diagonal step* $(1,-1)$ denoted by D, and the *horizontal step* $(1,0)$ denoted by H. Assign the weight of 1 to each U step, the weight of 1 to each D step, and the weight of the indeterminate t to each H step. The t-weight of a path P, denoted by $|P|$, is the product of the weights of its steps; the t-weight of a set of paths S, denoted by $|S|$, is the sum of the t-weights of the paths in S. Let $U(x,y)$ denote the set of all *unrestricted* paths running from $(0,0)$ to (x,y) and using the steps U, D, and H.

The set of *Motzkin paths* running from $(0,0)$ to $(n,0)$, denoted by M_n, consists of those paths in $U(n,0)$ that never run below the x-axis. Of particular interest is the set of *elevated Motzkin paths*, denoted by E_n and defined to be the set of those paths in M_n that never touch the x-axis except initially and finally. Figure 1 illustrates the elevated paths in E_5.

For $n \geq 2$ let f_n denote the t-weight of E_n; i.e., $f_n = |E_n|$. Hence, $(f_n)_{\geq 2} = (1, t, 1+t^2, 3t+t^3, 2+6t^2+t^4 \ldots)$ (see Table I). When $t = 1$, the sequence

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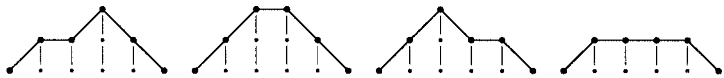

FIG. 1. The four elevated Motzkin paths of E_5 bound a total area of 20 units. Equivalently, the sum of the ordinates of their lattice points is equal to 20.

$(f_n)_{n\geq 2} = (1, 1, 2, 4, 9, 21, 51, 127, \ldots)$ is the well-known Motzkin numbers, named for Theodor Motzkin [6], who introduced them while counting all possible sets of nonintersecting chords joining some of n points on a circle. (In our notation, there are f_{n+2} sets of nonintersecting chords for n points.) Donaghey and Shapiro [2] made an early study of the Motzkin sequence in which they showed the sequence counting 14 different combinatorial objects, including Motzkin paths in M_n. Exercises 6.37, 6.38, and 6.46 of [8] give further information regarding this sequence. Putting $t = 0$ effectively disallows the horizontal steps and yields the Dyck (or Catalan) paths with $(f_n)_{n\geq 2}$ becoming the aerated Catalan numbers.

Given any lattice path P running from $(0, 0)$, express the path as a sequence of steps, $P = p_1 p_2 \cdots p_h \cdots p_n$. Define the *altitude*, or the *ordinate*, of the step p_h, denoted by $\text{ALT}(p_h)$, to be the ordinate of the step's terminus. For $n \geq 2$, we define the following sums of t-weighted moments for E_n that correspond respectively to the *cardinality*, the *sum of the mean altitudes*, and the *sum of the mean altitudes squared*:

$$f_n = \sum_{P\in E_n} |P| \tag{1}$$

$$g_n = \sum_{P\in E_n} \frac{|P|}{n-1} \sum_{h=1}^{n-1} \text{ALT}(p_h) \tag{2}$$

$$h_n = \sum_{P\in E_n} \frac{|P|}{n-1} \sum_{h=1}^{n-1} \text{ALT}(p_h)^2. \tag{3}$$

Some initial values for these sequences appear in Table II.

TABLE I
Contributions to $f_5 = 3t + t^3$, $g_5 = 4t + t^3$, $h_5 = 6t + t^3$ and the Total Area from the Four Paths of E_5

Path	Contribution to f_5	Contribution to g_5	Contribution to total area	Contribution to h_5
UHUDD	t	$5t/4$	5	$7t/4$
UUHDD	t	$6t/4$	6	$10t/4$
UUDHD	t	$5t/4$	5	$7t/4$
UHHHD	t^3	$4t^3/4$	4	$4t^3/4$

TABLE II
Some Initial Values for the Sequences (Recall That $g_n = A_n/(n-1)$)

t	n	2	3	4	5	6	7		
0	f_n	1	0	1	0	2	0	...	Aerated Catalan nos.
1	f_n	1	1	2	4	9	21	...	Motzkin nos.
2	f_n	1	2	5	14	42	132	...	Catalan nos.
0	A_n	1	0	4	0	16	0	...	Aerated powers of 4
1	A_n	1	2	7	20	61	182	...	
2	A_n	1	4	16	64	128	256	...	Powers of 4
0	h_n	1	0	2	0	6	0	...	Central binom. coefs.
1	h_n	1	1	3	7	19	51	...	Central trinom. coefs.
2	h_n	1	2	6	20	70	252	...	Central binom. coefs.

Our main purpose is to give bijective proofs of the recurrences for these moments as recorded in Proposition 1.1. As noted in Section 4, our proof of the third recurrences is not completely bijective.

PROPOSITION 1.1. *The sequences $(f_n)_{n\geq 2}$, $(g_n)_{n\geq 2}$, and $(h_n)_{n\geq 2}$ satisfy*

$$(n+1)f_{n+1} = (2n-1)tf_n + (n-2)(4-t^2)f_{n-1}, \tag{4}$$

$$ng_{n+1} = (2n-2)tg_n + (n-2)(4-t^2)g_{n-1}, \tag{5}$$

$$(n-1)h_{n+1} = (2n-3)th_n + (n-2)(4-t^2)h_{n-1}, \tag{6}$$

for $n \geq 3$, subject to $f_2 = g_2 = h_2 = 1$ and $f_3 = g_3 = h_3 = t$.

By the elementary formula for the area of a trapezoid, the area under any path P of E_n and above the x-axis, denoted $A(P)$, is equal to the sum of the ordinates of P. Thus, if A_n denotes $\sum_{P\in E_n} |P| A(P)$, i.e., the total t-weighted area under the paths of E_n, we have that $A_n = \sum_{P\in E_n} |P| \sum_{i=1}^{n} \mathrm{ALT}(p_i)$. Hence, the recurrence in (5) is equivalent to that in (7) below. Formula (8) follows by induction.

PROPOSITION 1.2. *The sequence $(A_n)_{n\geq 2}$ for the sum of the t-weighted areas under the paths of $(E_n)_{n\geq 2}$ satisfies*

$$A_{n+1} = 2tA_n + (4-t^2)A_{n-1}, \tag{7}$$

for $n \geq 3$, subject to $A_2 = 1$ and $A_3 = 2t$. Moreover, for $n \geq 2$,

$$A_n = \frac{(2+t)^{n-1} + (2-t)^{n-1}}{4}. \tag{8}$$

In Table II, when $t = 0$, we are surprised by the apparent equality of the sequence $(h_n)_{n\geq 2}$ with the familiar sequence of the central binomial coefficients. Likewise, when $t = 1$, we find that $(h_n)_{n\geq 2}$ appears to be the central trinomial coefficients, which clearly count the unrestricted paths terminating on the x-axis. Indeed, when $t = 1$, recurrence (6) has the same form as

$$(n+1)|U(n+1,0)| = (2n+1)|U(n,0)| + 3n|U(n-1,0)|.$$

This recurrence for the trinomial coefficients dates from 1764 when it was found by Euler [4]. In Section 4 we will show that h_n satisfies another recurrence that is also satisfied by $|U(n-2,0)|$. Hence it follows that:

PROPOSITION 1.3. *For any integer $n \geq 2$, the sum of the t-weighted means of the squared altitudes of the elevated Motzkin paths relates to the t-weighted cardinality of unrestricted paths by*

$$h_n = |U(n-2,0)|. \tag{9}$$

Foata and Zeilberger's [5] combinatorial proof of a recurrence for the Schröder numbers sparked this work. Our paper furthers the studies made in [7, 9, 10, 12] for other recurrences for the cardinality and for the total area of structures associated with the Catalan and Schröder numbers. Barcucci, Pinzani, and Sprugnoli [1] used generating functions to derive various recurrences related to Motzkin paths. The author [11] used generating functions to obtain a generalization of Proposition 1.1 by allowing the horizontal steps to have any fixed integer length. The resulting recurrences of [11] unify the recurrences for the moments associated with Dyck, Motzkin, and Schröder paths.

2. A BIJECTIVE RECURRENCE FOR THE CARDINALITY

The key to our combinatorial proof of the recurrence in (4) for the cardinality of elevated paths, namely,

$$(n+1)f_{n+1} = (2n-1)tf_n + (n-2)(4-t^2)f_{n-1},$$

for $n \geq 3$, is to reformulate this recurrence as

$$ntf_n + (n-1)tf_n + 4(n-2)f_{n-1} = (n+1)f_{n+1} + (n-2)t^2 f_{n-1}. \tag{10}$$

Correspondingly, we will define a map, and its domain and codomain, denoted as

$$\begin{aligned} \mu\colon [n] \times E_n \times \{H\} \cup [n-1] \times E_n \times \{H\} \cup [4(n-2)] \times E_{n-1} \\ \to [n+1] \times E_{n+1} \cup [n-2] \times E_{n-1} \times \{HH\} \end{aligned} \tag{11}$$

that is bijective and reduces to Eq. (10) when we account for weights of the sets of paths.

We begin by defining the set products appearing in the domain and codomain of (11). Define $[n] \times E_n \times \{H\}$ to be the set of the distinguishable copies of the paths in E_n where each copy has one step marked by the superscript a and has an appended t-weighted H step.

For example, since $E_5 = \{UHHHD, UHUDD, UUHDD, UUDHD\}$,

$$
\begin{aligned}
&[5] \times E_5 \times \{H\} \\
&\quad = \{U^aHHHDH, \quad U^aHUDDH, \quad U^aUHDDH, \quad U^aUDHDH, \\
&\qquad UH^aHHDH, \quad UH^aUDDH, \quad UU^aHDDH, \quad UU^aDHDH, \\
&\qquad UHH^aHDH, \quad UHU^aDDH, \quad UUH^aDDH, \quad UUD^aHDH, \\
&\qquad UHHH^aDH, \quad UHUD^aDH, \quad UUHD^aDH, \quad UUDH^aDH, \\
&\qquad UHHHD^aH, \quad UHUDD^aH, \quad UUHDD^aH, \quad UUDHD^aH\}.
\end{aligned}
$$

Define $[n-1] \times E_n \times \{H\}$ to be the set of the distinguishable copies of the paths in E_n where each copy has one nonfinal step marked with superscript c and has an appended t-weighted H step.

Define $[4(n-2)] \times E_{n-1}$ to be the set of the distinguishable copies of the paths in E_{n-1} where each copy has one nonfinal step marked by the superscript a, b, c, or d.

Define $[n+1] \times E_{n+1}$ to be the set of distinguishable copies of the paths in E_{n+1} where each copy has one U step marked by A, one D step marked by B, or one H step marked by C.

Define $[n-2] \times E_{n-1} \times \{HH\}$ to be the set of distinguishable copies of the paths in E_{n-1} where each copy has one nonfinal step marked by superscript D and has an appended pair of H steps.

Next we define the map μ of (11) in three cases, each of which corresponds to one of the sets in the domain. In Case 1 and Case 2, a horizontal step is deleted and then reinserted.

Case 1. Suppose $P = p_1 p_2 \cdots p_j^a \cdots p_k \cdots p_n H \in [n] \times E_n \times \{H\}$. There are three subcases:

Case 1.1. If $p_j^a = U^a$ and p_k is the first D step following p_j for which $\text{ALT}(p_k) = \text{ALT}(p_j) - 1$, define

$$\mu(P) = p_1 p_2 \cdots p_j^{\mathrm{A}} \cdots H p_k \cdots p_n \in [n+1] \times E_{n+1}.$$

Here the appended H step on P is moved so that it immediately precedes the step p_k. The top pair of Fig. 2 illustrates this case: there $\mu(UU^aHUDDDH) = UU^{\mathrm{A}}HUDHDD$, with $p_j^a = p_2^a = U^a$ and $p_k = p_6 = D$. In the figure, ignore the vertical segments until Section 3.

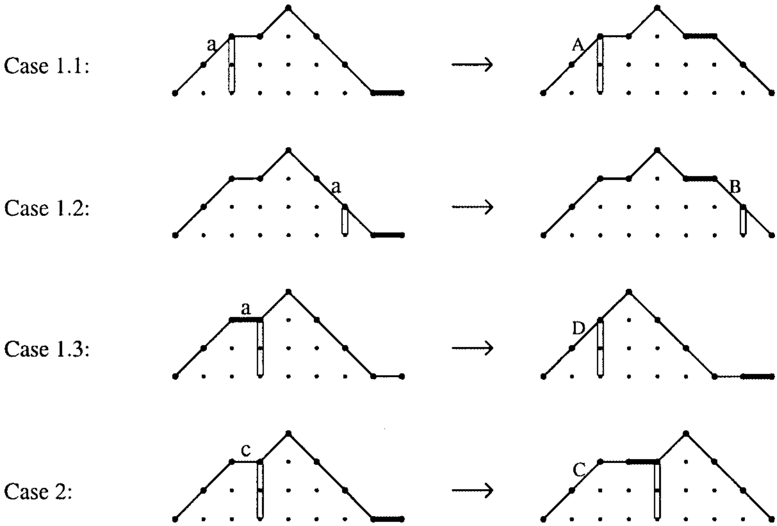

FIG. 2. This illustrates the map μ under Cases 1.1, 1.2, 1.3, and 2. The vertical segments are used in Section 3 where they show how area is transferred.

Case 1.2. If $p_j = D^a$, define

$$\mu(P) = p_1 p_2 \cdots p_{j-1} H p_j^{\mathrm{B}} \cdots p_n \in [n+1] \times E_{n+1}.$$

Case 1.3. If $p_j = H^a$, define

$$\mu(P) = p_1 p_2 \cdots p_{j-1}^{\mathrm{D}} p_{j+1} \cdots p_n HH \in [n-2] \times E_{n-1} \times \{HH\}.$$

Here the labeled H step moves to the end of the path and the labeling moves to the previous step.

Figure 2 show examples for Cases 1.2 and 1.3: For examples of Cases 1.2 and 1.3, $\mu(UUHUDD^aDH) = UUHUDHD^{\mathrm{B}}D$ and $\mu(UUH^aUDDDH) = UU^{\mathrm{D}}UDDDHH$.

Case 2. Suppose that $P = p_1 p_2 \cdots p_j^c \cdots p_k \cdots p_n H \in [n-1] \times E_n \times \{H\}$. Define

$$\mu(P) = p_1 p_2 \cdots p_j H^{\mathrm{C}} p_{j+1} \cdots p_n \in [n+1] \times E_{n+1}.$$

Here the appending H step on P is moved to the $(j+1)$th position of $\mu(P)$ and receives a label, as shown in the bottom correspondence of paths of Fig. 2.

Case 3. Suppose that $R = r_1 r_2 \cdots r_j^{\ell} \cdots r_k \cdots r_{n-1} \in [4(n-2)] \times E_{n-1}$.

Case 3.1. If $\ell = a$, define

$$\mu(R) = r_1 r_2 \cdots r_j U^{\mathrm{A}} D r_{j+1} \cdots r_{n-1} \in [n+1] \times E_{n+1}.$$

Case 3.2. If $\ell = b$, define

$$\mu(R) = r_1 r_2 \cdots r_j U D^{\mathrm{B}} r_{j+1} \cdots r_{n-1} \in [n+1] \times E_{n+1}.$$

Case 3.3. If $\ell = c$, let r_k be the first D step following r_j^c for which $\mathrm{ALT}(r_k) = \mathrm{ALT}(r_j) - 1$, and let r_i be the last U step preceding r_{j+1} for which $\mathrm{ALT}(r_i) = \mathrm{ALT}(r_j)$. (When $r_j = U$, $i = j$.) Define

$$\mu(R) = r_1 r_2 \cdots r_i^{\mathrm{A}} \cdots r_j U R' D r_k \cdots r_{n-1} \in [n+1] \times E_{n+1},$$

where $R' = r_{j+1} \cdots r_{k-1}$ if $j + 1 < k$ and R' is empty if $j + 1 = k$.

Case 3.4. If $\ell = d$, let r_k be the first D step following r_j^d for which $\mathrm{ALT}(r_k) = \mathrm{ALT}(r_j) - 1$. Define

$$\mu(R) = r_1 r_2 \cdots r_j U R' D r_k^{\mathrm{B}} \cdots r_{n-1} \in [n+1] \times E_{n+1},$$

where $R' = r_{j+1} \cdots r_{k-1}$ if $j + 1 < k$ and R' is empty if $j + 1 = k$.

For examples of Case 3 see Fig. 3, which illustrates the following four mappings:

$$\mu(UUUD^a HDD) = UUUDU^{\mathrm{A}} DHDD,$$
$$\mu(UUUD^b HDD) = UUUDUD^{\mathrm{B}} HDD,$$
$$\mu(UUUD^c HDD) = UUU^{\mathrm{A}} DUHDDD,$$
$$\mu(UUUD^d HDD) = UUUDUHDD^{\mathrm{B}} D.$$

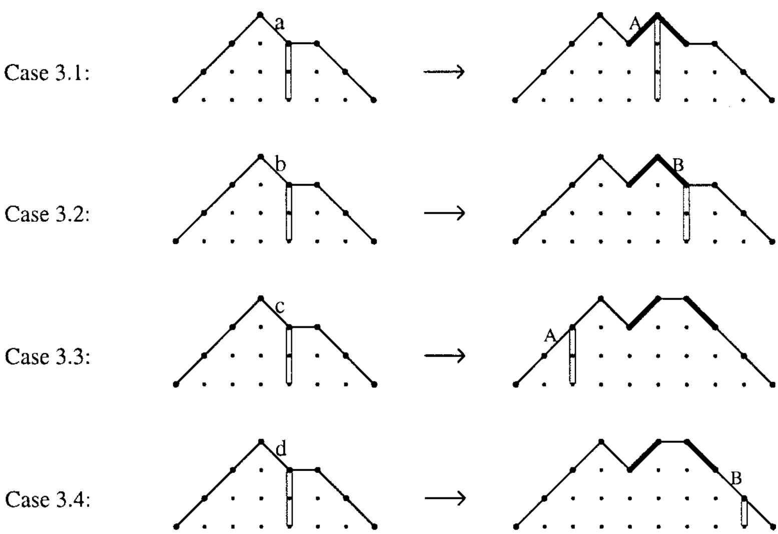

FIG. 3. This illustrates Case 3. The vertical segments are used in Section 3 to show how *total* area is transferred.

To see that μ is indeed bijective we note that both the domain and the codomain of (11) can be partitioned into collections of eight subsets as follows. The eight subsets of the domain correspond to Cases 1.1 through 3.4. The eight subsets of the codomain correspond Cases (i) to (viii) considered below. We then check that each of the subsets of the codomain can be bijectively matched to a unique subset from the partition of the domain.

First suppose that $Q = q_1 q_2 \cdots q_{n+1} \in [n+1] \times E_{n+1}$. If the labeled step is a U step, let q_m be the first D step following the labeled step q_i^{A} for which $\mathrm{ALT}(q_m) = \mathrm{ALT}(q_i) - 1$.

(i) If $q_{m-1} = H$, then $Q = \mu(P)$ under Case 1.1.

(ii) If $q_{m-1} = U$, then $m - 1 = i$ and $Q = \mu(R)$ under Case 3.1.

(ii) If $q_{m-1} = D$, let q_j be the last step preceding q_{m-1} such that $\mathrm{ALT}(q_j) = \mathrm{ALT}(q_{m-1})$. Then, under Case 3.3, we have $\mu(R) = Q$ where R satisfies $r_1 \cdots r_j = q_1 \cdots q_j$ with r_j being the labeled step of R.

If the labeled step of Q is a D step q_m, we observe that

(iv) If $q_{m-1} = H$, then $Q = \mu(P)$ under Case 1.2.

(v) If $q_{m-1} = U$, then $Q = \mu(R)$ under Case 3.2.

(vi) If $q_{m-1} = D$, let q_j be the last step preceding q_{m-1} such that $\mathrm{ALT}(q_j) = \mathrm{ALT}(q_{m-1})$. Then, under Case 3.4, we have $\mu(R) = Q$ where P satisfies $r_1 \cdots r_j = q_1 \cdots q_j$ with r_j being the labeled step of R.

If the labeled step of Q is an H step,

(vii) then $Q = \mu(P)$ under Case 2.

For the remaining case:

(viii) if $R \in [n-2] \times E_{n-1} \times \{HH\}$, then $R = \mu(P)$ under Case 1.3.

3. THE RECURRENCE FOR THE AREA

We will use the map of the previous section to construct a bijective recurrence related to the total weighted area $A_n = \sum_{P \in E_n} |P| \sum_{i=1}^{n} \mathrm{ALT}(p_i)$. Our t-weight preserving bijection is based on the observation that, in most cases, the altitude of the labeled step on a path P is equal to the altitude of the labeled step on $\mu(P)$. The exceptions are the labeled steps r^a and r^d under Cases 3.1 and 3.4.

For each step p_j of each path P, let $[p_j]$ denote the line segment from $(j, 0)$ to $(j, \mathrm{ALT}(p_i))$. The vertical segments in Figs. 2 and 3 represent such segments. For any segment $[p_j]$, define the *t-weight of a segment* to be $|P|\mathrm{ALT}(p_j)$. The *t-weight of a set of segments* will be the sum of the t-weights of its members. The distributive law ensures that this definition agrees with the definition of the weighted area A_n. We define the following collections of segments:

$$V_\ell = \{[p_j^\ell]: 1 \le j \le n-1, P \in E_n \times \{H\}\} \quad \text{for } \ell \in \{a, c\},$$
$$W_\ell = \{[r_j^\ell]: 1 \le j \le n-2, R \in E_{n-1}\} \quad \text{for } \ell \in \{a, b, c, d\},$$
$$X = \{[q_j]: 1 \le j \le n, Q \in E_{n+1}\},$$
$$Y = \{[r_j]: 1 \le j \le n-2, R \in E_{n-1} \times \{HH\}\}.$$

For example, referring to Figs. 2 and 3 we find 16 vertical segments which can be represented sequentially as they appear by $[p_2^a] \in V_a, [q_2] \in X, [p_6^a] \in V_a, [q_7] \in X, [p_3^a] \in V_a, [r_2] \in Y, [p_3^c] \in V_c, [q_4] \in X, [r_4^a] \in W_a, [q_5] \in X, [r_4^b] \in W_b, [q_6] \in X, [r_4^c] \in W_c, [q_2] \in X, [r_4^d] \in W_d, [q_8] \in X$.

We will usually omit the exponent labeling in our segment notation as the context will indicate when $[p_j]$ connotes $[p_j^a]$ or $[p_j^c]$, etc. For $n \ge 3$, the recurrences in (5) and (7) are equivalent to

$$2tA_n + 4A_{n-1} = A_{n+1} + t^2 A_{n-1}. \tag{12}$$

Using the notation for the map μ, we define a map

$$\phi: V_a \cup V_c \cup W_a \cup W_b \cup W_c \cup W_d \to X \cup Y \tag{13}$$

that is bijective, t-weight transferring, and hence establishes (12).

Case 1. For $[p_j] = [p_j^a] \in V_a$,

$$\phi([p_j]) = [q_j] \qquad \text{if } p_j = U,$$
$$\phi([p_j]) = [q_{j+1}] \qquad \text{if } p_j = D,$$
$$\phi([p_j]) = [q_{j-1}] \qquad \text{if } p_j = H,$$

where $Q = \mu(p_1 \cdots p_j^a \cdots p_n H)$. Here ϕ preserves the t-weight since

$$\text{ALT}(q_j) = \text{ALT}(p_j) \qquad \text{if } p_j = U,$$
$$\text{ALT}(q_{j+1}) = \text{ALT}(p_j) \qquad \text{if } p_j = D,$$
$$\text{ALT}(q_{j-1}) = \text{ALT}(p_{j-1}) = \text{ALT}(p_j) \qquad \text{if } p_j = H.$$

(The vertical segments of Fig. 2 shows how area is preserved.)

Case 2. For $[p_j] = [p_j^c] \in V_c$,

$$\phi([p_j]) = [q'_{j+1}],$$

where $Q' = \mu(p_1 \cdots p_j^c \cdots p_n H)$. For this case it is easily checked that ϕ preserves the t-weight.

Case 3. For segments in $W_a \cup W_b \cup W_c \cup W_d$, in order to prove that ϕ preserves the total t-weight, it is convenient to map segments as quadruples. For the quadruple

$$([r_j],[r_j],[r_j],[r_j]) = ([r_j^a],[r_j^b],[r_j^c],[r_j^d]) \in W_a \times W_b \times W_c \times W_d,$$

define

$$(\phi([r_j]),\phi([r_j]),\phi([r_j]),\phi([r_j])) = ([q_{j+1}],[q'_{j+2}],[q''_i],[q'''_{k+2}]), \qquad (14)$$

where

$$Q = \mu(r_1 \cdots r_j^a \cdots r_{n-1}), \qquad Q' = \mu(r_1 \cdots r_j^b \cdots r_{n-1}),$$
$$Q'' = \mu(r_1 \cdots r_j^c \cdots r_{n-1}), \qquad Q''' = \mu(r_1 \cdots r_j^d \cdots r_{n-1}).$$

Here the map ϕ *preserves the total* t-weight since $\mathrm{ALT}(q_{j+1}) = \mathrm{ALT}(r_j)+1$, $\mathrm{ALT}(q'_{j+2}) = \mathrm{ALT}(r_j)$, $\mathrm{ALT}(q''_i) = \mathrm{ALT}(r_j)$, and $\mathrm{ALT}(q'''_{k+2}) = \mathrm{ALT}(r_j)-1$, and thus the deficit of $\mathrm{ALT}(q'''_{k+2})$ cancels the surplus of $\mathrm{ALT}(q_{j+1})$. (See Fig. 3.)

4. THE RECURRENCE FOR THE SECOND MOMENT

To obtain recurrence (6) for the second moment we will first establish the mixed recurrence

$$nh_{n+1} = 2(n-1)th_n + (n-2)(4-t^2)h_{n-1} + 2(n-2)f_{n-1}, \qquad (15)$$

for $n \geq 3$, subject to the aforementioned initial conditions, by considering the rearranged form:

$$2(n-1)th_n + 4(n-2)h_{n-1} + 2(n-2)f_{n-1} = nh_{n+1} + (n-2)t^2h_{n-1}. \qquad (16)$$

The *t-altitude squared* of a segment $[p_j]$ will be defined as $|P|\mathrm{ALT}(p_j)^2$. Observe that $(n-1)h_n = \sum_{P\in E_n}\sum_{1\leq h\leq n-1}|P|\mathrm{ALT}(p_j)^2$ and hence is a sum of t-altitudes squared. In order to engage the term $2(n-2)f_{n-1}$ into the argument, we will introduce a collection of pairs consisting of a segment and a step: let

$$T = \{([r_j],r_j)\colon 1 \leq j \leq n-2, R \in E_{n-1}\}.$$

The *t-altitude squared of any step* (as opposed to a *segment*) of the path R will be defined to be the constant $2|R|$.

Keeping the notions of the previous sections, we now define a map

$$\psi\colon V_a \cup V_c \cup W_a \cup W_b \cup W_c \cup T \to X \cup Y \qquad (17)$$

that is bijective and transfers the sum of the t-altitudes squared.

On $V_a \cup V_c$ define the map ψ as the map ϕ in Cases 1 and 2 of Section 3 with $\text{ALT}(\cdot)^2$ replacing $\text{ALT}(\cdot)$. Since there is no change in the altitudes under ϕ on $V_a \cup V_c$, the map ψ preserves the sum of the t-altitudes squared on $V_a \cup V_c$.

On $W_a \cup W_b \cup W_c \cup T$, define

$$(\psi([r_j]), \psi([r_j]), \psi([r_j]), \psi(([r_j], r_j))) = ([q_{j+1}], [q''_{j+2}], [q''_j], [q'''_{k+2}]),$$

where the right side is defined in (14). Here the map ψ preserves the sum of the t-altitudes squared for each quintuple, $([r_j], [r_j], [r_j], [r_j], r_j)$, since

$$\begin{aligned} &4 \cdot \text{ALT}(r_j)^2 + 2 \\ &\quad = (\text{ALT}(r_j) + 1)^2 + (\text{ALT}(r_j))^2 + (\text{ALT}(r_j))^2 + (\text{ALT}(r_j) - 1)^2 + 2 \\ &\quad = \text{ALT}(q_{j+1})^2 + \text{ALT}(q'_{j+2})^2 + \text{ALT}(q''_j)^2 + \text{ALT}(q'''_{k+2})^2. \end{aligned}$$

The bijection of line (17) reduces to

$$\begin{aligned} &2 \sum_{P \in E_n} \sum_{j=1}^{n-1} |P| t(\text{ALT}(p_j))^2 + 4 \sum_{R \in E_{n-1}} \sum_{j=1}^{n-2} |R| (\text{ALT}(r_j))^2 + \sum_{R \in E_{n-1}} \sum_{j=1}^{n-2} 2|R| \\ &\quad = \sum_{Q \in E_{n+1}} \sum_{j=1}^{n} |Q| (\text{ALT}(q_j))^2 + \sum_{R \in E_{n-1}} \sum_{j=1}^{n-2} |R| t^2 (\text{ALT}(r_j))^2 \end{aligned}$$

which in turn reduces to (16) and hence (15).

We consider the proof of the recurrence for $(h_n)_{n \geq 2}$ in (6), which we restate for convenience by

$$(n-1)h_{n+1} = (2n-3)th_n + (n-2)(4-t^2)h_{n-1}. \tag{18}$$

Lacking a purely bijective proof of this recurrence, we use the mixed recurrence of (15) with the recurrence for $(f_n)_{n \leq 2}$ in (4) to obtain (18) by induction. Recurrence (18) and its initial conditions are easily checked for n, $2 \leq n \leq 3$. Assume, as the induction hypothesis, that (18) holds for all n where $4 \leq n < m$ for some $m > 4$. We verify the case for $n = m$, i.e., precisely the formula stated in (18), by a sequence of substitutions that use the recurrences (4), (15), and (18), where the last holds for $n < m$. We use a computer algebra program to manipulate our substitutions. Below we list the sequence of input statements for the substitution steps in *Maple*, with "`h(n)`" representing "h_n," etc. Notice that the first two lines correspond to the induction hypothesis. The sequence of statements for *Mathematica* is essentially the same.

```
h(n-3) := ((n-3)*h(n-1) - (2*n-7)*t*h(n-2))/(n-4)
          /(4-t*t):
```

```
h(n-2) := ((n-2)*h(n) - (2*n-5)*t*h(n-1))/(n-3)
          /(4-t*t):
f(n-3) := ((n-2)*h(n-1) - 2*(n-3)*t*h(n-2)
          - (n-4)*(4-t*t)*h(n-3))/(n-4)/2:
f(n-2) := ((n-1)*h(n) - 2*(n-2)*t*h(n-1)
          - (n-3)*(4-t*t)*h(n-2))/(n-3)/2:
f(n-1) := ((2*n-5)*t*f(n-2) + (n-4)*(4-t*t)
          *f(n-3))/(n-1):
print(simplify(f(n-1))):
h(n+1) := (2*(n-1)*t*h(n) + (n-2)*(4-t*t)*h(n-1)
          + 2*(n-2)*f(n-1))/n:
print(simplify(h(n+1))):
```

The statement `print(simplify(h(n+1)))` yields recurrence (18) for $n = m$, completing its proof. Parenthetically, as an unexpected by-product, the intermediate step corresponding to `print(simplify(f(n-1)))` yielded a recurrence found in [1], namely:

$$f_{n-1} = \frac{th_n + (4 - t^2)h_{n-1}}{2(n-1)}.$$

Now with recurrence in (18) established, we can use the following substitutions:

```
h(n-3) := ((n-3)*h(n-1) - (2*n-7)*t*h(n-2))
          /(n-4)/(4-t*t):
h(n-2) := ((n-2)*h(n) - (2*n-5)*t*h(n-1))
          /(n-3)/(4-t*t):
h(n-1) := ((n-1)*h(n+1) - (2*n-3)*t*h(n))
          /(n-2)/(4-t*t):
f(n-3) := ((n-2)*h(n-1) - 2*(n-3)*t*h(n-2)
          - (n-4)*(4-t*t)*h(n-3))/(n-4)/2:
f(n-2) := ((n-1)*h(n) - 2*(n-2)*t*h(n-1)
          - (n-3)*(4-t*t)*h(n-2))/(n-3)/2:
f(n-1) := ((2*n-5)*t*f(n-2) + (n-4)*(4-t*t)
          *f(n-3))/(n-1):
print(simplify(f(n-1))):
```

The statement `print(simplify(f(n-1)))` supplies the recurrence

$$f_{n-1} = \frac{h_{n+1} - th_n}{2(n-2)} \tag{19}$$

which suffices for Proposition 1.3, as we will see in the next section.

5. RELATING ALTITUDE SQUARED TO CENTRAL TRINOMIAL NUMBERS

In this section we complete the proof of Proposition 1.3. Our main combinatorial tool is the cycle lemma, introduced by Dvoretzky and Motzkin [3]. (Alternatively, one can use the André reflection principle as was done in [11]. However, it is appropriate that the counting of Motzkin paths use a result bearing Motzkin's name.) For completeness we establish a version of this lemma for cyclic permutation classes of unrestricted paths in $U(n,1)$. For any path $P = p_1 p_2 \cdots p_n \in U(n,1)$, let $\langle P\rangle$ denote the equivalence class of cyclic permutations of P.

LEMMA 5.1. *For any path P in $U(n,1)$ there is a unique Q in $\langle P\rangle$ such that the concatenation QD is an elevated path in E_{n+1}. Furthermore, the cyclic permutations of $\langle P\rangle$ are distinct, and hence $\langle P\rangle$ has n paths, each having the same t-weight.*

Proof. Let P be any path in $U(n,1)$. Since the ordinate of the terminal point of P is one greater than its starting point, the rightmost lowest point of the concatenation PP must occur at the initial point of a step, say p_i, in the first factor of the concatenation. Thus the path $Q = p_i p_{i+1} \cdots p_{i-1}$, which is a cyclic permutation of P and which we initiate at $(0,0)$, satisfies $QD \in E_{n+1}$.

Next we observe that, for this Q, there cannot be another path Q' in $\langle P\rangle = \langle Q\rangle$ such that $Q'D$ belongs to E_{n+1}. Let $Q = q_1 \cdots q_n$ and $Q' = q_j \cdots q_n \cdots q_{j-1}$ for $1 < j < n$. (Here $j \neq n$, since it must be that $q_j = U$, and $q_n \neq U$.) When Q' is superimposed on the concatenation QQ running from $(0,0)$, its initial ordinate must exceed 0, while the ordinate of q_n is 1. If we translate Q' to start at $(0,0)$, we see that Q' is not above the x-axis at q_n and hence $Q'D \notin E_{n+1}$.

Finally we show that the permutations in $\langle P\rangle = \langle Q\rangle$ are distinct. Suppose to the contrary, that $P = q_i \ldots q_{i-1}$ and $P' = q_j \ldots q_{j-1}$ belong to $\langle Q\rangle$, for $1 \leq i < j \leq n$ and that they are equal as paths when we initiate each at $(0,0)$. Then $Q = q_1 \ldots q_{i-1} q_i \ldots q_n$ and $q_{j-i+1} \ldots q_{j-1} q_j \ldots q_{j-i}$ are equal and belong to $\langle Q\rangle$, contradicting the uniqueness of Q. ∎

The lemma implies $(n-2)|E_{n-1}| = |U(n-2,1)|$, and hence

$$f_{n-1} = \frac{|U(n-2,1)|}{n-2} = \frac{|U(n-1,0)| - t|U(n-2,0)|}{2(n-2)}, \tag{20}$$

where the second equality follows from the elementary recurrence, $|U(x,y)| = |U(x-1,y-1)| + t|U(x-1,y)| + |U(x-1,y+1)|$. The equality of the left and the right ends of (20) matches the recurrence in (19). Hence Proposition 1.3 follows.

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