Comp. & Maths. with Appls. Vol. 6, pp. 339–347
Pergamon Press Ltd., 1980. Printed in Great Britain

THE REPETITION FACTOR AND NUMERICAL STABILITY OF VOLTERRA INTEGRAL EQUATIONS

S. McKee
Hertford College and the Computing Laboratory, Oxford, England

and

H. Brunner
Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada

Communicated by E. Y. Rodin

(*Received for publication October* 1979)

Abstract—In this paper we prove that direct linear multistep methods for Volterra integral equations of the second kind with repetition factor equal to one are always stable. We show trivially that this result is not true for first kind equations. We also demonstrate constructively that direct linear multistep methods for both first and second kind Volterra integral equations can have repetition factors greater than one, and indeed of arbitrary high order, and be numerically stable. Finally we explain why the first form of Simpson's rule for second kind equations is stable while the second form is unstable.

1. INTRODUCTION

Consider the Volterra integral equation

$$\gamma y(t)+\int_0^t k(t,s,y(s))\,\mathrm{d}s = g(t), \quad 0\le s\le t\le T, \tag{1.1}$$

with $\gamma \in \{0, 1\}$; for $\gamma = 0$, (1.1) is a Volterra integral equation of the first kind whereas for $\gamma = 1$, (1.1) is a Volterra integral equation of the second kind. For the first kind equations we require $g(0) = 0$ while for the second kind equations we have $y(0) = g(0)$.

We shall assume that the kernel $k(t, s, y(s))$ and the forcing function $g(t)$ have sufficient continuity at least to guarantee the existence of a unique continuous solution of the continuous problem and when necessary to permit consistency of the discrete problem of the required order through the usual Taylor series expansions.

This paper has been largely motivated by a conjecture of Linz[1] (see also[2, 3]). With reference to second kind Volterra integral equations he stated, "We may conjecture that methods with a repetition factor of one tend to be numerically stable, those with repetition factor greater than one numerically unstable". In this paper we show that this conjecture is only partially correct by proving that all linear multistep methods with repetition factor equal to one are indeed stable while those with repetition factor greater than one can be stable. This done by constructing stable methods with arbitrary high repetition factors. Similarly we demonstrate that the repetition factor has nothing whatever to do with stability or otherwise of methods for first kind Volterra integral equations. (This result is not surprising in view of the methods of Gladwin[4]). We further explain why the first form of Simpson's rule is stable while the second form is unstable for Volterra second kind equations. Some numerical results are presented as verification.

2. PRELIMINARIES

For $h \in (0,h_0]$, $h_0 > 0$ and N such that $T = Nh$ define the discretisation algorithm for (1.1)

$$\Phi_N(y) = 0 \quad \text{where} \quad \Phi_N: \mathbf{R}^{N+1}\to\mathbf{R}^{N+1},$$

with

$$[\Phi_N(y)]_i = \begin{cases} y_i - \tilde{y}_i & ,i = 0, 1, \ldots, r-1 \\ \gamma y_i + h\sum_{j=0}^{i} w_{ij}k(t_i,t_j,y_j) - g_i & ,i = r, r+1, \ldots, N, \end{cases} \tag{2.1}$$

where $\bar{y}_i$, $i = 0,1,\ldots,r-1$, are given starting values. For first kind equations we assume that $w_{ii} \neq 0$.

In matrix notation

$$\Phi_N(y) = \gamma I_N y + h\Psi_N(y) = g^{(\gamma)} \tag{2.2}$$

where Ψ_N is a (nonlinear) lower triangular matrix, $y = (y_0, y_1, \ldots, y_N)^T$, and $g^{(0)} = (h\bar{y}_0, \ldots, h\bar{y}_{r-1}, g_r, \ldots, g_N)^T$ and $g^{(1)} = (\bar{y}_0, \ldots, \bar{y}_{r-1}, g_r, \ldots, g_N)^T$ where $g_i = g(t_i)$.

The total quadrature expression

$$h \sum_{j=0}^{i} w_{ij} k(t_i, t_j, y_j)$$

is the sum of a starting formula, an end formula, and a main repeated rule. For further details the reader is referred to Holyhead *et al.* [5].

We shall say that Ψ_N has a repetition factor l if l is the smallest positive integer such that there exists ν_1, ν_2 and ν_3, independent of N, with

$$w_{ij} = w_{i-l,j}, \; i \geq \nu_1, \; j \geq \nu_2, \; i - j \geq \nu_3.$$

Note that Holyhead and McKee [6] call this a rowwise repetition factor in order to distinguish it from the columnwise repetition factor which they also introduce. This however for our needs will not be necessary.

We introduce the idea of a differentiation matrix. We define

$$D_N^{(l)} = \frac{1}{h} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & -1,0,\ldots,0,1 & & & & \\ & & & & -1,0,\ldots,0,1 & & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -1,0,\ldots,0,1 & \end{pmatrix}_{(N+1)\times(N+1)} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} r \text{ rows} \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} l \text{ rows} \end{matrix}$$

consisting of 1's on the diagonal and -1's on the lth subdiagonal starting at the $(r+1)$st column. All other elements are zero.

3. NUMERICAL STABILITY

Consider the trivial Volterra integral equation

$$\gamma y(t) + \int_0^t \lambda y(s)\,ds = \gamma + (1-\gamma)t, \quad \lambda > 0. \tag{3.1}$$

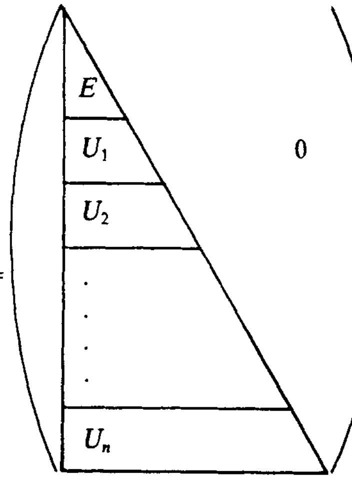
This is essentially the "test equation" used by Linz [1], Noble [2], Mayers [7] and Baker and Keech [8]. We shall say that a numerical method is numerically stable if when applied to (3.1) the discretised solution tends to zero as $N \to \infty$ for some fixed h. (Clearly this is really only a necessary condition for stability (e.g. Van der Houwen and Wolkenfelt[9] and Williams *et al.*[10]).

Assume that ψ_N (where $\Psi_N(y) = \psi_N y$ when Ψ_N is linear) has a recepition factor l and apply (2.2) to (3.1) to obtain

$$\Phi_N(y) = (\gamma I_N + \bar{h}\psi_N)y - g^{(\gamma)}, \quad \bar{h} = h\lambda. \tag{3.2}$$

Note that ψ_N will have the structure

$\psi_N =$



$(N+1)\times(N+1)$ (3.3)

where $k + ln = N + 1$, E is a $k \times k$ matrix, and

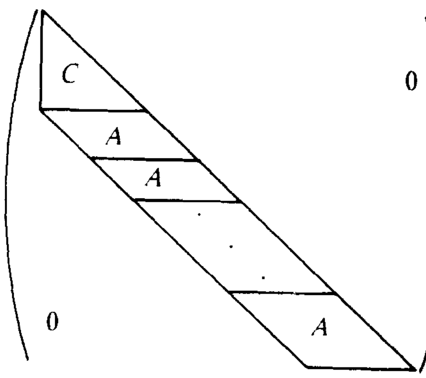
$$U_q = \left[\, B_0' \mid B_1 \mid \dots \mid B_{q-1} \mid B_q \,\right], \quad q = 1,2,\dots,n, \tag{3.4}$$

Here the B_i's are $l \times l$ matrices defined by starting from the right and adding zero columns on the left if necessary. Thus B_0' is B_0 with (possibly) some of its columns removed. Premultiplication by the appropriate differentation matrix results in

$$\begin{aligned} D_N^{(l)}\Phi_N(y) &= (\gamma D_N^{(l)} + \bar{h}D_N^{(l)}\psi_N)y - D_N^{(l)}g^{(\gamma)} \\ &= (\gamma D_N^{(l)} + \lambda A_N)y - D_N^{(l)}g^{(\gamma)} \end{aligned}$$

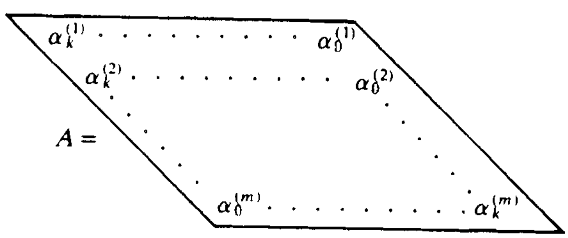
where

$A_N =$



$(N+1)\times(N+1)$ (3.5)

with



where m can be greater or less than l, $g^{(\gamma)} = ({}^{1-\gamma}\bar{y}_0, \ldots, h^{1-\gamma}\bar{y}_{r-1}.g_r, \ldots, g_N)^T$, and C is a $k \times k$ matrix.

The matrix $h(\gamma D_N^{(j)} + \lambda A_N) = (\gamma h D_N^{(j)} + \bar{h} A_N)$ can now be characterised by the comparison matrices

$$G_j = \begin{pmatrix} 0 & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \\ \dfrac{-\bar{h}\alpha_k^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}}, & \ldots, & \dfrac{-\bar{h}\alpha_{l+1}^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}}, & \dfrac{\gamma - \bar{h}\alpha_l^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}}, & \dfrac{-\bar{h}\alpha_{l-1}^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}} & \ldots, & \dfrac{-\bar{h}\alpha_1^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}} \end{pmatrix}_{k \times k} \tag{3.6}$$

$$j = 1, 2, \ldots, m,$$

when $k \geq l$ and

$$\begin{pmatrix} 0 & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & \cdot & \\ & & & & & & 1 \\ \dfrac{\gamma}{\gamma + \bar{h}\alpha_0^{(j)}}, & 0 & \ldots, & \dfrac{-\bar{h}\alpha_k^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}} & \ldots\ldots, & & \dfrac{-\bar{h}\alpha_1^{(j)}}{\gamma + \bar{h}\alpha_0^{(j)}} \end{pmatrix}_{l \times l} \tag{3.7}$$

$$j = 1, 2, \ldots, m,$$

when $k < l$.

Now since

$$y = (\gamma h D_N^{(j)} + \bar{h} A_N)^{-1} (h D_N^{(j)}) g^{(\gamma)},$$

$y_N \to 0$ as $N \to \infty$ for a fixed $\bar{h}$ if

$$\|(\gamma h D_N^{(j)} + \bar{h} A_N)^{-1}\|_\infty \leq M, \text{ independent of } N. \tag{3.8}$$

The linear algebra results of McKee[11] (theorem 3.2) then say that (3.8) holds if and only if the eigenvalues of

$$G = G_m G_{m-1} \ldots G_1 \tag{3.9}$$

all lie strictly inside the unit circle.

4. A REPETITION FACTOR OF ONE

We shall first consider linear multistep methods for Volterra integral equations of the second kind with repetition factor equal to 1. Let such an algorithm be represented by

$$\Phi_N(y) = I_N y + h \Psi_N(y) - g^{(1)}. \tag{4.1}$$

Apply (4.1) to (3.1) to obtain

$$\Phi_N(y) = (I_N + \bar{h}\psi_N) y - g^{(1)}, \ \bar{h} = h\lambda.$$

Premultiplying by $hD_N^{(1)}$ results in

$$hD_N^{(1)}\Phi_N(y) = (hD_N^{(1)} + \bar{h}A_N)y - hD_N^{(1)}g^{(1)}, \tag{4.2}$$

that is, (4.2) is a consistent approximation to

$$y' = -\lambda y. \tag{4.3}$$

Let us assume that (4.1) is a consistent method of order p. Then we see that (4.2) is a consistent method to (4.3) of order p (i.g. see Andrade[12]). We note further that (4.1) is a stable method if

$$\|(hD_N^{(1)} + \bar{h}A_N)^{-1}\|_\infty$$

is uniformly bounded. But this is simply the condition that the cyclic linear multistep method (4.2) for solving (4.3) be (absolutely) stable. The cyclic linear multistep method takes the form

$$y_{n-1} - y_n = -\lambda h \sum_{i=0}^{k} \alpha_{k-i}^{(j)} y_{n+i},\ j = 1, 2, \ldots, m. \tag{4.4}$$

Define, in the usual way, the auxiliary stability polymomials associated with each indicidual method:

$$\begin{aligned}\pi_j(r,\bar{h}) &= \rho^{(j)}(r) + \bar{h}\sigma^{(j)}(r)\\ &= r^k - r^{k-1} + \bar{h}\sum_{i=0}^{k} \alpha_{k-i}^{(j)} r^i.\end{aligned}$$

Now from Lambert (1973) p. 66 we have that the principal root of $\pi_j(r,\bar{h})$, say $r_1^{(j)}$, satisfies

$$r_1^{(j)} = \exp(-\bar{h}) + 0(\bar{h}^{p+1}). \tag{4.5}$$

The characteristic equation associated with the m cyclic linear multistep method is

$$\det|G - rI| = 0, \tag{4.6}$$

where G is defined by (3.6) and (3.9) with $l = 1$. Let r_1 be the principal root of (4.6). We now appeal to the lemma:

Lemma 1 (Andrade and McKee[13])

Let the m-cyclic linear multistep method $D_N^{(1)}\Phi_N$ be composed of individual methods of order of consistency p_ν, $\nu = 1,2,\ldots,m$ and set $p = \min_\nu p_\nu$. If the first characteristic polynomial, say $\rho(r)$, of $D_N^{(1)}\Phi_N$ has a simple root at 1, then the principal root r_1 of the stability polynomial (4.6) satisfies

$$r_1 = \exp(-m\bar{h}) + 0(\bar{h}^{p+1}).$$

□

We can now prove

Theorem 1

All consistent linear multistep methods for second kind Volterra integral equations with repetition factor one are numerically stable.

Proof

All consistent linear multistep methods with repetition factor 1 can be expressed as

$$\Phi_N(y) = I_N y + h\Psi_N(y) - g^{(1)}, \tag{4.1}$$

and (4.1) will be numerically stable if

$$\|(hD_N^{(1)} + \bar{h}\psi_N)^{-1}\|_\infty \le M, \text{ independent of } N. \tag{3.8}$$

For $\bar{h} = 0$, the characteristic equation associated with (3.8) is

$$\det\left|\begin{pmatrix} 0 & 1 & & & \\ & & 1 & . & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & 1 \\ 0 & , & \cdot & \cdot & , & 0 & 1 \end{pmatrix}^m\right| = 0$$

which has roots 1, 0 ($k-1$ times). Thus for $\bar{h}$ small the characteristic equation given by (4.5) (4.5) will have, by continuity, roots $1 + 0(\bar{h})$, $0(\bar{h})$ ($k-1$ times). Thus we observe that all consistent linear multistep methods with repetition factor 1 will be numerically stable if the constant corresponding to the $0(\bar{h})$ is negative. We now call upon Lemma 1 and note that $\bar{h} > 0$ to complete the proof. □

To demonstrate that there exist unstable linear multistep methods with repetition factor equal to 1 for Volterra integral equations of the first kind we need only construct a trivial example (applied to (3.1))

$$h\lambda\begin{pmatrix} 1 & & & & \\ \alpha & 1-\alpha & & & \\ \alpha & 1 & 1-\alpha & & \\ \alpha & 1 & 1 & 1-\alpha & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \end{pmatrix} y = \begin{pmatrix} h\bar{y}_0 \\ h \\ 2h \\ 3h \\ \cdot \\ \cdot \end{pmatrix} \quad \text{with } \alpha > 1.$$

5. REPETITION FACTOR GREATER THAN ONE

Using the forward rectangular rule

$$\int_{t_j}^{t_{j+\nu}} f(t)\,\mathrm{d}t \doteqdot \nu h f(t_{j+\nu}), \quad \nu = 1,2,\ldots,l,$$

we can write down the order 1 family of methods defined by (3.3) with

$$B_0' = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}_{l\times 1}, \quad B_1 = \begin{pmatrix} 1 & & & & & \\ & 2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & l \end{pmatrix}_{l\times l}, \quad B_j = \gamma I_l,$$

$j = 2,3,\ldots,l+1, \quad E = I_l.$

Thus for the two values of γ we have two families of discretisation methods with repetition factor l. Premultiplying by $D_N^{(l)}$ results in a matrix A_N which is characterised by the single companion matrix

$$G_l = \begin{pmatrix} 0 & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & 1 \\ \frac{\gamma}{\gamma + \bar{h}l}, & 0, & \cdot & \cdot & \cdot & \cdot \;, \; 0 \end{pmatrix}_{l \times l}$$

with eigenvalues 0 (l times) for first kind Volterra integral equations and the l roots of

$$\lambda^l - \frac{1}{1 - \bar{h}l} = 0$$

for second kind Volterra integral equations.

Lest the reader might think that these (somewhat artificial) methods are in any way unique we give another simple example.

Consider the order 2 method for second kind equations,

$$(I_N + \bar{h}\psi_N)y = g^{(1)}, \tag{5.1}$$

where

$$\psi_N = \begin{pmatrix} 0 & & & & & & \\ \frac{1}{2} & \frac{1}{2} & & & & & \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & & & & \\ \frac{1}{2} & 1 & 1 & \frac{1}{2} & & & \\ \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} & & \\ \frac{1}{2} & 1 & 1 & 1 & 1 & \frac{1}{2} & \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & & \cdot \end{pmatrix} \tag{5.2}$$

and $g^{(1)} = (\bar{y}_0, g_1, g_2, \ldots, g_N)^T$. This method has repetion factor 2 and so $(hD_N^{(2)} + \bar{h}A_N)$ can be characterized by

$$G_1 = \begin{pmatrix} 0 & 1 \\ \frac{1 - 2/3\bar{h}}{1 + 2/3\bar{h}} & \frac{-2/3\bar{h}}{1 + 2/3\bar{h}} \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & 1 \\ \frac{1 - 1/2\bar{h}}{1 + 1/2\bar{h}} & \frac{-\bar{h}}{1 + 1/2\bar{h}} \end{pmatrix}.$$

It is an easy matter to demonstrate that the eigenvalues of $G = G_2G_1$ are

$$1 - \bar{h}/3, \quad 1 - 2\bar{h}$$

to within order $\bar{h}^2$ and so there exists some nonempty stability region.

6. THE TWO FORMS OF SIMPSON'S RULE FOR SECOND KIND EQUATIONS

The first form of Simpson's rule uses the 3/8th rule alternately on the upper end $[t_{i-3}, t_i]$ while the second form uses the 3/8th rule alternately at the lower end $[t_0, t_3]$ (for fuller details see Linz [1] or Noble [2]).

We observe that the first form has repetition factor 1. Thus premultiplying by $D_N^{(1)}$ results in a matrix

$$(hD_N^{(1)} + \bar{h}A_N)$$

which can be characterized by the two companion matrices

$$G_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(1/3-3/8)\bar{h}}{(1+1/3\bar{h})}, & \frac{-(4/3-9/8)\bar{h}}{(1+1/3\bar{h})}, & \frac{-(2/3-9/8)\bar{h}}{(1+1/3\bar{h})}, & \frac{1-(4/3-3/8)\bar{h}}{(1+1/3\bar{h})} \end{pmatrix}$$

and

$$G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-(3/8-1/3)\bar{h}}{(1+3/8\bar{h})} & \frac{-(9/8-4/3)\bar{h}}{(1+3/8\bar{h})} & \frac{1-(9/8-4/3)\bar{h}}{(1+3/8\bar{h})} \end{pmatrix}$$

A tedious calculation yields of $G = G_2G_1$ to be

$$0, 0 + 0(\bar{h}^2)\,(\text{twice}), \quad 1 - 3\bar{h} + 0(\bar{h}^2).$$

Thus this method is reminiscent of the Adams–Moulton methods for ordinary differential equations in having all the roots of its first characteristic polynomial at the origin except for one at +1. It is therefore optimally stable (with respect to the test equation (3.1)), a fact that was reflected in Linz's numerical results.

Although the second form of Simpson's rule has repetition factor 2 it is essentially simpler to analyse. Permultiplying by $D_N^{(2)}$ results in a matrix $(hD_N^{(2)} + \bar{h}A_N)$ characterized by the single companion matrix

$$G_1 = \begin{pmatrix} 0 & 1 \\ \frac{1-1/3\bar{h}}{1+1/3\bar{h}} & \frac{-4/3\bar{h}}{1+1/3\bar{h}} \end{pmatrix}$$

whose characteristic equation is easily recognised as that of Simpson's method for ordinary differential equations. It therefore has a root outside the unit circle, independent of the sign of $\bar{h}$, indicating unconditional instability.

7. NUMERICAL VERIFICATION

The linear multistep method (5.1) with repetition factor 2 for second kind Volterra integral equations was used to solve the test equation (3.1) with λ equal to 1 and 5. The errors for different values of N for $h = 0.1$ and 0.01 are presented in Table 1. It is clear that the method is numerically stable.

8. CONCLUSIONS

In this paper we have proved that Linz's conjecture[1] is in part correct, that is, that all linear multistep methods for Volterra second kind integral equations with repetition factor 1 are numerically stable while those with repetition factor greater than 1 can be either stable or unstable. We showed that for first kind equations stability or instability is independent of the repetition factor. We also explained why Simpson's rule in the first form is stable while in the second form is unstable. Finally numerical results were presented to verify our conclusions.

Table 1.

N	h = 0.1		h = 0.01	
	$\lambda = 1$	$\lambda = 5$	$\lambda = 1$	$\lambda = 5$
10	6.8×10^{-4}	5.4×10^{-3}	1.5×10^{-6}	1.3×10^{-4}
20	5.9×10^{-4}	1.9×10^{-3}	2.7×10^{-6}	1.7×10^{-4}
25	7.3×10^{-5}	1.9×10^{-3}	1.5×10^{-6}	4.1×10^{-5}
100	7.6×10^{-5}	2.2×10^{-6}	6.8×10^{-6}	5.4×10^{-5}
200	1.4×10^{-5}	5.0×10^{-10}	5.9×10^{-6}	1.9×10^{-5}
210	1.2×10^{-5}	2.0×10^{-10}	5.7×10^{-6}	1.7×10^{-5}
220	1.0×10^{-5}	1.0×10^{-10}	5.5×10^{-6}	1.6×10^{-5}
230	8.6×10^{-6}	1.0×10^{-10}	5.4×10^{-6}	1.5×10^{-5}
240	7.3×10^{-6}	0.0×10^{-10}	5.2×10^{-6}	1.3×10^{-5}
250	6.1×10^{-6}	0.0×10^{-10}	5.0×10^{-6}	1.2×10^{-5}

The reader perhaps should be reminded that, whereas the repetition factor has no bearing on numerical stability for first kind equations, its existence is crucial for the different and in some ways more important question of convergence of the multistep methods to the continuous problem. The reader is also referred to Holyhead and McKee[6] and for a generalised concept of repetition factor to McKee and Holyhead[14].

Acknowledgements—The first author is the C.E.G.B. Research Fellow in Numerical Analysis at the University of Oxford and is grateful to them for their financial support. The second author was supported by the National Research Council of Canada under Grant A-4805.

REFERENCES

1. P. Linz, Numerical methods for Volterra integral equations with applications to certain boundary value problems, Ph.D. Thesis, University of Wisconsin, Madison (1968).
2. B. Noble, Instability when solving Volterra integral equations of the second kind by multistep methods, *Conference on the Numerical Solution of Differential Equations* (Ed. J. Ll. Morris). Dundee, Scotland (1969).
3. C. T. H. Baker, *The Numerical Treatment of Integral Equations.* Oxford University Press (1977).
4. C. J. Gladwin, Numerical solution of Volterra integral equations of the first kind. Ph.D. thesis, Dalhousie University (1975).
5. P. A. W. Holyhead, S. McKee and P. J. Taylor, Multistep methods for solving linear Volterra integral equations of the first kind, *SINUM* **12**(5), 698–711 (1975).
6. P. A. W. Holyhead and S. McKee, Stability and convergence of multistep methods for linear Volterra integral equations of the first kind, *SINUM* **13**(2), 269–292 (1976).
7. D. F. MAyers, *Numerical Solution of Ordinary Differential Equations,* (Edited by L. Fox). Pergamon Press, Oxford (1962).
8. C. T. H. Baker and M. S. Keech, Stability regions in the numerical treatment of Volterra integral equations. *SINUM* **15**, 394–417 (1978).
9. P. J. Van der Houwen and P. H. M. Wolkenfelt, On the stability of multistep formulas for Volterra integral equations of the second kind. *Comput.* (to appear) (1980).
10. H. Williams, S. McKee and H. Brunner, Numerical stability and the relative merits of multistep methods for singular and nonsingular Volterra integral equations. (in preparation).
11. S. McKee, The linear algebra of discretisation methods (submitted to *J. Lin. Algebra*), (1980).
12. Celia Andrade, Métodos lineares de passo múltiplo estáveis de alta precisão aplicados a equações diferenciais ordinárias, equações integrais, equações diferenciais parciais, Ph.D. Thesis, University of São Paulo (1977).
13. Celia Andrade and S. McKee, Sobre estabilidade numérica de méthodos cíclicos lineares de passo múltiplo para equações diferenciais ordinárias. *Anais do Primeiro Simpósio Nacional de Cálculo Numérico.* Universidade Federal de Belo Horizonte, M. G., Brazil (1978).
14. S. McKee and P. A. W. Holyhead, Linear multistep methods for the generalized Abel's integral equation (in preparation).