Graphs of nonsingular threshold transformations

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Abstract
After the graph structures of self-dual nonsingular (i.e. one-to-one) transformations of \( \{0, 1\}^n \) are described, a construction method of generating minimal nonsingular threshold transformations from lower-dimensional ones is presented. Theorems which concern nonsingular threshold transformations and support that procedure are also proved. Then, in addition to circular, nonsingular transformations in the author's previous paper, five classes of noncircular, nonsingular threshold transformations are given with their graph structures.

1. Introduction

Let \( Q = \{0, 1\} \) be the minimal Boolean algebra with \( 0 = 1 \) and \( 1 = 0 \). Functions from \( Q^n \) into \( Q \) are called Boolean functions, and mappings from \( Q^n \) into itself are called Boolean transformations. If \( F \) is a transformation of \( Q^n \), then \( F \) defines its (directed) graph as \( (Q^n, E) \), where

\[
E = \{(x, y) \mid x \in Q^n, F(x) = y\}.
\]

If \( F \) is a threshold transformation, its graph drew a considerable interest in the 1960s and early 70s [1–4], etc.). These studies provided some construction procedures for finding a threshold transformation that satisfies given conditions for its graph. However, they were mainly concerned with proper subgraphs of \( (Q^n, E) \) and did not bring out concrete results for the whole graph structure of a transformation except the one given in [1, 4] (see Remarks 4.6). Since what characterizes a transformation is its (whole) graph structure rather than the structure of its particular proper subgraph, our goal here is to fill part of that vacancy. In order to simplify the problem, we limit our scope to graphs of nonsingular (i.e. one-to-one) transformations and use our recent paper [6], which dealt with circular nonsingular threshold transformations, as a new point of departure.

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We call a Boolean transformation trivial if it permutes or/and complements variables. Transformations \( F \) and \( G \) are equivalent if there exist trivial transformations \( S \) and \( T \) such that \( G = SFT \). Let \( \text{Var}(F) \), the variation of \( F \), denote the total number of coordinates that change under \( F \), i.e.

\[
\text{Var}(F) = \sum_{x \in \mathbb{Q}^n} d(x, F(x)),
\]

where \( d(x, y) \) denotes the Hamming distance, i.e.

\[
d(x, y) = |\{ i \mid x_i \neq y_i \}|.
\]

We call a Boolean transformation \( F \) minimal, if \( \text{Var}(F) \leq \text{Var}(TF) \) for every trivial transformation \( T \). Since a great number of different graphs can be produced from a single transformation by applying trivial transformations to it, we are primarily concerned with minimal transformations. If \( \sigma \) is a permutation of the set \( \{1, 2, \ldots, n\} \), then \( \sigma \) defines the trivial transformation of \( \mathbb{Q}^n \) as \( \sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \). We call a transformation \( F \) circular if there exists an \( n \)-cyclic permutation \( \tau = (r_1, r_2, \ldots, r_n) \) such that \( F \circ \tau = \tau F \).

For each point \( x = (x_1, \ldots, x_n) \) of \( \mathbb{Q}^n \), \( \bar{x} \) denotes \( (\bar{x}_1, \ldots, \bar{x}_n) \). Then for each function \( f \), \( \bar{f} \) and \( \bar{f} \) are defined by \( f(x) = (f(x)) \) and \( \bar{f}(x) = f(x) \). \( f \) is called self-dual if \( \bar{f} = \bar{f} \).

Similarly, \( \bar{f} \) and \( \bar{f} \) are defined by \( \bar{f}(x) = (\bar{f}(x)) \) and \( \bar{f}(x) = \bar{f}(x) \) for a transformation \( F \). We refer to the set \( f \) meaning the set \( f^{-1}(1) \). Therefore, the set \( \bar{f} \) is \( f^{-1}(0) \), and the set \( f \) is the set of complements of all points of \( f^{-1}(1) \). A transformation \( F \) is called self-dual, if \( \bar{F} = F \). Let \( F \) be a Boolean transformation such that \( F(x) = (F_1(x), \ldots, F_n(x)) \). Then \( F \) is self-dual if and only if \( F_i \) is self-dual for every \( i \).

Assume \( F \) is self-dual. Then if \( f_i \) is defined as

\[
f_i = x_i \cdot \bar{F}_i,
\]

then

\[
F_i = x_i \cdot \bar{f}_i \vee f_i.
\]

Conversely, for any Boolean function \( f_i \) such that \( f_i(x_1, \ldots, x_n) = x_i \cdot g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \) for a \( g_i \), if \( F_i \) is defined by (1.2), then \( F(x) = (F_1(x), \ldots, F_n(x)) \) is a self-dual transformation. Therefore, as in [6], any self-dual transformation \( F \) such that \( F(x) = (F_1(x), \ldots, F_n(x)) \) will be expressed by

\[
F = [f_1, \ldots, f_n],
\]

where if \( p = (p_1, \ldots, p_n) \) is a point of \( f_i \) then \( p_i = 1 \), and the relations between \( F_i \) and \( f_i \) are given above. In this expression, a point \( p \) of \( \mathbb{Q}^n \) such that its \( i \)th coordinate is 1 is transformed into a point such that its \( i \)th coordinate is 0, if and only if \( p \) is a point of the set \( f_i \).

A Boolean function \( f \) is called a threshold function, if the sets \( f \) and \( \bar{f} \) are separated by a hyperplane in \( \mathbb{R}^n \). A Boolean transformation \( F(x) = (F_1(x), \ldots, F_n(x)) \) is called a threshold transformation, if each \( F_i \) is a threshold function. By Theorem 2.2, [6],
a self-dual transformation $F = [f_1, \ldots, f_n]$ is a threshold transformation, if and only if each $f_i$ is a threshold function.

If $f$ is a Boolean function and $p$ is a point, then $f \cup p$ denotes the function obtained by adding $p$ to the set $f$, and $f \cap p$ denotes the function obtained by excluding $p$ from the set $f$. A complete set is a set $C$ such that if $p \in C$ then $f \in C$. The number of elements of a set $A$ will be denoted by $|A|$. Let $A \subset Q^n$ and $B \subset Q^m$. The direct product $A \times B$ is the subset of $Q^{n+m}$ such that $(p_1, \ldots, p_n) \in A$ and $(p_{n+1}, \ldots, p_{n+m}) \in B$.

If $f$ is a function from $Q^n$ into $Q$, $f \times B$ denotes the function from $Q^{n+m}$ into $Q$ such that $(f \times B)^{-1}(1) = f^{-1}(1) \times B$. Conversely, for a function $g$ from $Q^{n+m}$ into $Q$, $g + B$ is the function from $Q^n$ into $Q$ such that $(p_1, \ldots, p_n) \in (g + B)^{-1}(1)$ if and only if $(p_1, \ldots, p_n, q_1, \ldots, q_m) \in g^{-1}(1)$ for every $(q_1, \ldots, q_m) \in B$. For a complete set $C \subset Q^m$ and a self-dual transformation $F = [f_1, \ldots, f_n]$, $F \times C$ defined as

$$F \times C = [f_1 \times C, \ldots, f_n \times C, \phi_1, \ldots, \phi]$$

is a direct sum of $|C|$ transformations whose graphs are isomorphic to that of $F$, where $\phi(p) = 0$ for every $p \in Q^{n+m}$. If $C$ is $Q^m$, we call $F \times Q^m$ the $(n+1, \ldots, n+m)$th face copy of $F$.

2. Graph structure of nonsingular self-dual transformations

Since nonsingular threshold transformations are self-dual (Theorem 2.1, [6]), let us first describe easily-established facts about the graphs of nonsingular self-dual transformations, though at least parts of them are well-known [2].

Let $F$ be a Boolean transformation of $Q^n$ and $x$ be a point of $Q^n$. If $F(x) = x$, then $x$ is called a fixed point of $F$; if $F(x) \neq x$, then $x$ is called a nonfixed point of $F$. If the set of all nonfixed points of a transformation $F$ of $Q^n$ and the set of all nonfixed points of a transformation $G$ of $Q^n$ are disjoint, we can define a transformation $H$ of $Q^n$ such that $H(p) = F(p)$ if $p$ is a nonfixed point of $F$, $H(p) = G(p)$ if $p$ is a nonfixed point of $G$, and $H(p) = p$ if $p$ is a fixed point of both $F$ and $G$. We call $H$ the sum of $F$ and $G$ and write $H = F + G$. If, in particular, the set of all nonfixed points and their images of $F$ and the set of all nonfixed points and their images of $G$ are disjoint, we call $H$ the direct sum of $F$ and $G$ and write $H = F \oplus G$. We call a self-dual transformation $H$ elementary, when if $H = F + G$, and both $F$ and $G$ are self-dual, then $F$ or $G$ is the identity. From these definitions the following theorem is clear.

**Theorem 2.1.** Any self-dual transformation is a direct sum of one or several elementary self-dual transformations.

The graph of any nonsingular transformation of a finite set consists of a set of disjoint cycles. Let $G_\text{s} = \{(r_1, s_1), \ldots, (r_k, s_k)\}$, where $r_i > 0$ and $s_i > 1$ are integers for every $i$, $s_i \neq s_j$ for $i \neq j$, and $r_1 \cdot s_1 + \cdots + r_k \cdot s_k \leq 2^n$. We call $G_\text{s}$ the graph structure of a nonsingular transformation of $Q^n$. We say the graph structure of a transformation
The graph structure of $F$ is $\{(1, s)\}$ or $\{(2, s)\}$ for an $s > 1$. In particular, if it is $\{(1, s)\}$, then $s$ is even. Conversely, the graph structure $\{(1, s)\}$, where $s$ is even, and the graph structure $\{(2, s)\}$, where $s$ is arbitrary, are realized by some elementary self-dual transformations of $Q^n$, so far as $s < 2^n$ and $2s < 2^n$, respectively.

Proof. Let $p$ be a point of an $s$-cycle such that $s > 1$ of an elementary nonsingular self-dual transformation $F$. If $p$ is in the same cycle, then $(F^{-1}(p)) = F(p)$ and $F(p)$ are in the same cycle. Inductively, if $q$ is any point of this cycle, then $\bar{q}$ is also in the same cycle. Therefore, in this case, $s$ is even, and this cycle and fixed points form the graph of $F$. If $\bar{p}$ is in another cycle, then $F^{-m}(p)$ is in the first cycle and $(F^{-m}(p)) = F^{-m}(\bar{p})$ is in the second cycle for every $m$. The two cycles and fixed points form the graph of $F$. Conversely, if $s$ is a positive integer, consider a complete set $C$ such that $|C| = 2s$, and let $C = A \cup B$ such that $A \cap B = \emptyset$ and if $p \in A$ then $\bar{p} \in B$ and if $p \in B$ then $\bar{p} \in A$. Construct a transformation $F$ composed of fixed points and one $s$-cycle ranging over $A$. $G = F \oplus \bar{F}$ is elementary and self-dual, and its graph structure is $\{(2, s)\}$. If $s$ is even, consider $G$ defined above for a complete set $C$ such that $|C| = s$. For a point $p$ of $C$, define $H$ as $H(p) = G(\bar{p})$, $H(\bar{p}) = G(p)$, and $H(q) = G(q)$ for every other $q$. $H$ is elementary and self-dual and realizes $\{(1, s)\}$.

Theorem 2.3. The necessary and sufficient condition for the graph structure $\{(r_1, s_1), \ldots, (r_k, s_k)\}$ to be realized by a self-dual transformation of $Q^n$ is that $r_1 \cdot s_1 + \cdots + r_k \cdot s_k < 2^n$ and if $s_i$ is odd then $r_i$ is even.

Proof. By decomposing nonsingular self-dual transformations into elementary self-dual transformations, we obtain the necessary part. For the sufficiency, express $Q^n$ as a mutually disjoint union of $r_i$ complete sets with $s_i$ elements for even $s_i$ and $r_i/2$ complete sets with $2s_i$ elements for odd $s_i$ and the rest of $Q^n$. Following the proof of Theorem 2.2, construct self-dual elementary transformations composed of fixed points and one $s_i$-cycle for even $s_i$ and two $s_i$-cycles for odd $s_i$ ranging over each complete set. The direct sum of these elementary transformations realizes the given graph structure.

3. Face copies and face connections of threshold transformations

Theorem 2.3 completely determines the graph structures of nonsingular self-dual transformations. However, such a condition for nonsingular threshold transformations is unknown. Therefore, it is necessary to try to construct a threshold transformation individually in order to determine whether a given graph structure is realized by
a threshold transformation or not. In this section we prove two properties concerning face copies of nonsingular threshold transformations. Then using these results, we discuss a method of constructing higher-dimensional nonsingular threshold transformations from lower-dimensional ones.

**Definition 3.1.** Let \((i)\) denote the complementation of the \(i\)th coordinate.

**Lemma 3.2.** If \(g\) is a threshold function and \(g = f \times C\) for a function \(f\) from \(Q^n\) into \(Q\) and a complete set \(C \subseteq Q^m\), then \(C = Q^m\).

**Proof.** Let \(p\) be a point of \(g\); then \(q = (n \mp 1) \cdots (n \mp m)p\) is also a point of \(g\). If \(p' = (p_1, \ldots, p_n, a_{n+1}, \ldots, a_{n+m}) \notin g\) for a point \((a_{n+1}, \ldots, a_{n+m}) \in Q^m\), then \(q' = (n \mp 1) \cdots (n \mp m)p' \notin g\). Since \(p + q = p' + q'\), we have a contradiction to the linear separability of \(g\) and \(\tilde{g}\). \(\Box\)

**Theorem 3.3.** If \(G\) is a nonsingular threshold transformation of \(Q^{n+m}\), and none of the \((n+1)\)th, \(\ldots\), \((n+m)\)th coordinates of any point of \(Q^{n+m}\) changes under \(G\), then \(G\) is the \((n+1), \ldots, n+m)\)th face copy of a nonsingular threshold transformation of \(Q^n\).

**Proof.** Let \(p\) be a point of a cycle of any transformation \(H\) of \(Q^{n+m}\). For any \(i\), if \(p\) changes its \(i\)th coordinate from 1 to 0 under \(H\), then another point in the same cycle must change its \(i\)th coordinate from 0 to 1 under \(H\). Moreover, in the same cycle, the number of points that change their \(i\)th coordinates from 1 to 0 must be the same as the number of points that change their \(i\)th coordinates from 0 to 1. Let \(a = (a_1, \ldots, a_m)\) be a point of \(Q^m\). From the given condition, each cycle of \(G\) is contained either in \(Q^n \times \{a\}\) or outside \(Q^n \times \{a\}\). Therefore, the number of \(p \in Q^n \times \{a\}\) such that \(p\) changes its \(i\)th coordinate from 1 to 0 under \(G\) and the number of \(q \in Q^n \times \{a\}\) such that \(q\) changes its \(i\)th coordinate from 0 to 1 under \(G\) are the same. Let \(G = \{g_1, \ldots, g_n, a, \ldots, a\}\), then the number of \(p \in g_i\) such that \(p_{n+1} = a_{n+1}, \ldots, p_{n+m} = a_{n+m}\), and the number of \(q \in g_i\) such that \(q_{n+1} = a_{n+1}, \ldots, q_{n+m} = a_{n+m}\) are the same. Let \(p\) be an arbitrary point of \(g_i\), and \(q = (n \mp 1)(n \mp 2) \cdots (n \mp m)p\). If \(q \in \tilde{g}_i\), then there exist \(p' \in g_i\) and \(q' \in \tilde{g}_i\) such that \(p'_{n+1} = \tilde{p}_{n+1}, \ldots, p'_{n+m} = \tilde{p}_{n+m}\) and \(q' = (n \mp 1) \cdots (n \mp m)p'\). Hence, \(p + p' = q + q'\), \(p\) and \(p'\) are points of \(g_i\), and \(q\) and \(q'\) are points of \(\tilde{g}_i\), contrary to the linear separability of \(g_i\) and \(\tilde{g}_i\). Hence, there exist an \(f_i\) and a complete set \(C \subseteq Q^m\) such that \(g_i = f_i \times C\). Since \(C = Q^m\) by Lemma 3.2 we have \(G = F \times Q^m\), where \(F = [f_1, \ldots, f_n]\), and \(F\) is a nonsingular threshold transformation. \(\Box\)

**Lemma 3.4** (Muroga et al. [5]). If \(f\) is a threshold function, then either \(f \div \{0\}\) or \(f \div \{1\}\) is a subset of the other, and both are threshold functions.

**Proof.** Suppose \(p = (p_1, \ldots, p_n, 0) \in f\), \(p' = (p_1, \ldots, p_n, 1) \notin f\), \(q = (q_1, \ldots, q_n, 0) \notin f\), and \(q' = (q_1, \ldots, q_n, 1) \notin f\). Then \(p + q' = p' + q\), which contradicts the linear separability of the sets \(f\) and \(f\). It is clear that both \(f \div \{0\}\) and \(f \div \{1\}\) are threshold functions. \(\Box\)
Lemma 3.5. If a point \( p \) and \( \bar{p} \) belong to a threshold function \( f \) from \( Q^n \) into \( Q \), then \( |f| \geq 2^{n-1} + 1 \).

Proof. If both \( p \) and \( \bar{p} \) belong to \( f \), then the linear separability of \( f \) and \( \bar{f} \) requires that for any \( q \in Q^n \), either \( q \) or \( \bar{q} \) or both belong to \( f \).

Theorem 3.6. Let \( G \) be a minimal nonsingular threshold transformation of \( Q^{n+1} \). Then there exists a self-dual threshold transformation \( F \) of \( Q^n \) such that if both \( p \in Q^{n+1} \) and \( (n+1)p \) change or neither changes their \( i \)th coordinate under \( G \) for \( i = 1, \ldots, n \), then \( G(p) = (F \times Q)(p) \).

Proof. Let \( G = [g_1, \ldots, g_{n+1}] \). Let

\[
 f_i = g_i + \{0\} \cup g_i + \{1\} \quad \text{or} \quad f_i = g_i + \{0\} \cap g_i + \{1\}
\]

for \( i = 1, \ldots, n \). Then \( F = [f_1, \ldots, f_n] \) is a threshold transformation of \( Q^n \) by Lemma 3.4.

If both a point \( p \in Q^{n+1} \) and \( p' = (n+1)p \) change or neither changes their \( i \)th coordinate under \( G \) for \( i = 1, \ldots, n \), but only one of them changes its \( (n+1) \)th coordinate, then \( G(p) = G(p') \), contrary to the nonsingularity of \( G \). Suppose both \( p \) and \( p' \) change their \( (n+1) \)th coordinates. Then \( p \in g_{n+1} \) and \( p' \notin g_{n+1} \), or \( p \notin g_{n+1} \) and \( p' \in g_{n+1} \). By Lemma 3.5, \( |g_{n+1}| > 2^{n-1} + 1 \); hence, \( G \) is not minimal, because \( \text{Var}((n+1)G) < \text{Var}(G) \). Therefore, neither \( p \) nor \( p' \) changes its \( (n+1) \)th coordinate, and \( F \) satisfies the desired condition.

Face connections

Theorem 3.6 shows that any minimal nonsingular threshold transformation \( G = [g_1, \ldots, g_{n+1}] \) of \( Q^{n+1} \) can be constructed from a self-dual threshold transformation \( F = [f_1, \ldots, f_n] \) through its \( (n+1) \)th face copy \( F \times Q = [f_1 \times Q, \ldots, f_n \times Q, \phi] \); \( g_i \) for \( i = 1, \ldots, n \) is obtained by adding or deleting a subset of either \( Q^n \times \{0\} \) or \( Q^n \times \{1\} \) to or from \( f_i \times Q \). \( g_{n+1} \) is composed of \( p, \bar{p}, (n+1)p \) or \( (n+1)\bar{p} \) for some of these added or deleted points \( p \). The \( (n+1) \)th coordinates of two of the four are 1, but \( g_{n+1} \) cannot contain both because of the minimality of \( G \). By Theorem 3.3, \( g_{n+1} \neq \phi \) for \( G \) to be non-singular, unless \( G = F \times Q \). We call \( G \) a face connection of \( F \times Q \). Similarly, we can define a face connection of \( F \times Q^n \).

Simple face connections

A special face connection used in the proof of Theorem 4.9 is as follows: Consider the \( (n+1, \ldots, n+m) \)th face copy \( F \times Q^m = [f_1 \times Q^n, \ldots, f_n \times Q^n, \phi, \ldots, \phi] \) of a self-dual transformation \( F \) of \( Q^n \). Let \( p \) be a point of \( f_k \times Q^n \cup f_k \times Q^m \) for a \( k \leq n \). If \( p_k = 1 \), then let \( g_i = f_i \times Q^m \cup p \) if \( p \in f_i \times Q^m \), and let \( g_i = f_i \times Q^m / p \) if \( p \notin f_i \times Q^m \). If \( p_k = 0 \) then let \( g_i = f_i \times Q^m \cup \bar{p} \) if \( \bar{p} \notin f_i \times Q^m \), and let \( g_i = f_i \times Q^m / \bar{p} \) if \( \bar{p} \in f_i \times Q^m \). \( G = [g_1, \ldots, g_{n+1}] \) is nonsingular, if \( F \) is nonsingular. We have \( G(p) = F \times Q^m(\bar{p}), G(\bar{p}) = F \times Q^m(p) \), and
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$G(q) = F \times Q^m(q)$ for every other $q$. We call the self-dual transformation $G$ the simple face connection of $F \times Q^m$ at $p$.

The simple face connection was first introduced by Arimoto [1] and was part of the more general construction procedures given by Ishii and Miyazaki [3]. Here we represented it as a special case of face connections in terms of $F[f_1, \ldots, f_n]$.

4. Graph structures of nonsingular threshold transformations

In this section we seek graph structures realized by nonsingular threshold transformations constructed by face connections discussed in Section 3. Although Theorem 3.6 shows a method of constructing every minimal nonsingular threshold transformation from lower-dimensional self-dual threshold transformations, we limit its application to construction from nonsingular threshold transformations. Also, we exclude from our list face copies of lower-dimensional transformations and three special 6-dimensional transformations given in [6], because the former are too obvious, and the latter are too special. In the following, $n$ is the dimension of the domain $Q^n$ of the transformations. First, [6, Examples 1–6] give the following theorem.

**Theorem 4.1** (Ueda [6]). The following graph structures are realized by threshold transformations:

1. $\{(1, 2)\}$.
2. $\{(1, n/r)\}$, where $n/r$ is even.
3. $\{(1, 2n/r)\}$, where $n/r$ is odd.
4. $\{(3, n)\}$, where $n \geq 4$ is even.
5. $\{(2, n)\}$, where $n \geq 3$ is odd.
6. $\{(1, 2n), (1, 2)\}$, where $n \geq 3$ is odd.
7. $\{(2, n), (1, 2)\}$, where $n \geq 4$ is even.

**Open Question.** Determine the graph structure of [6, Example 7].

**Definition 4.2.** In the following, $\tau$ denotes the cyclic permutation

$\tau = (1 \ 2 \ \ldots \ n)$.

**Theorem 4.3.** If $n$ is odd, then $\{(1, n+1)\}$ is realized by a threshold transformation.

**Proof.** Consider the transformation $F$ of [6, Example 2] for $Q^{n-1}$. Specifically, $F = [f, f\tau, \ldots, f\tau^{n-1}]$, where

$f = x_1 \cdots x_m, \bar{x}_m + 1 \cdots \bar{x}_{2m}$ and $m = (n-1)/2$. 
From the $n$th face copy of $F$ through a face connection, we obtain $G=[g_1, \ldots, g_n]$, which is threshold, where

$g_i=f^i \cdot \bar{x}_n$ for $i=1, \ldots, m,$

$g_i=f^i \cdot x_n$ for $i=m+1, \ldots, 2m,$

$g_n=f \cdot x_n.$

The graph structure of $G$ is $\{(1, n+1)\}$, for example,

$(11000) \rightarrow (01100) \rightarrow (00110) \rightarrow (10011) \rightarrow (11001) \rightarrow (11000)$. □

Theorem 4.3. If $n \geq 4$ is even, then $\{(1, n), (2, n-2)\}$ is realized by a threshold transformation.

Proof. Let $n=2m+2$. Consider the transformation of [6, Example 2] for $Q^{2m}$. Specifically, $F=[f, f_1, \ldots, f_{2^{2m-1}}]$, where

$f=x_1 \cdot x_2 \cdots x_m \cdot \bar{x}_{m+1} \cdots \bar{x}_{2m}.$

From the $(2m+1, 2m+2)$th face copy of $F$ through a face connection, we obtain $G=[g_1, \ldots, g_{2m+2}]$, where

$g_i=(f^{i-1}) \cdot (x_{2m+1} \lor x_{2m+2})$ for $i=1, \ldots, m,$

$g_i=(f^{i-1}) \cdot (\bar{x}_{2m+1} \lor x_{2m+2})$ for $i=m+1, \ldots, 2m,$

$g_{2m+1}=g_{2m+2}=f \cdot x_{2m+1} \cdot x_{2m+2}.$

The threshold transformation $G$ realizes the desired graph structure. For example,

$(1000) \rightarrow (0100) \rightarrow (0111) \rightarrow (1011) \rightarrow (1000),$

$(1010) \rightarrow (0110) \rightarrow (1010),$

$(1001) \rightarrow (0101) \rightarrow (1001).$ □

Theorem 4.4. If $n$ is even, then $\{(1, n), (2, n-1)\}$ is realized by a threshold transformation.

Proof. Let $n=2m+2$. Consider the $n$th face copy of the $(n-1)$-dimensional transformation $F$ of [6, Example 3]. Specifically, $F=[f, f_1, \ldots, f_{2^{n-2}}]$, where

$f=f_1 \lor f_2,$

$f_1=x_1 \cdots x_{m+1} \cdot \bar{x}_{m+2} \cdots \bar{x}_{2m+1},$

$f_2=x_1 \cdots x_m \cdot \bar{x}_{m+1} \cdots \bar{x}_{2m+1}.$
From the \(n\)th face copy of \(F\) through a face connection, we obtain the threshold transformation \(G = [g_1, \ldots, g_n]\), where
\[
g_i = f_i \tau_t^{-1} \lor f_2 \tau_t^{-1} \cdot x_n \quad \text{for} \quad i = 1, \ldots, m + 1, \\
g_i = f_1 \tau_t^{-1} \cdot x_n \lor f_2 \tau_t^{-1} \quad \text{for} \quad i = m + 2, \ldots, 2m + 1, \\
g_n = f_2 \cdot x_n.
\]

\(G\) realizes the desired graph structure. For example,
\[
(1100) \rightarrow (0110) \rightarrow (0011) \rightarrow (1001) \rightarrow (1100), \\
(1000) \rightarrow (0100) \rightarrow (0010) \rightarrow (1000), \\
(0111) \rightarrow (1011) \rightarrow (1101) \rightarrow (0111).
\]

**Theorem 4.5.** For any \(n\) and even \(k\) such that \(k \leq 2n\), \(\{(1, k)\}\) is realized by a threshold transformation.

**Proof.** Let
\[
f_i = x_1 \cdots x_i \cdot x_{i+1} \cdots x_n \quad \text{for} \quad i \leq k/2 - 1, \\
f_i = x_1 \cdots x_n \quad \text{for} \quad i \geq k/2.
\]

Then \(F = [f_1, \ldots, f_n]\) realizes the desired graph structure. For example,
\[
(11111) \rightarrow (11000) \rightarrow (10000) \rightarrow (00000) \rightarrow (00111) \rightarrow (01111) \rightarrow (11111).
\]

**Remark 4.6.** For \(k = 2n\), this is a theorem of Masters and Mattson [4]. The \(n\)-dimensional \(F\) of Theorem 4.5 is obtained from the \(n\)th face copy of the \((n-1)\)-dimensional \(F\) through a face connection. The graph structure of Theorem 4.2 is the same as the one in Theorem 4.5, but the corresponding two transformations that realize them are not equivalent. Theorem 4.5 can be further generalized into the following theorem.

**Theorem 4.7.** For any \(n, m\) and even \(k\) such that \(1 \leq m \leq n\) and \(k \leq 2m\), \(\{(1, k+2), (2^{n-m} - 2, k)\}\) is realized by a threshold transformation.

**Proof.** Let the \(m\)-dimensional threshold transformation of Theorem 4.5 be \([f_1, \ldots, f_m]\). If \(G = [g_1, \ldots, g_n]\) is defined as
\[
g_i = f_i \cdot (x_{m+1} \lor \cdots \lor x_n) \quad \text{for} \quad i = 1, \ldots, m \quad \text{and} \\
g_i = x_1 \cdots x_n \quad \text{for} \quad i = m + 1, \ldots, n,
\]

then \(G\) realizes the desired graph structure. For example, if \(n = 5, m = 3\) and \(k = 4\), then
\[
(10000) \rightarrow (00000) \rightarrow (00011) \rightarrow (01111) \rightarrow (11111) \rightarrow (11100) \rightarrow (10000), \\
(10001) \rightarrow (00001) \rightarrow (01101) \rightarrow (11101) \rightarrow (10001), \\
(10010) \rightarrow (00010) \rightarrow (01110) \rightarrow (11110) \rightarrow (10100).
\]
The following lemma is clear.

**Lemma 4.8.** If \( f(x_1, \ldots, x_n) \) is a threshold function, then \( f \cdot x_{n+1} \) and \( f \lor x_{n+1} \) are also threshold functions.

We now prove the following theorem which gives one of the two main classes of nonsingular threshold transformations; the other is given by Theorem 4.7.

**Theorem 4.9.** The following graph structure is realized by a threshold transformation, where \( n_1 + \cdots + n_k < n \), \( k \geq 0 \) and \( n_i \geq 1 \):
\[
\{(1, 2^{k+1}), (2^{n_k} - 2, 2^k), (2^{n_k-1} - 2) \cdot 2^{n_k}, 2^{k-1}), \ldots, (2^{n_1} - 2) \cdot 2^{n_2} \cdots + n_k, 2)\}.
\]

**Proof.** (1) If \( k = 0 \), then \( \{(1, 2)\} \) is realized by the transformation \( F = [f_1, \ldots, f_n] \) of [6, Example 1], where
\[
f_i = x_1 \cdots x_n \quad \text{for } i = 1, \ldots, n.
\]

(2) \( \{(2^n, 2)\} \), where \( n = m + n_1 \), is realized by the \((m + 1, \ldots, n)\)th face copy of (1) for \( n = m \).

(3) \( \{(1, 4), (2^n-2, 2)\} \), where \( n = m + n_1 \), is realized by the simple face connection \( F = [f_1, \ldots, f_n] \) of (2) at \((1, \ldots, 1)\), where
\[
f_i = x_1 \cdots x_m / (1 \ldots 1) = x_1 \cdots x_m \cdot (\bar{x}_{m+1} \lor \cdots \lor \bar{x}_n) \quad \text{for } i = 1, \ldots, m
\]
and
\[
f_i = x_1 \cdots x_n \quad \text{for } i = m + 1, \ldots, n
\]
are clearly threshold functions.

(4) \( \{(2^n, 4), (2^n - 2, 2^n, 2)\} \), where \( n = m + n_1 + n_2 \), is realized by the \((m + n_1 + 1, \ldots, n)\)th face copy of (3) for \( n = m + n_1 \).

(5) \( \{(1, 8), (2^n-2, 4), (2^n-2) \cdot 2^n, 2)\} \), where \( n = m + n_1 + n_2 \), is realized by the simple face connection \( F = [f_1, \ldots, f_n] \) of (4) at \((1, \ldots, 1)\), where
\[
f_i = x_1 \cdots x_m \cdot (\bar{x}_{m+1} \lor \cdots \lor \bar{x}_{m+n_1}) \lor (1 \ldots 1)
\]
\[
= x_1 \cdots x_m \cdot (\bar{x}_{m+1} \lor \cdots \lor \bar{x}_{m+n_1} \lor x_{m+n_1+1} \cdots x_n) \quad \text{for } i = 1, \ldots, m
\]
\[
f_i = x_1 \cdots x_{m+n_1} / (1 \ldots 1)
\]
\[
= x_1 \cdots x_{m+n_1} \cdot (\bar{x}_{m+n_1+1} \lor \cdots \lor \bar{x}_n) \quad \text{for } i = m + 1, \ldots, m + n_1
\]
and
\[
f_i = x_1 \cdots x_n \quad \text{for } i = m + n_1 + 1, \ldots, n
\]
are threshold functions by the lemma.

(6) \( \{(2^n, 8), (2^n-2, 2^n, 4), (2^n-2) \cdot 2^n, 2)\} \), where \( n = m + n_1 + n_2 + n_3 \), is realized by the \((m + n_1 + n_2 + 1, \ldots, n)\)th face copy of (5) for \( n = m + n_1 + n_2 \).
The iterative procedure above leads to the desired graph structure, which is realized by a threshold transformation. □

**Corollary 4.10.** For any $n$ and $k$ such that $1 \leq k \leq n$, $\{(1,2^k)\}$ is realized by a threshold transformation.

**Remark 4.11.** For $k=n$, this corollary is a theorem of Arimoto [1]. He also proved that for any positive integer $s$ such that $s \leq 2^n$, there exists a threshold transformation $F$ of $Q^n$ (not necessarily nonsingular) such that the graph of $F$ has an $s$-cycle as its subgraph.

**References**