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Approximation of fixed points of weakly contractive nonself maps in Banach spaces

C.E. Chidume, H. Zegeye,[1] and S.J. Aneke[2]

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

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Abstract

Let K be a closed convex subset of a real uniformly smooth Banach space E. Suppose K is a nonexpansive retract of E with P as the nonexpansive retraction. Let $T : K \to E$ be a d-weakly contractive map such that a fixed point $x^* \in \text{int}(K)$ of T exists. It is proved that a descent-like approximation sequence converges strongly to x^*. Furthermore, if K is a nonempty closed convex subset of an arbitrary real Banach space and $T : K \to K$ is a uniformly continuous d-weakly contractive map with $F(T) := \{x \in K\colon\ Tx = x\} \neq \emptyset$, it is proved that a descent-like approximation sequence converges strongly to $x^* \in F(T)$.
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1. Introduction

Let E be a real normed linear space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{f^* \in X^*\colon\ \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\right\},$$

E-mail addresses: chidume@ictp.trieste.it (C.E. Chidume), habz@ictp.trieste.it (H. Zegeye), syma@infoweb.abs.net (S.J. Aneke).

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[2] Permanent address: Department of Mathematics, University of Nigeria, Nsukka.

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where $\langle .,. \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued and if E^* is uniformly convex then J is uniformly continuous on bounded subsets of E (see, e.g., [1]). We shall denote the single-valued duality mapping by j. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1\colon \|x\| = 1,\ \|y\| = \tau \right\}, \quad \tau > 0.$$

E is said to be uniformly smooth if $\lim \rho_E(\tau)/\tau = 0$ as $\tau \to 0$. Typical examples of such spaces are the *Lebesgue* L_p, *the sequence* l_p, and the *Sobolev* W_p^m spaces, $1 < p < \infty$.

Let $K \subseteq E$ be closed convex and let P be a mapping of E onto K. Then P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geqslant 0$. A mapping P of E to E is said to be a *retraction* if $P^2 = P$. A subset K of E is said to be a *sunny nonexpnsive retract* of E if there exists a sunny nonexpansive retraction of E onto K. *If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.*

A mapping T with domain $D(T)$ and range $R(T)$ in E is called *d-weakly contractive* if there exists a continuous and nondecreasing function $\Phi : [0,\infty) := \Re^+ \to \Re^+$ such that Φ is positive on $\Re^+ \setminus \{0\}$, $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$ and for $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\left|\left\langle Tx - Ty, j(x-y)\right\rangle\right| \leqslant \|x-y\|^2 - \Phi\left(\|x-y\|^2\right). \tag{1.1}$$

It is called *weakly contractive* (see, e.g., [2–4]) if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and Φ as above such that

$$\|Tx - Ty\| \leqslant \|x-y\| - \Phi\left(\|x-y\|\right). \tag{1.2}$$

If $F(T) \neq \emptyset$ and inequalities (1.1) and (1.2) hold for $x \in D(T)$ and $x^* \in F(T)$, then the operators will be called *d-weakly hemi-contractive* and *weakly hemi-contractive*, respectively. Note that, if we set $\Phi(t^2) = \psi(t)$, then ψ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that ψ is positive on $\Re^+ \setminus \{0\}$, $\psi(0) = 0$, $\lim_{t\to\infty} \psi(t) = \infty$. Thus, the above definition of d-weakly contractive map can be restated as follows: for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and ψ as above such that

$$\left|\left\langle Tx - Ty, j(x-y)\right\rangle\right| \leqslant \|x-y\|^2 - \psi\left(\|x-y\|\right). \tag{1.3}$$

The d-weakly contractive operators were first introduced and studied by Alber and Guerre-Delabriere [3] and include several important classes of nonlinear operators. In particular, they include the weakly contractive operators.

In [3], Alber and Guerre-Delabriere proved the following theorem.

Theorem AG. *Let $T : G \to H$ be a d-weakly contractive map, G a closed convex* bounded *subset of a Hilbert space H and suppose that a fixed point $x^* \in \operatorname{int}(G)$ of T exists. Then the sequence $\{x_n\}$ defined by*

$$x_1 \in G;\ x_{n+1} := P_G\big(x_n - \alpha_n(x_n - Tx_n)\big), \quad n = 1, 2, \ldots, \tag{1.4}$$

where P_G is the metric projection onto the set G, $\{\alpha_n\}$ is a sequence of positive numbers such that $\sum_1^\infty \alpha_n = \infty$ and $\lim_{n\to 0} \alpha_n = 0$ converges strongly to x^*. Moreover, there exist a constant $K > 0$ and a bounded sequence $\{x_{n_l}\} \subset \{x_n\}$, $l = 1, 2, \ldots$ such that

$$\|x_{n_l} - x^*\|^2 \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + K^2\alpha_{n_l}\right), \tag{1.5}$$

Furthermore,

$$\|x_{n_l+1} - x^*\|^2 \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + K^2\alpha_{n_l}\right) + K^2\alpha_{n_l}^2, \tag{1.6}$$

$$\begin{aligned} &\|x_n - x^*\|^2 \leqslant \|x_{n_l+1} - x^*\|^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^{m-1} \alpha_j}, \\ &\quad n_l + 1 \leqslant n < n_l + 1, \end{aligned} \tag{1.7}$$

$$\begin{aligned} &\|x_{n+1} - x^*\|^2 \leqslant \|x_1 - x^*\|^2 - \sum_1^n \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leqslant \|x_1 - x^*\|^2, \\ &\quad 1 \leqslant n \leqslant n_1 - 1, \end{aligned} \tag{1.8}$$

$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{s\colon \sum_1^s \frac{\alpha_m}{\sum_1^m \alpha_j} \leqslant \|x_1 - x^*\|^2\right\}. \tag{1.9}$$

From Theorem AG, two questions arise quite naturally.

Question 1. Can the boundedness condition on G in Theorem AG be dropped?

Question 2. Can Theorem AG be extended to Banach spaces more general than Hilbert spaces?

It is our purpose in this paper to give affirmative answers to these questions. In particular, we prove that Theorem AG remains true in *real uniformly smooth Banach spaces* and *without the boundedness condition imposed on G*. Furthermore, we prove a related convergence theorem in our more general setting when the fixed point x^* of T exists but is not necessarily in the interior of G. Finally, we prove a convergence theorem for approximating a fixed point of a uniformly continuous d-weakly contractive and bounded *self map T* of G with $F(T) \neq \emptyset$, *in arbitrary real Banach spaces*.

2. Preliminaries

In the sequel we shall use the following well known lemmas.

Lemma 2.1 (see, e.g., [5]). *Let E be a real Banach space and J the normalized duality map on E. Then for any given $x, y \in E$, the following inequality holds*:

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma AG [2]. *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ be a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\gamma_n/\alpha_n \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leqslant \lambda_n - \alpha_n \phi(\lambda_n) + \gamma_n, \quad n = 1, 2, \ldots, \tag{2.1}$$

be given where ϕ is a continuous and nondecreasing function from $\Re^+$ to $\Re^+$ such that it is positive on $\Re^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Then

(a) $\lambda_n \to 0$, *as* $n \to \infty$*;*
(b) *there exists a subsequence* $\{\lambda_{n_k}\} \subset \{\lambda_n\}$, $l = 1, 2, \ldots,$ *such that*

$$\lambda_{n_l} \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\gamma_{n_l}}{\alpha_{n_l}}\right), \tag{2.2}$$

$$\lambda_{n_l+1} \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + \frac{\gamma_{n_l}}{\alpha_{n_l}}\right) + \gamma_{n_l}, \tag{2.3}$$

$$\lambda_n \leqslant \lambda_{n_l+1} - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{A_m}, \quad n_l + 1 \leqslant n < n_l + 1, \quad A_m = \sum_1^{m-1} \alpha_i, \tag{2.4}$$

$$\lambda_{n+1} \leqslant \lambda_1 - \sum_1^n \frac{\alpha_m}{A_m} \leqslant \lambda_1, \quad 1 \leqslant n \leqslant n_1 - 1, \tag{2.5}$$

$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{s\colon \sum_1^s \frac{\alpha_m}{A_m} \leqslant \lambda_1\right\}. \tag{2.6}$$

We shall also need the following lemma whose proof is identical with the proof of Lemma 5.6 of [3]. However, for completeness, we give a sketch of the proof.

Lemma 2.2. *Let E be an arbitrary real Banach space and let $T : D(T) \subseteq E \to E$ be a d-weakly contractive map, and suppose that a fixed point $x^* \in \operatorname{int}(K)$ of T exists. Then $A := I - T$ is bounded.*

Proof. Clearly A is accretive. Then by Lemma 5.5 of [3] (see also [6]) there exists a constant $r_0 > 0$ and a closed ball $S(r_0, x^*) \subset D(A)$ such that *for all* $x \in D(A)$ we have

$$\langle Ax - Ax^*, j(x - x^*)\rangle \geqslant r_0\|Ax\| - c_0\big(\|x - x^*\| + r_0\big), \tag{2.7}$$

where $c_0 = \sup_{\eta\in S(r_0,x^*)} \|A(\eta)\| < \infty$. On the other hand, for some $j(x - x^*) \in J(x - x^*)$ we have that

$$\begin{aligned}\langle Ax - Ax^*, j(x - x^*)\rangle &= \langle x - x^*, j(x - x^*)\rangle - \langle Tx - Tx^*, j(x - x^*)\rangle\\ &\leqslant \|x - x^*\|^2 + \big|\langle Tx - Tx^*, j(x - x^*)\rangle\big|\\ &\leqslant 2\|x - x^*\|^2.\end{aligned} \tag{2.8}$$

Thus from (2.7) and (2.8) we get that

$$\|Ax\| \leqslant r_0^{-1}\big(2\|x - x^*\|^2 + c_0\big(\|x - x^*\| + r_0\big)\big). \tag{2.9}$$

Hence the conclusion holds. □

3. Main results

Now, we state and prove the following theorems.

Theorem 3.1. *Let E be a real uniformly smooth Banach space. Suppose K is a closed convex subset of E which is a nonexpansive retract of E with P as the nonexpansive retraction. Suppose $T : K \to E$ is a d-weakly contractive map such that a fixed point $x^* \in \operatorname{int}(K)$ of T exists. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := P\big(x_n - \alpha_n(x_n - Tx_n)\big), \quad n \geqslant 1, \tag{3.1}$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leqslant d_0$, $\{x_n\}$ converges strongly to $x^* \in F(T)$. Moreover, there exist a constant $d > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$\|x_{n_l} - x^*\|^2 \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l}\alpha_m} + d\overline{\gamma}_{n_l}\right), \tag{3.2}$$

where $\overline{\gamma}_n := \|j(p_n - x^*) - j(x_n - x^*)\|$ and $p_n := x_n - \alpha_n Ax_n$. Furthermore,

$$\|x_{n_l+1} - x^*\|^2 \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l}\alpha_m} + d\overline{\gamma}_{n_l}\right) + d\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{3.3}$$

$$\begin{aligned}&\|x_n - x^*\|^2 \leqslant \|x_{n+1} - x^*\|^2 - \sum_{n_l+1}^{n-l}\frac{\alpha_m}{\sum_1^{m-1}\alpha_j},\\ &\quad n_l + 1 \leqslant n < n_l + 1,\end{aligned} \tag{3.4}$$

$$\|x_{n_l+1} - x^*\|^2 \leqslant \|x_1 - x^*\|^2 - \sum_1^n \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leqslant \|x_1 - x^*\|^2,$$
$$1 \leqslant n \leqslant n_1 - 1, \tag{3.5}$$
$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{ s\colon \sum_1^s \frac{\alpha_m}{\sum_1^m \alpha_j} \leqslant \|x_1 - x^*\|^2 \right\}. \tag{3.6}$$

Proof. Observe that the recursion formula (3.1) can be written as follows:

$$x_{n+1} = P(x_n - \alpha_n A x_n), \quad n \geqslant 0, \text{ where } A := (I - T). \tag{3.7}$$

Moreover, we have that $\langle Ax - Ax^*, j(x - x^*)\rangle \geqslant \Phi(\|x - y\|^2)$, where Φ is as in (1.1). Now, choose r sufficiently large such that $x_1 \in B_r(x^*)$. Let $G := B_r(x^*) \cap K$, then since by Lemma 2.2 A is bounded we have that $A(G)$ is bounded. Let diam $A(G) = \sigma$. As j is uniformly continuous on bounded subsets of E, for $\varepsilon = \Phi((r/2)^2)/(2\sigma)$ there exists a $\delta > 0$ such that $x, y \in D(T)$, $\|x - y\| < \delta$ implies $\|j(x) - j(y)\| < \varepsilon$. Set $d_0 = \min\{1, \delta/(2\sigma), r/(2\sigma)\}$.

Claim: $\{x_n\}$ *is bounded*. Suffices to show that x_n is in G for all $n \geqslant 1$. The proof is by induction. By our assumption $x_1 \in G$. Suppose $x_n \in G$. We prove that $x_{n+1} \in G$. Assume for contradiction that $x_{n+1} \notin G$. Then, since $x_{n+1} \in K$ $\forall n \geqslant 1$, we have that $\|x_{n+1} - x^*\| > r$. Thus we have the following estimates:

$$\begin{aligned}\|x_{n+1} - x^*\| &= \big\|P(x_n - \alpha_n A x_n) - P x^*\big\| \\ &\leqslant \big\|x_n - x^* - \alpha_n(Ax_n - Ax^*)\big\|\end{aligned}$$

and hence

$$\begin{aligned}\|x_n - x^*\| &\geqslant \|x_{n+1} - x^*\| - \alpha_n\|Ax_n - Ax^*\| \\ &> r - \alpha_n\sigma \geqslant r - \frac{r}{2} = \frac{r}{2}.\end{aligned}$$

Set $p_n := x_n - \alpha_n A x_n$. Then from (3.1), Lemma 2.1 and the above estimates we have that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \big\|P(x_n - \alpha_n(Ax_n - Ax^*)) - Px^*\big\|^2 \\ &\leqslant \big\|x_n - x^* - \alpha_n(Ax_n - Ax^*)\big\|^2 \\ &\leqslant \|x_n - x^*\|^2 - 2\alpha_n\big\langle Ax_n - Ax^*, j(p_n - x^*)\big\rangle \\ &= \|x_n - x^*\|^2 - 2\alpha_n\big\langle Ax_n - Ax^*, j(x_n - x^*)\big\rangle \\ &\quad - 2\alpha_n\big\langle Ax_n - Ax^*, j(p_n - x^*) - j(x_n - x^*)\big\rangle \\ &\leqslant \|x_n - x^*\|^2 - 2\alpha_n\Phi\big(\|x_n - x^*\|^2\big) \\ &\quad + 2\alpha_n\|Ax_n\|\big\|j(p_n - x^*) - j(x_n - x^*)\big\|.\end{aligned} \tag{3.8}$$

Since $\|p_n - x_n\| \leqslant \alpha_n\|Ax_n\| \leqslant \alpha_n\sigma < \delta$ we have that $\|j(p_n - x^*) - j(x_n - x^*)\| \leqslant \Phi((r/2)^2)/(2\sigma)$. Thus (3.8) gives that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leqslant \|x_n-x^*\|^2-2\alpha_n\Phi\left(\left(\frac{r}{2}\right)^2\right)+2\alpha_n\sigma\frac{\Phi((\frac{r}{2})^2)}{2\sigma}\\ &=\|x_n-x^*\|^2-2\alpha_n\Phi\left(\left(\frac{r}{2}\right)^2\right)+\alpha_n\Phi\left(\left(\frac{r}{2}\right)^2\right)\\ &=\|x_n-x^*\|^2-\alpha_n\Phi\left(\left(\frac{r}{2}\right)^2\right)<r^2, \end{aligned}\tag{3.9}$$

i.e., $\|x_{n+1}-x^*\|<r$, a contradiction. Therefore $x_{n+1}\in G$. Thus by induction $\{x_n\}$ is bounded. Now we show that $x_n\to x^*$. Note that $p_n-x_n\to 0$ as $n\to\infty$ and hence by the uniform continuity of j on bounded subsets of E we have that

$$\overline{\gamma}_n := \big\| j(p_n-x^*)-j(x_n-x^*)\big\|\to 0 \quad \text{as } n\to\infty. \tag{3.10}$$

Let $\lambda_n := \|x_n-x^*\|^2$ and $\gamma_n := 2\alpha_n\sigma\overline{\gamma}_n$, then from inequality (3.8) we obtain that

$$\lambda_{n+1}\leqslant \lambda_n-2\alpha_n\Phi(\lambda_n)+\gamma_n, \tag{3.11}$$

where $\gamma_n/\alpha_n\to 0$ as $n\to\infty$. Thus, the conclusions of the theorem follow from Lemma AG, completing the proof of the theorem. □

If $x^*\in F(T)$ is an arbitrary point of $D(T)$ then we have the following theorem.

Theorem 3.2. *Let K be a closed convex subset of a real uniformly smooth Banach space. Suppose K is a nonexpansive retract of E with P as the nonexpansive retraction. Let $T: K\to E$ be a d-weakly contractive bounded map with $F(T):=\{x\in K\colon Tx=x\}\neq\emptyset$. For arbitrary $x_1\in K$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := P\big(x_n-\alpha_n(x_n-Tx_n)\big),\quad n\geqslant 1, \tag{3.12}$$

where $\lim\alpha_n=0$ and $\sum\alpha_n=\infty$. Then, there exists a constant $d_0>0$ such that if $0<\alpha_n\leqslant d_0$, then, $\{x_n\}$ converges strongly to $x^*\in F(T)$. Moreover, there exist a constant $d>0$ and a subsequence $\{x_{n_l}\}\subseteq\{x_n\}$ such that

$$\|x_{n_l}-x^*\|^2\leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l}\alpha_m}+d\overline{\gamma}_{n_l}\right), \tag{3.13}$$

where $\overline{\gamma}_n$ is as defined in (3.10). *Furthermore,*

$$\|x_{n_l+1}-x^*\|^2\leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l}\alpha_m}+d\overline{\gamma}_{n_l}\right)+d\alpha_{n_l}\overline{\gamma}_{n_l}, \tag{3.14}$$

$$\begin{aligned}&\|x_n-x^*\|^2\leqslant \|x_{n+1}-x^*\|^2-\sum_{n_l+1}^{n-l}\frac{\alpha_m}{\sum_1^{m-1}\alpha_j},\\ &\quad n_l+1\leqslant n<n_l+1,\end{aligned}\tag{3.15}$$

$$\|x_{n_l+1} - x^*\|^2 \leqslant \|x_1 - x^*\|^2 - \sum_1^n \frac{\alpha_m}{\sum_1^{m-1} \alpha_j} \leqslant \|x_1 - x^*\|^2,$$
$$1 \leqslant n \leqslant n_1 - 1, \tag{3.16}$$
$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{s\colon \sum_1^s \frac{\alpha_m}{\sum_1^m \alpha_j} \leqslant \|x_1 - x^*\|^2\right\}. \tag{3.17}$$

Proof. Since we have by hypothesis that A is bounded, the proof follows as in the proof of Theorem 3.1 without the use of Lemma 2.2. □

If T is a *self map* and $0 \leqslant \alpha_n < 1$, the use of the operator P will not be necessary. To present our next theorem, we shall need the following lemma.

Lemma 3.3. *Let $\{\lambda_k\}$ and $\{\gamma_k\}$ be sequences of nonnegative numbers and $\{\alpha_k\}$ a sequence of positive numbers satisfying the conditions $\sum_1^\infty \alpha_n = \infty$ and $\gamma_n/\alpha_n \to 0$, as $n \to \infty$. Let the recursive inequality*

$$\lambda_{n+1} \leqslant \lambda_n - 2\alpha_n \phi(\lambda_{n+1}) + \gamma_n, \quad n = 1, 2, \ldots, \tag{3.18}$$

be given where ϕ is a nondecreasing function from $\Re^+$ to $\Re^+$ such that it is positive on $\Re^+ \setminus \{0\}$, $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$. Then $\lambda_n \to 0$, as $n \to \infty$.

Proof. Let $\liminf \lambda_n = a \geqslant 0$. *Claim*: $a = 0$. Suppose not. Then there exists $N_1 > 0$ such that $\lambda_n \geqslant a/2$ $\forall n \geqslant N_1$. Since $\gamma_n/\alpha_n \to 0$, there exists $N_2 > 0$ such that $\gamma_n/\alpha_n \leqslant \phi(a/2)$ which implies $\gamma_n \leqslant \alpha_n \phi(a/2)$ $\forall n \geqslant N_2$. Then for $n \geqslant N = \max\{N_1, N_2\}$ we have from (3.18) that

$$\lambda_{n+1} \leqslant \lambda_n - 2\alpha_n \phi\left(\frac{a}{2}\right) + \alpha_n \phi\left(\frac{a}{2}\right) = \lambda_n - \alpha_n \phi\left(\frac{a}{2}\right), \quad \forall n > N,$$

which implies that $\phi(a/2) \sum \alpha_n < \infty$, a contradiction. Therefore, $a = 0$. Thus, there exists a subsequence $\{\lambda_{n_j}\} \subset \{\lambda_n\}$ such that $\lim \lambda_{n_j} = 0$. For arbitrary $\varepsilon > 0$ let $N_3 > 0$ such that $\lambda_{n_j} < \varepsilon/4$ $\forall j \geqslant N_3$ and $N_4 > 0$ such that $\gamma_n \leqslant 2\alpha_n \phi(\varepsilon/4)$. Let $N_* := \max\{N_3, N_4\}$ and fix $j_* > N_*$. Then we show that $\lambda_{n_{j_*}+k} < \varepsilon/4$ $\forall k \in N \cup \{0\}$. For $k = 0$ the result clearly holds. Suppose it holds for any $k > 0$. Then we show that it holds for $k + 1$. Suppose not. Then we have $\lambda_{n_{j_*}+k+1} > \varepsilon/4$ and hence from (3.18) we get that

$$\begin{aligned}\frac{\varepsilon}{4} < \lambda_{n_{j_*}+k+1} &\leqslant \lambda_{n_{j_*}+k} - 2\alpha_n \phi(\lambda_{n_{j_*}+k+1}) + 2\alpha_n \phi\left(\frac{\varepsilon}{4}\right) \\ &\leqslant \lambda_{n_{j_*}+k} - 2\alpha_{n_{j_*}+k} \phi\left(\frac{\varepsilon}{4}\right) + 2\alpha_{n_{j_*}+k} \phi\left(\frac{\varepsilon}{4}\right) = \lambda_{n_{j_*}+k},\end{aligned}$$

a contradiction. Therefore, $\lambda_{n_{j_*}+k} < \varepsilon/4$ $\forall k \in N \cup \{0\}$ and hence $\lambda_n \to 0$ as $n \to \infty$. □

Theorem 3.4. *Let K be a closed convex subset of a real Banach space. Suppose $T: K \to K$ is a uniformly continuous d-weakly contractive map with $F(T) := \{x \in K\colon Tx = x\} \neq \emptyset$. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := x_n - \alpha_n(x_n - Tx_n), \quad n \geqslant 1, \tag{3.19}$$

where $\lim \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, there exists a constant $d_0 > 0$ such that if $0 < \alpha_n \leqslant d_0$, then, $\{x_n\}$ converges strongly to $x^* \in F(T)$. Moreover, there exist a constant $d > 0$ and a subsequence $\{x_{n_l}\} \subseteq \{x_n\}$ such that

$$\|x_{n_l} - x^*\|^2 \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l} \alpha_m} + d\overline{\gamma}_{n_l}\right). \tag{3.20}$$

where $\overline{\gamma}_n := \|(I-T)x_{n+1} - (I-T)x_n\|$. Furthermore,

$$\|x_{n_l+1} - x^*\|^2 \leqslant \phi^{-1}\left(\frac{1}{\sum_1^{n_l+1} \alpha_m} + d\overline{\gamma}_{n_l}\right) + d\overline{\gamma}_{n_l}, \tag{3.21}$$

$$\|x_n - x^*\|^2 \leqslant \|x_{n+1} - x^*\|^2 - \sum_{n_l+1}^{n-l} \frac{\alpha_m}{\sum_1^m \alpha_j}, \quad n_l + 1 \leqslant n < n_l + 1, \tag{3.22}$$

$$\|x_{n_l+1} - x^*\|^2 \leqslant \|x_1 - x^*\|^2 - \sum_1^n \frac{\alpha_m}{\sum_1^m \alpha_j} \leqslant \|x_1 - x^*\|^2,$$
$$1 \leqslant n \leqslant n_1 - 1, \tag{3.23}$$

$$1 \leqslant n_1 \leqslant s_{\max} = \max\left\{s\colon \sum_1^s \frac{\alpha_m}{\sum_1^m \alpha_j} \leqslant \|x_1 - x^*\|^2\right\}. \tag{3.24}$$

Proof. Let $x^* \in F(T)$ and let G, r and σ be as in the proof of Theorem 3.1. By uniform continuity of A, for $\varepsilon = \Phi(r^2)/(4r)$, there exists $\delta_* > 0$ such that $\|x - y\| < \delta_*$ implies $\|Ax - Ay\| < \varepsilon$ for all $x, y \in D(T)$. Choose any $0 < \delta \leqslant \delta_*$ and set $d_0 := \min\{1, \delta/2\sigma, r/\sigma\}$.

Claim: $x_n \in G$ $\forall n \geqslant 1$. We show this by induction. By our choice $x_1 \in G$. Suppose $x_n \in G$. We show that $x_{n+1} \in G$. Suppose not, then $\|x_{n+1} - x^*\| > r$ and from (3.19) we have $\|x_{n+1} - x^*\| \leqslant \|x_n - x^*\| + \alpha_n\|Ax_n\| \leqslant r + d_0\sigma \leqslant 2r$.

Now, by Lemma 2.1 and the above estimates we have that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leqslant \|x_n - x^*\|^2 - 2\alpha_n\langle Ax_n - Ax^*, j(x_{n+1} - x^*)\rangle \\
&= \|x_n - x^*\|^2 - 2\alpha_n\langle Ax_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n\langle Ax_{n+1} - Ax_n, j(x_{n+1} - x^*)\rangle \\
&\leqslant \|x_n - x^*\|^2 - 2\alpha_n\Phi(\|x_{n+1} - x^*\|^2) \\
&\quad + 2\alpha_n\|Ax_{n+1} - Ax_n\|\,\|x_{n+1} - x^*\|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \|x_n - x^*\|^2 - 2\alpha_n \Phi\big(\|x_{n+1} - x^*\|^2\big)\frac{\|x_{n+1} - x^*\|}{2r} \\
&\quad + 2\alpha_n \|Ax_{n+1} - Ax_n\|\,\|x_{n+1} - x^*\| \\
&\leqslant \|x_n - x^*\|^2 \\
&\quad - 2\alpha_n\left(\frac{\Phi(r^2)}{2r} - \|Ax_{n+1} - Ax_n\|\right)\|x_{n+1} - x^*\| \\
&\leqslant \|x_n - x^*\|^2 - 2\alpha_n\left(\frac{\Phi(r^2)}{2r} - \varepsilon\right)\|x_{n+1} - x^*\| \\
&\leqslant \|x_n - x^*\|^2 - \alpha_n\frac{\Phi(r^2)}{2r}\|x_{n+1} - x^*\|, \quad \text{since } \varepsilon = \frac{\Phi(r^2)}{4r} \\
&\leqslant \|x_n - x^*\|^2 < r^2, \qquad (3.25)
\end{aligned}$$

and hence $\|x_{n+1} - x^*\| < r$, a contradiction. Therefore, the claim holds. Now we show that $x_n \to x^*$. Since $x_{n+1} - x_n \to 0$, by the uniform continuity of A we have that

$$\overline{\gamma}_{n_l} := \|Ax_{n+1} - Ax_n\| \to 0 \quad \text{as } n \to \infty.$$

Let $\lambda_n := \|x_n - x^*\|^2$, $\gamma_n := 2\alpha_n\sigma\overline{\gamma}_{n_l}$. Then, (3.24) gives

$$\lambda_{n+1} \leqslant \lambda_n - 2\alpha_n\Phi(\lambda_{n+1}) + \gamma_n.$$

Thus, the conclusion follows from Lemma 3.3. □

Remark 3.5. Theorem 3.1 extends Theorem AG from real Hilbert spaces to the more general real uniformly smooth Banach spaces. Furthermore, the *boundedness* assumption imposed on K in Theorem AG is not needed in our more general setting.

Remark 3.6. Theorems 3.1 and 3.2 also hold, without any modification in the proofs, for d-weakly hemi-contractive maps.

Remark 3.7. Observe that if T is weakly contractive then clearly it is uniformly continuous and is hence bounded. Moreover, it is d-weakly contractive and in Hilbert spaces, $F(T) \neq \emptyset$ (see, e.g., [4,7]). Therefore, Theorem 3.4 extends Theorem 6.1 of [3] from the class of weakly contractive maps to the class of d-weakly contractive maps.

References

[1] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II, Springer, Berlin, 1979.

[2] Ya. Alber, S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, Operator Theory, Advances and Applications 98 (1997) 7–22.

[3] Ya. Alber, S. Guerre-Delabriere, On the projection methods for fixed point problems, Analysis 21 (2001) 17–39.
[4] Ya. Alber, S. Guerre-Delabriere, L. Zalenko, The principle of weakly contractive maps in metric spaces, Commun. Applied Nonlinear Analysis 5 (1) (1998) 45–68.
[5] C.E. Chidume, H. Zegeye, B. Ntatin, A generalized steepest descent approximation for the zeros of m-accretive operators, J. Math. Anal. Appl. 261 (1) (1999) 48–73.
[6] A.I. Perov, Y.V. Trubnikov, Differential Equations with Monotone Nonlinearities, Nauka i Technika, Minsk, 1986.
[7] Ya. Alber, S. Guerre-Delabriere, Problems of fixed point theory in Hilbert and Banach spaces, Functional Differential Equations 2 (1994) 5–10.