



Modified zeta functions as kernels of integral operators

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Abstract

The modified zeta functions $\sum_{n \in K} n^{-s}$, where $K \subset \mathbb{N}$, converge absolutely for $\operatorname{Re} s > 1$. These generalise the Riemann zeta function which is known to have a meromorphic continuation to all of \mathbb{C} with a single pole at $s = 1$. Our main result is a characterisation of the modified zeta functions that have pole-like behaviour at this point. This behaviour is defined by considering the modified zeta functions as kernels of certain integral operators on the spaces $L^2(I)$ for symmetric and bounded intervals $I \subset \mathbb{R}$. We also consider the special case when the set $K \subset \mathbb{N}$ is assumed to have arithmetic structure. In particular, we look at local L^p integrability properties of the modified zeta functions on the abscissa $\operatorname{Re} s = 1$ for $p \in [1, \infty]$.

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1. Introduction

We consider the behaviour of the modified zeta functions defined by

$$\zeta_K(s) = \sum_{n \in K} \frac{1}{n^s}, \quad K \subset \mathbb{N}, \quad (1.1)$$

near the point $s = 1$. Here $s = \sigma + it$ denotes the complex variable. The infinite series defining these functions converge absolutely in the half-plane $\sigma > 1$. We refer to these as K -zeta functions. Note that for $K = \mathbb{N}$, the formula (1.1) defines the Riemann zeta function.

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The main objective of this paper is, for general $K \subset \mathbb{N}$, to find an operator-theoretic generalisation of the classical result, due to B. Riemann, that the Riemann zeta function can be expressed as

$$\zeta(s) = \frac{1}{s-1} + \psi(s), \tag{1.2}$$

where ψ is an entire function. (See [5] for an extensive discussion, as well as an English translation, of Riemann’s original paper.)

To motivate this approach, we note that although N. Kurokawa [16] found sufficient conditions on the sets K for ζ_K to have an analytic continuation across the abscissa $\sigma = 1$, it was shown by J.-P. Kahane and H. Queffelec [11,26] that for most choices of the subset K , in the sense of Baire categories, the K -zeta functions have the abscissa $\sigma = 1$ as a natural boundary. So, instead of looking at the formula (1.2) as a statement about analytic continuation, we consider it as saying that the local behaviour of $\zeta(s)$ at $s = 1$ is an analytic, and therefore small, perturbation of a pole with residue one.

To interpret this in operator-theoretic terms, we define, for $K \subset \mathbb{N}$ and intervals I of the form $(-T, T)$ with finite $T > 0$, the family of operators

$$\mathcal{Z}_{K,I} : g \in L^2(I) \mapsto \lim_{\delta \rightarrow 0} \frac{\chi_I(t)}{\pi} \int_I g(\tau) \operatorname{Re} \zeta_K(1 + \delta + i(t - \tau)) \, d\tau \in L^2(I).$$

Here the characteristic function χ_I is applied to emphasise that we look at $L^2(I)$ as a subspace of $L^2(\mathbb{R})$. To understand these operators, we consider the example $K = \mathbb{N}$. The formula (1.2) implies

$$\operatorname{Re} \zeta_{\mathbb{N}}(1 + \delta + it) = \frac{\delta}{\delta^2 + t^2} + \operatorname{Re} \psi(1 + \delta + it),$$

whence

$$\mathcal{Z}_{\mathbb{N},I} = \operatorname{Id} + \Psi_{\mathbb{N},I}, \tag{1.3}$$

for a compact operator $\Psi_{\mathbb{N},I}$ and the identity operator Id . Indeed, the term $\pi^{-1}\delta/(\delta^2 + t^2)$ is the Poisson kernel which, under convolution, gives rise to the identity operator, while convolution with continuous kernels give compact operators (see Lemma 2). Hence, $\mathcal{Z}_{\mathbb{N},I}$ is a compact, and therefore a small perturbation of the identity operator.

In Theorem 1, we generalise the above formula in the following manner. We show that given $K \subset \mathbb{N}$, and a bounded and symmetric interval $I \subset \mathbb{R}$, there exist a subset $L \subset \mathbb{R}$ and a compact operator $\Phi_{K,I}$ such that

$$\mathcal{Z}_{K,I} = \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} + \Phi_{K,I}.$$

We remark that, intuitively, large $K \subset \mathbb{N}$ should correspond to large $L \subset \mathbb{R}$. In fact, it follows from our construction (see (2.1) below) that if $K = \mathbb{N}$ then $L = \mathbb{R}$. Hence, $\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} = \operatorname{Id}$ and we obtain again the formula (1.3).

For general K we present two additional results. In Theorem 2, we characterise for which K the operator $Z_{K,I}$ is bounded below. By a stability theorem of semi-Fredholm theory, it turns out that this takes place if and only if the principal term of (2.2), the operator $\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F}$, is bounded below. So, heuristically, a lower norm bound may be thought of as detecting the presence of a mass in the kernel of the integral operator $Z_{K,I}$. Along with a result by B. Panejah [25], this enables us to show that the operator $Z_{K,I}$ is bounded below in norm exactly for the sets K for which

$$\liminf_{x \rightarrow \infty} \frac{\pi_K(x)}{x} > 0, \tag{1.4}$$

where $\pi_K(x)$ is the counting function of K . Note that this condition is independent of the interval I . In Theorem 3, we obtain a complete characterisation of when $Z_{K,I}$ satisfies a formula of the type (1.3), in the sense that it is a compact perturbations of a scalar multiple of the identity operator; this happens if and only if the limit

$$\lim_{x \rightarrow \infty} \frac{\pi_K(x)}{x} \tag{1.5}$$

exists. Theorem 3 closely mirrors a recent generalisation, due to J. Korevaar, of a classical tauberian result of S. Ikehara [10]. Indeed, a formula of the type (1.3) holds if and only if a formula of the type (1.2) holds, where the appropriate substitute for ψ extends to a nicely behaved distribution on the abscissa $\sigma = 1$.

We also consider K -zeta functions for which $K \subset \mathbb{N}$ is assumed to have arithmetic structure. More specifically, we look at the case when K consists exactly of the integers whose prime number decomposition contain only factors belonging to some specified subset Q of the prime numbers \mathbb{P} . In this setting, the limit (1.5) always exists, implying that Theorems 1 to 3 are simplified. We state this as Theorem 4. The condition (1.4) is now expressed as

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{1}{p} < \infty.$$

Also, more detailed information about the K -zeta functions, in terms of certain L^p estimates on the abscissa $\sigma = 1$, is obtained. This is stated in Theorem 5. Finally, in Theorem 6, we show that a prime number theorem for the subset of the primes Q , in a sense to be defined, neither implies, nor is implied by, a lower norm bound for the operators $Z_{K,I}$.

We make some remarks on related work. The operator $Z_{\mathbb{N},I}$ appears in the context of Hilbert spaces of Dirichlet series in a paper by J.-F. Olsen and E. Saksman [23] (see also Section 7). In work done by J.-P. Kahane [12], a closely related functional is used to give a proof of the classical prime number theorem. Also, F. Moricz [21] found precise estimates of the L^p norms of a class of functions containing the K -zeta functions, along the segment $s \in (1, 2)$ in terms of the counting functions π_K .

The structure of this paper is as follows. In Sections 2 and 3 we state and prove, respectively, Theorems 1 and 2. In the latter section, we also make some comments relating our results to frame theory. Next, the Ikehara–Korevaar theorem is stated along with Theorem 3 and its proof in Section 4. In Sections 5 and 6 we present results obtained under the additional assumption that K has arithmetic structure. These are Theorems 4 and 5 and 6. Finally, in Section 7, we give some concluding remarks.

2. A generalisation of Riemann’s formula

In this section we give a generalisation of the formula (1.2). Note that we use the convention

$$\mathcal{F} : g \mapsto \hat{g}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) e^{-i\xi t} d\xi$$

for the Fourier transform on $L^2(\mathbb{R})$. Moreover, for $I \subset \mathbb{R}$ we identify the space $L^2(I)$ with the subspace of $L^2(\mathbb{R})$ consisting of functions with support in I . In this way, we give sense to the expression $\mathcal{F}g$ for $g \in L^2(I)$.

Theorem 1. *Let $K \subset \mathbb{N}$ be arbitrary, $I \subset \mathbb{R}$ be a bounded and symmetric interval, and*

$$L = \bigcup_{n \in K} ((-\log(n + 1), -\log n] \cup [\log n, \log(n + 1))). \tag{2.1}$$

Then there exists a compact operator $\Phi_{K,I}$ such that

$$\mathcal{Z}_{K,I} = \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} + \Phi_{K,I}. \tag{2.2}$$

Proof. The theorem is essentially the observation that by expanding the Dirichlet series of $2 \operatorname{Re} \zeta_K(s) = \zeta_K(s) + \overline{\zeta_K(s)}$, we get

$$\mathcal{Z}_{K,I} g(t) = \frac{\chi_I(t)}{\sqrt{2\pi}} \sum_{n \in K} \left(\frac{\hat{g}(\log n)}{n} n^{it} + \frac{\hat{g}(-\log n)}{n} n^{-it} \right). \tag{2.3}$$

Indeed, the formula (2.2) follows if we show that for $g \in C_0^\infty(I)$ the difference

$$\sum_{n \in K} \left(\frac{\hat{g}(\log n)}{n} n^{it} + \frac{\hat{g}(-\log n)}{n} n^{-it} \right) - \int_L \hat{g}(\xi) e^{it\xi} d\xi$$

is given by a compact operator Φ . Note that for $g \in C_0^\infty(I)$ this sum converges absolutely since $\hat{g}(\xi) = \mathcal{O}((1 + \xi^2)^{-1})$. In particular, since the Fourier transform is bounded, this implies that $\mathcal{Z}_{K,I}$ is bounded as an operator on $L^2(I)$. (This also follows from the fact that $\mathcal{Z}_{\mathbb{N},I}$ is bounded and the remarks following Theorem 2 relating these operators to frame theory.)

Recall that for $K \subset \mathbb{N}$,

$$L = \bigcup_{n \in K} ((-\log(n + 1), -\log n] \cup [\log n, \log(n + 1))).$$

In order to simplify notation, we set $L_+ = L \cap (0, \infty)$ and consider the difference of only the positive frequencies,

$$\sum_{n \in K} \frac{\hat{g}(\log n)}{n} n^{it} - \int_{L_+} \hat{g}(\xi) e^{it\xi} d\xi. \tag{2.4}$$

It suffices to show that this is given by a compact operator, say $2\pi\Phi_+$. The same argument then works on the negative frequencies by taking complex conjugates, giving us a compact operator $2\pi\Phi_-$. With the choice $\Phi = \Phi_+ + \Phi_-$, the proof is complete.

By adding and subtracting intermediate terms, we see that the difference (2.4) can be expressed as

$$\underbrace{\sum_{n \in K} \frac{1}{n \log(1 + \frac{1}{n})} \int_{L_n} (\hat{g}(\log n)n^{it} - \hat{g}(\xi)e^{it\xi}) d\xi}_{(I)} + \underbrace{\sum_{n \in K} \left(\frac{1}{n \log(1 + \frac{1}{n})} - 1 \right) \int_{L_n} \hat{g}(\xi)e^{it\xi} d\xi}_{(II)}.$$

We want to interchange the integral and sum signs in these expressions. For (I), it suffices to show that

$$\sum_{n \in K} \int_{L_n} |\hat{g}(\log n)n^{it} - \hat{g}(\xi)e^{it\xi}| d\xi \leq C \|g\|_{L^2(I)}. \tag{2.5}$$

for some constant $C > 0$. Note that by expressing the difference inside the absolute value as a definite integral, we have

$$\int_{L_n} |n^{it} - e^{it\xi}| d\xi \leq |t| \frac{1}{n^2}.$$

Pulling the absolute value sign inside of the expression for the Fourier transforms in combination with this inequality, gives us the bound

$$\begin{aligned} \int_{L_n} |\hat{g}(\log n)n^{it} - \hat{g}(\xi)e^{it\xi}| d\xi &\leq \int_I |g(\tau)| \int_{L_n} |n^{i(t-\tau)} - e^{i(t-\tau)\xi}| d\xi d\tau \\ &\leq \frac{1}{n^2} \int_I |t - \tau| |g(\tau)| d\tau \\ &\leq \frac{2|I|}{n^2} \left(\int_I |g(\tau)|^2 d\tau \right)^{1/2}. \end{aligned}$$

Taking the sum, and using the Cauchy–Schwarz inequality, we get (2.5) with constant $C = 2|I|\zeta(4)^{1/2}$. Interchanging the integral and sum signs, we get

$$(I) = \int_I g(\tau)\alpha(t - \tau) d\tau,$$

where

$$\alpha(\tau) = \frac{1}{\sqrt{2\pi}} \sum_K \frac{1}{n \log(1 + 1/n)} \int_{L_n} (n^{i\tau} - e^{i\xi\tau}) d\xi.$$

By the same bound we used above, this sum converges absolutely and therefore the function $\alpha(t)$ is continuous on I . Similar arguments show that

$$(II) = \int_I g(\tau)\beta(t - \tau) d\tau,$$

where

$$\beta(\tau) = \sum_K \left(\frac{1}{n \log(1 + 1/n)} - 1 \right) \int_{L_n} e^{i\xi\tau} d\xi$$

is a continuous function on I . Hence,

$$2\pi \Phi_+ g(t) = \frac{1}{\sqrt{2\pi}} \int_I g(\tau)(\alpha(t - \tau) + \beta(t - \tau)) d\tau,$$

and so the compactness of Φ_+ follows from Lemma 2. By the comments of the first half of the proof this implies that Φ is also a compact operator. \square

3. Characterisation of $\mathcal{Z}_{K,I}$ that are bounded below in norm

The following theorem explains when the operator $\mathcal{Z}_{K,I}$ is bounded below.

Theorem 2. *Let $K \subset \mathbb{N}$ be arbitrary, $I \subset \mathbb{R}$ be a bounded and symmetric interval, and $L \subset \mathbb{R}$ be given by the relation (2.1). Then the following conditions are equivalent.*

$$\mathcal{Z}_{K,I} \text{ is bounded below on } L^2(I), \tag{a}$$

$$\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} \text{ is bounded below on } L^2(I), \tag{b}$$

$$\text{there exists } \delta \in (0, 1) \text{ such that } \liminf_{x \rightarrow \infty} \frac{\pi_K(x) - \pi_K(\delta x)}{x} > 0, \tag{c}$$

$$\liminf_{x \rightarrow \infty} \frac{\pi_K(x)}{x} > 0. \tag{d}$$

Before we give the proof, we mention a corollary of Theorem 2. Recall that a sequence of vectors (f_n) in some Hilbert space H is called a frame if for all $f \in H$ there exists constants such that $\sum |\langle f | f_n \rangle|^2 \simeq \|f\|^2$. It is a basic result of frame theory that such a sequence of vectors is a frame if and only if the operator defined by

$$g \mapsto \sum \langle g | f_n \rangle f_n$$

is bounded and bounded below in norm (see e.g. [4]). With this in mind, we define the sequence

$$\mathcal{G}_K = \left(\dots, \frac{3^{it}}{\sqrt{3}}, \frac{2^{it}}{\sqrt{2}}, 1, 1, \frac{2^{-it}}{\sqrt{2}}, \frac{3^{-it}}{\sqrt{3}}, \dots \right), \tag{3.1}$$

where n is understood to run through $K \cup (-K)$. It is readily checked that Eq. (2.3) says exactly that $2\pi \mathcal{Z}_{K,I}$ is the frame operator of the sequence \mathcal{G}_K when restricted to the space $L^2(I)$. Hence, we get the following.

Corollary 1. *Let $K \subset \mathbb{N}$ be arbitrary and $I \subset \mathbb{R}$ be a bounded and symmetric interval. Then the sequence of vectors \mathcal{G}_K given by (3.1), restricted to the interval I , forms a frame for $L^2(I)$ if and only if any of the conditions of Theorem 2 holds.*

Our proof of Theorem 2 is based on the two following results (an alternative approach based on Corollary 1 is outlined in Section 7). The first is a theorem due to B. Panejah [25].

Theorem (Panejah, 1966). *Let $L \subset \mathbb{R}$ be measurable and $I \subset \mathbb{R}$ an interval. Then the operator $\chi_L \mathcal{F}$ is bounded below in norm from $L^2(I)$ to $L^2(\mathbb{R})$ if and only if there exists a $\delta > 0$ such that*

$$\inf_{\xi \in \mathbb{R}} |L \cap (\xi - \delta, \xi)| > 0. \tag{3.2}$$

The second result is a simple lemma (see e.g. [13, p. 238, Theorem 5.26]), for which we provide a simple proof.

Lemma 1. *Let X, Y be Banach spaces, let $Z : X \rightarrow Y$ a continuous linear operator that is bounded below and $\Phi : X \rightarrow Y$ be a compact operator. If $Z + \Phi$ is injective, then $Z + \Phi$ is bounded below.*

Proof. If $Z + \Phi$ is not bounded below, there exists a sequence of numbers x_n such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ while $\|(Z + \Phi)x_n\| \rightarrow 0$. Since Φ is a compact operator, we may assume that Φx_n is a Cauchy sequence. This implies that Zx_n is also a Cauchy sequence, whence x_n itself is a Cauchy sequence since Z is assumed to be bounded below in norm. Therefore, x_n converges to some element x with $\|x\| = 1$, whereas $(Z + \Phi)x = 0$. This contradicts the injectivity of $Z + \Phi$. \square

Proof of Theorem 2. We proceed to show the equivalences (b) \Leftrightarrow (c), (c) \Leftrightarrow (d) and (a) \Leftrightarrow (b).

(b) \Leftrightarrow (c): Condition (b) says that $\mathcal{F}^{-1} \chi_L \mathcal{F}$ is bounded below on $L^2(I)$. We begin by establishing that this is equivalent to $\chi_L \mathcal{F}$ being bounded below from $L^2(I)$ to $L^2(\mathbb{R})$. Indeed, one direction is clear since

$$\|\chi_L \mathcal{F}g\|_{L^2(\mathbb{R})} = \|\mathcal{F}^{-1} \chi_L \mathcal{F}g\|_{L^2(\mathbb{R})} \geq \|\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F}g\|_{L^2(I)}.$$

To prove the converse, assume that there exists some $\delta > 0$ such that for $g \in L^2(I)$

$$\|\chi_L \mathcal{F}g\|_{L^2(\mathbb{R})} \geq \delta \|g\|_{L^2(I)}. \tag{3.3}$$

Moreover, assume that for all $\epsilon > 0$ there exists a $g_\epsilon \in L^2(I)$ such that

$$\|\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} g_\epsilon\|_{L^2(I)} \leq \epsilon^2 \|g_\epsilon\|_{L^2(I)}.$$

This implies that

$$\begin{aligned} \|\chi_I \mathcal{F}^{-1} \chi_{L^c} \mathcal{F} g_\epsilon\|_{L^2(I)} &\geq \|g_\epsilon\|_{L^2(I)} - \|\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} g_\epsilon\|_{L^2(I)} \\ &\geq (1 - \epsilon^2) \|g_\epsilon\|_{L^2(I)}. \end{aligned}$$

On the other hand, the inequality (3.3) implies that

$$\begin{aligned} \|\chi_I \mathcal{F}^{-1} \chi_{L^c} \mathcal{F} g_\epsilon\|_{L^2(I)}^2 &\leq \|\mathcal{F}^{-1} \chi_{L^c} \mathcal{F} g_\epsilon\|_{L^2(\mathbb{R})}^2 \\ &= \|g_\epsilon\|_{L^2(I)}^2 - \|\mathcal{F}^{-1} \chi_L \mathcal{F} g_\epsilon\|_{L^2(\mathbb{R})}^2 \\ &\leq (1 - \delta^2) \|g_\epsilon\|_{L^2(I)}^2. \end{aligned}$$

Combining these two inequalities, we find that $\epsilon \geq \delta$. This leads to a contradiction since we may choose $\epsilon = \delta/2$.

We now invoke Panejah’s theorem. For our set L , the condition (3.2) is equivalent to

$$\liminf_{\xi \rightarrow \infty} \frac{\pi_K(e^\xi) - \pi_K(e^{\xi-\delta})}{e^\xi} > 0,$$

which is exactly condition (c). Indeed, this is just a matter of observing that

$$\frac{\pi_K(e^\xi) - \pi_K(e^{\xi-\delta})}{e^\xi} \leq \sum_{\log k \in (\xi-\delta, \xi)} \frac{1}{k} \leq e^\delta \frac{\pi_K(e^\xi) - \pi_K(e^{\xi-\delta})}{e^\xi}.$$

(c) \Leftrightarrow (d): It is clear that (c) implies (d). To see the converse, we need to use the fact that $\pi_K(x)/x \leq 1$. So assume that (d) holds, but not (c). Then for all $\delta \in (0, 1)$ there exists a sequence x_n such that

$$\lim_{n \rightarrow \infty} \frac{\pi_K(x_n) - \pi_K(\delta x_n)}{x_n} = 0,$$

whence

$$\limsup_{n \rightarrow \infty} \frac{\pi_K(x_n)}{x_n} = \lim_{n \rightarrow \infty} \frac{\pi_K(x_n) - \pi_K(\delta x_n)}{x_n} + \limsup_{n \rightarrow \infty} \delta \frac{\pi_K(\delta x_n)}{\delta x_n} \leq \delta.$$

Since $\delta \in (0, 1)$ is arbitrary, this contradicts (d).

(a) \Leftrightarrow (b): This equivalence follows essentially from the result from Lemma 1 and the identity $\mathcal{Z} = \mathcal{F}^{-1}\chi_L\mathcal{F} + \Phi$, where Φ is a compact operator on $L^2(I)$ and L is given by (2.1). What needs to be checked is that the lower bound of \mathcal{Z} implies the injectivity of $\chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}$, and vice versa.

By the equivalence of (b) and (c), we know that if the operator $\mathcal{F}^{-1}\chi_L\mathcal{F}$ is bounded below, then there exists $\delta \in (0, 1)$ such that $\inf_{x \in \mathbb{R}}(\pi_K(x) - \pi_K(\delta x))/x > 0$. This is readily seen to imply that $\sum_{n \in K} n^{-1} = \infty$. We show that this is sufficient for the operator \mathcal{Z} to be injective. Indeed, define the operator

$$R : g \longmapsto \left(\dots, \frac{\hat{g}(-\log 3)}{\sqrt{3}}, \frac{\hat{g}(-\log 2)}{\sqrt{2}}, \hat{g}(0), \hat{g}(0), \frac{\hat{g}(\log 2)}{\sqrt{2}}, \frac{\hat{g}(\log 3)}{\sqrt{3}}, \dots \right).$$

By an easy computation we have $\mathcal{Z} = (2\pi)^{-1}R^*R$. Since an operator is always injective on the image of its adjoint it suffices to check that the hypothesis implies that R is injective, i.e. that for $g \in L^2(I)$ then $\hat{g}(\pm \log n) = 0$ for all $n \in K$ implies $g = 0$. To get a contradiction, assume that the function g is non-zero. The function \hat{g} is entire and of exponential type $|I|/2$. In particular, as follows from Jensen’s inequality, the number of zeroes with modulus less than $r > 0$, which we denote by $\lambda(r)$, has to satisfy

$$\limsup_{r \rightarrow \infty} \frac{\lambda(r)}{r} < \infty.$$

Let $\pi_K(x)$ be the counting function for K . Then $\lambda(r) \geq \pi_K(e^r)$. The existence of the limit implies that $\pi_K(n) \leq C \log n$ for some $C > 0$. Summing by parts and using this estimate, we see that

$$\sum_{n \in K} \frac{1}{n} = \frac{\pi_K(N)}{N} + \sum_{n=1}^{N-1} \frac{\pi_K(n)}{n(n+1)} \leq 1 + C \sum_{n=1}^N \frac{\log n}{(n+1)^2}, \tag{3.4}$$

which converges as $N \rightarrow +\infty$. Hence, we have a contradiction and so g has to equal zero, as was to be shown. We can now apply Lemma 1 to conclude that \mathcal{Z} is bounded below on all of $L^2(I)$.

The same argument holds if we reverse the roles of \mathcal{Z} and $\chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}$ since the latter operator is injective whenever K is non-empty. Indeed, assume that $K \neq \emptyset$ and let $g \in L^2(I)$ be such that $g \neq 0$. It is clear that neither $\chi_L\mathcal{F}g$ nor $\mathcal{F}^{-1}\chi_L\mathcal{F}g$ can be equal to zero almost everywhere as functions in $L^2(\mathbb{R})$. To conclude, we use the Plancherel–Parseval formula. For suppose that $\chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}g = 0$. Since $g = \chi_I\mathcal{F}^{-1}\chi_{L^c}\mathcal{F}g + \chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}g$, this implies $\chi_I\mathcal{F}^{-1}\chi_{L^c}\mathcal{F}g = g$. And so

$$\begin{aligned} \|g\|_{L^2(I)}^2 &= \|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|_{L^2(\mathbb{R})}^2 + \|\mathcal{F}^{-1}\chi_{L^c}\mathcal{F}g\|_{L^2(\mathbb{R})}^2 \\ &\geq \|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|_{L^2(\mathbb{R})}^2 + \|\chi_I\mathcal{F}^{-1}\chi_{L^c}\mathcal{F}g\|_{L^2(I)}^2 \\ &= \|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|_{L^2(\mathbb{R})}^2 + \|g\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

But from what is already established $\|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|_{L^2(\mathbb{R})} > 0$, which leads to a contradiction. This concludes the proof of the theorem. \square

4. Characterisation of $\mathcal{Z}_{K,I}$ that behave like the identity operator

The following result describes when $\mathcal{Z}_{K,I}$ is a compact perturbation of a scalar multiple of the identity operator.

Theorem 3. *Suppose $K \subset \mathbb{N}$ and $A \geq 0$. Then for all bounded and symmetric intervals $I \subset \mathbb{R}$, the operator defined by*

$$\Psi_{K,I} = \mathcal{Z}_{K,I} - A \text{Id}$$

is compact if and only if

$$\lim_{x \rightarrow \infty} \frac{\pi_K(x)}{x} = A. \quad (4.1)$$

As mentioned in the introduction, this theorem should be compared to the following tauberian result due to S. Ikehara [10] and J. Korevaar [15]. Indeed, the sufficiency of the condition (4.1) in Theorem 3 follows directly from it. Note that we call the distributional Fourier transform of L^∞ functions which decay to zero at infinity pseudo-functions. These are in general distributions.

Theorem (Ikehara 1931, Korevaar 2005). *Let $S(t)$ be a non-decreasing function with support in $(0, \infty)$, and suppose that the Laplace transform*

$$F(s) = \mathcal{L}f(s) = \int_0^\infty \frac{S(u)}{e^u} e^{-(s-1)u} du$$

exists for $\sigma > 1$. For some constant A , let

$$g(s) = F(s) - \frac{A}{s-1}.$$

If $g(s)$ coincides with a pseudo-function on every bounded interval on the abscissa $\sigma = 1$ then

$$\lim_{t \rightarrow \infty} \frac{S(u)}{e^u} = A.$$

Conversely, if this limit holds, then g extends to a pseudo-function on $\sigma = 1$.

We remark that it follows from the Ikehara–Korevaar theorem that g extends to a pseudo-function on $\sigma = 1$ if and only if $e^{-u}S(u)$ tends to A .

The significance of pseudo-functions is that they are the class of distributions that satisfy, by definition, the Riemann–Lebesgue lemma. In particular, this implies that the convolution-type operators they give rise to are compact operators. To make this more precise, we give the following lemma.

Lemma 2. Let $I \subset \mathbb{R}$ be a bounded and symmetric interval and $k \in L^1(2I)$. Then the operator defined by

$$\Lambda : g \in L^2(I) \mapsto \chi_I \int_I g(\tau)k(t - \tau) \, d\tau \in L^2(I),$$

is a compact operator on $L^2(I)$. More generally, if $(k_\delta)_{\delta \in (0,1)}$ is a net of functions in $L^1_{\text{loc}}(\mathbb{R})$ converging in the sense of distributions to a pseudo-function k , then the operator

$$\tilde{\Lambda} : g \in L^2(I) \mapsto \lim_{\delta \rightarrow 0} \chi_I \int_I g(\tau)k_\delta(t - \tau) \, d\tau$$

is bounded and compact on $L^2(I)$.

Proof. Let $e_n(t)$ denote the Fourier characters of $L^2(2I)$, and let the Fourier expansion of k on $L^2(2I)$ be given by

$$k(t) = \sum_{n \in \mathbb{Z}} c_n e_n(t).$$

Hence, for $g \in L^2(I)$,

$$\Lambda g(t) = |2I|^{1/2} \sum_{n \in \mathbb{Z}} c_n (g, e_n)_{L^2(I)} e_n(t).$$

By the Riemann–Lebesgue lemma it follows that $|c_n| \rightarrow 0$ as $|n| \rightarrow \infty$ and the operator Λ is seen to be compact.

We turn to the second part of the statement. Let $g \in C_0^\infty(I)$. Then

$$\lim_{\delta \rightarrow 0} \int_{\mathbb{R}} g(t - \tau)k_\delta(\tau) \, d\tau = (g(t - \cdot), k) = \int_{\mathbb{R}} \hat{g}(\xi)\hat{k}(\xi)e^{it\xi} \, d\xi.$$

By the dual expression of the $L^2(I)$ norm this is seen to be bounded by some constant times the L^2 norm of g . To see that it is compact, define an operator on $C_0^\infty(I)$ by

$$\Lambda_N g(t) = \int_{\mathbb{R}} \hat{g}(\xi)\hat{k}_N(\xi)e^{it\xi} \, d\xi,$$

where the functions k_N are defined by setting $\hat{k}_N = \chi_N \hat{k}$. Since $\mathcal{F}\hat{k}_N \in L^1(2I)$ this is a compact operator by the first part of the lemma. Moreover,

$$\|\tilde{\Lambda}g - \Lambda_N g\|_{L^2} \leq \|g\|_{L^2} \|\hat{k}\|_{L^\infty(|\xi| > N)}.$$

Hence the sequence of compact operators Λ_N approximates $\tilde{\Lambda}$ in the uniform operator topology as $N \rightarrow \infty$. \square

Proof of Theorem 3. We use Lemma 2 to check the sufficiency of the condition (4.1). Recall that $\pi_K(x)$ is the counting function of the integers K . If we set $S(u) = \pi_K(e^u)$ then Korevaar’s result says that the density condition (4.1) implies that the function

$$\psi_K(s) = \frac{1}{s} \zeta_K(s) - \frac{A}{s-1}, \tag{4.2}$$

extends to a pseudo-function on $\sigma = 1$. In fact, it is straight-forward to check this implication directly since

$$\frac{1}{s} \zeta_K(s) - \frac{A}{s-1} = \int_0^\infty e^{-itu} e^{-(\sigma-1)u} \left(\frac{\pi_K(e^u)}{e^u} - A \right) du.$$

In any case, by (4.2) it follows that

$$\operatorname{Re} \zeta_K(1 + \delta + it) = \frac{A\delta}{\delta^2 + t^2} + \operatorname{Re} \phi_K(1 + \delta + it),$$

where $\phi_K = \nu(s)\psi_K(s)$, with $\nu(s)$ being a smooth function with fast decay such that $\nu(s) = s$ in the strip $t \in (-2, 2)$. This ensures that $\operatorname{Re} \phi_K$ extends to a pseudo-function on $\sigma = 1/2$. Since convolution operators with pseudo-functions as kernels give compact operators, the sufficiency now follows.

The converse is more delicate since it is possible for a convolution operator to be compact with a kernel that is not a pseudo-function. For instance, the indicator function χ_Y , where $Y \subset \mathbb{R}$ is unbounded but has finite Lebesgue measure, gives rise to such an operator. By Theorem 1 we have the identity

$$\mathcal{Z}_{K,I} - A \operatorname{Id} = \underbrace{\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} - A \operatorname{Id}}_{(*)} + \Phi_{K,I},$$

for some compact operator $\Phi_{K,I}$. Since the identity operator on $L^2(I)$ can be expressed as $\operatorname{Id} = \chi_I \mathcal{F} \mathcal{F}^{-1}$, it follows from the hypothesis that

$$\sqrt{2\pi} \cdot (*) = \chi_I \int_{\mathbb{R}} (\chi_L - A) \hat{g}(\xi) e^{i\xi t} d\xi$$

defines a compact operator on $L^2(I)$ for all bounded and symmetric $I \subset \mathbb{R}$. We denote it by $\tilde{\Psi}$. It is known that compact operators map sequences that converge weakly to zero to sequences that converge to zero in norm. We use this to show that for all $\delta > 0$,

$$\frac{|L \cap (\xi - \delta, \xi)|}{\delta} - A \rightarrow 0, \quad \text{as } \xi \rightarrow \infty. \tag{4.3}$$

To that effect, let $\epsilon > 0$, write $I = (-T, T)$, for some $T > 0$, and for $\xi \in \mathbb{R}$ define the $L^2(-T, T)$ functions

$$g_\xi(t) = \chi_{(-T, T)} \mathcal{F}^{-1}\{\chi_{(\xi-\delta, \xi)}\}(t) = \sqrt{\frac{2}{\pi}} e^{it(\xi-\frac{\delta}{2})} \frac{\sin(\frac{\delta}{2}t)}{t}.$$

It is clear that for $T > 0$ large enough, the real valued functions \hat{g}_ξ approximate the characteristic functions $\chi_{(\xi-\delta, \xi)}$ to an arbitrary degree of accuracy in $L^2(\mathbb{R})$. This approximation is uniform in ξ . In particular, we may choose $T > 0$ so that

$$\frac{1}{2}\delta \leq \|g_\xi\|_{L^2(I)} \leq 2\delta.$$

Fix some sequence $|\xi_n| \rightarrow \infty$. It follows readily that the functions g_{ξ_n} converge weakly to zero in $L^2(I)$, whence $\|\Psi g_{\xi_n}\| \rightarrow 0$ as $n \rightarrow \infty$. To obtain the connection to the set L , we use the dual expression for the norm of $\tilde{\Psi} g_{\xi_n}$.

$$\begin{aligned} \|\tilde{\Psi} g_{\xi_n}\|_{L^2(I)} &\geq \frac{1}{\|g_n\|_{L^2(I)}} \left| \int_{\mathbb{R}} (\chi_L - A) \hat{g}_{\xi_n}(\xi)^2 d\xi \right| \\ &\geq \frac{1}{2\delta} \underbrace{\left| \int_{\mathbb{R}} (\chi_L - A) \chi_{(\xi_n-\delta, \xi_n)}(\xi) d\xi \right|}_{(**)} \\ &\quad - \frac{1}{2\delta} \underbrace{\left| \int_{\mathbb{R}} (\chi_L - A) (\hat{g}_{\xi_n}(\xi)^2 - \chi_{(\xi_n-\delta, \xi_n)}(\xi)) d\xi \right|}_{(***)}. \end{aligned}$$

It is clear that

$$(**) = \frac{1}{2} \left| \frac{|L \cap (\xi_n - \delta, \xi_n)|}{\delta} - A \right|.$$

Since $|\chi_L - A| \leq 1$ and $\chi_{(\xi_n-\delta, \xi_n)} = \chi_{(\xi_n-\delta, \xi_n)}^2$, we use the formula $(a^2 - b^2) = (a + b)(a - b)$ and the Cauchy–Schwarz inequality to find

$$\begin{aligned} (***) &\leq \frac{1}{2\delta} \|\hat{g}_{\xi_n} + \chi_{(\xi_n-\delta, \xi_n)}\|_{L^2(I)} \|\hat{g}_{\xi_n} - \chi_{(\xi_n-\delta, \xi_n)}\|_{L^2(I)} \\ &\leq \frac{3}{2} \|\hat{g}_{\xi_n} - \chi_{(\xi_n-\delta, \xi_n)}\|_{L^2(I)}. \end{aligned}$$

By choosing $T > 0$ large enough, we have $(***) \leq \epsilon/6$. Hence,

$$\left| \frac{|L \cap (\xi_n - \delta, \xi_n)|}{\delta} - A \right| \leq 2\|\tilde{\Psi} g_{\xi_n}\|_{L^2(I)} + \frac{\epsilon}{2}.$$

Since $\|\tilde{\Psi} g_{\xi_n}\|_{L^2(I)} < \epsilon/4$ for large enough n , this establishes (4.3).

To get a contradiction, we assume that $\pi_K(x)/x$ does not tend to the limit A . Without loss of generality, we assume that there exists a number $\kappa > 0$ such that

$$\limsup_{x \rightarrow \infty} \frac{\pi_K(x)}{x} = A + \kappa.$$

This means that for any number $\eta \in (0, 1)$ we may find a strictly increasing sequence of positive numbers ξ_n , with arbitrarily large separation, such that $\xi_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\frac{\pi_K(e^{\xi_n})}{e^{\xi_n}} > A + \eta\kappa \quad \text{for } n \in \mathbb{N}.$$

Moreover, since the counting function π_K changes slowly, there exists a number $\delta_0 > 0$ such that for $n \in \mathbb{N}$ and $\xi \in (\xi_n - \delta_0, \xi_n)$ we have

$$\frac{\pi_K(e^\xi)}{e^\xi} - A > \kappa/2.$$

Next, for $\xi_n > 2$,

$$\begin{aligned} |L \cap (\xi_n - \delta_0, \xi_n)| &\gtrsim \sum_{\substack{n \in K \\ n \in (e^{\xi_n - \delta_0}, e^{\xi_n} - 1)}} \log\left(1 + \frac{1}{n}\right) \\ &\gtrsim \sum_{\substack{n \in K \\ n \in (e^{\xi_n - \delta_0}, e^{\xi_n})}} \frac{1}{n} \\ &= \int_{e^{\xi_n - \delta_0}}^{e^{\xi_n}} \frac{1}{x} d\pi_K(x) \\ &= \frac{\pi_K(e^{\xi_n})}{e^{\xi_n}} - \frac{\pi_K(e^{\xi_n - \delta_0})}{e^{\xi_n - \delta_0}} + \int_{e^{\xi_n - \delta_0}}^{e^{\xi_n}} \frac{1}{x} \frac{\pi_K(x)}{x} dx. \end{aligned}$$

The last line follows from partial integration, and the implicit constants are absolute. By the properties of ξ_n , this implies that

$$|L \cap (\xi_n - \delta_0, \xi_n)| \gtrsim -(1 - \eta)\kappa + \left(A + \frac{\kappa}{2}\right)\delta = A\delta + \left(\eta + \frac{\delta}{2} - 1\right)\kappa.$$

By choosing $\eta = (4 - \delta)/4$, we find that for $\xi_n > 2$,

$$\frac{|L \cap (\xi_n - \delta_0, \xi_n)|}{\delta_0} - A > \frac{\kappa}{4}.$$

This contradicts (4.3). \square

5. Two results in the case when K has arithmetic structure

Assume that $K \subset \mathbb{N}$ has arithmetic structure in the sense that it is the semi-group generated by a subset Q of the prime numbers, which we denote by \mathbb{P} , i.e. K consists of the integers that are only divisible by primes in Q . It follows that we may write

$$\zeta_K(s) = \prod_{p \in Q} \left(\frac{1}{1 - p^{-s}} \right).$$

In other words, ζ_K admits an Euler product. A fundamental fact is that such K always admit an asymptotic density. We give a proof of this fact, no doubt well-known to specialists, before turning to Theorems 4 and 5.

Lemma 3. *Let $Q \subset \mathbb{P}$ generate the integers $K \subset \mathbb{N}$, and J be the integers generated by the primes not in Q . Then*

$$\lim_{x \rightarrow \infty} \frac{\pi_K(x)}{x} = \lim_{\sigma \rightarrow 1^+} \frac{1}{\zeta_J(\sigma)}.$$

Proof. This lemma seems to be folklore, indeed for finite $\mathbb{P} \setminus Q$ it is readily known that it holds. See for instance [20, Theorem 3.1]. An immediate consequence is that for infinite $\mathbb{P} \setminus Q$, then

$$\limsup_{x \rightarrow \infty} \frac{\pi_K(x)}{x} \leq \lim_{\sigma \rightarrow 1^+} \frac{1}{\zeta_J(\sigma)}.$$

In particular, if $\zeta_J(\sigma)$ diverges as $\sigma \rightarrow 1^+$, then $\pi_K(x)/x$ tends to zero. However, the remaining part of the lemma seems to be more difficult, and no analytic proof, or indication thereof, seems to be readily available in the literature. Therefore we show how one follows from the Ikehara–Korevaar theorem above.

Assume that $\zeta_J(1) < \infty$ and recall that $\zeta(s) = (s - 1)^{-1} + \psi(s)$ for some entire function ψ . By the Ikehara–Korevaar theorem, it suffices to show that the following function coincides with a pseudo-function on finite intervals along the abscissa $\sigma = 1$,

$$\begin{aligned} \frac{\zeta_K(s)}{s} - \frac{1}{\zeta_J(1)} \frac{1}{s-1} &= \frac{\zeta(s)}{s\zeta_J(s)} - \frac{1}{\zeta_J(1)} \frac{1}{s-1} \\ &= \frac{1}{s-1} \left(\frac{1}{s\zeta_J(s)} - \frac{1}{\zeta_J(1)} \right) + \frac{\psi(s)}{s\zeta_J(s)}. \end{aligned}$$

Since $\zeta_J(1) < \infty$ it is not hard to use the Euler product formula to see that $\zeta_J(1 + it)$ is bounded above and below in absolute value for all \mathbb{R} . This means that the last term coincides with a pseudo-function on finite intervals along the abscissa $\sigma = 1$. Hence, the same is true for the left-hand side if and only if it holds true for the first term on the right-hand side. It is readily seen that this function extends to a pseudo-function on $\sigma = 1$ if and only if the same is true for

$$\frac{1}{s-1} \left(\frac{\zeta_J(s)}{s} - \zeta_J(1) \right). \tag{5.1}$$

We calculate its distributional Fourier transform. Let ϕ be a test function. Since we may write

$$\frac{\zeta_J(s)}{s} = \frac{1}{s} \int_1^\infty x^{-s} d\pi_J(x) = \int_0^\infty \frac{\pi_J(e^u)}{e^u} e^{-(\sigma-1)-itu} du,$$

it follows that

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} \hat{\phi}(t) \frac{1}{\delta + it} \left(\frac{\zeta_J(1 + \delta + it)}{1 + \delta + it} - \zeta_J(1) \right) dt \\ &= \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} \frac{1}{\delta + it} \hat{\phi}(t) \int_0^\infty g(u) (e^{-\delta u - iut} - 1) du dt, \end{aligned}$$

where $g(u) = \pi_J(e^u)e^{-u}$. Using the smoothness of ϕ , we change the order of integration,

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_0^\infty g(u) \int_{\mathbb{R}} \hat{\phi}(t) \frac{e^{-\delta u - iut} - 1}{\delta + it} dt du &= \int_0^\infty g(u) \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} \hat{\phi}(t) \frac{e^{-\delta u - iut} - 1}{\delta + it} dt du \\ &= \int_0^\infty g(u) \Phi(u) du, \end{aligned}$$

where $\Phi'(u) = -2\pi\phi(u)$ and $\Phi(0) = 0$. This means that

$$\Phi(u) = -2\pi \int_0^u \phi(x) dx \quad \text{for } u \geq 0.$$

So,

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\mathbb{R}} \hat{\phi}(t) \frac{1}{\delta + it} \left(\frac{\zeta_J(1 + \delta + it)}{1 + \delta + it} - \zeta_J(1) \right) dt &= -2\pi \int_0^\infty g(u) \int_0^u \phi(x) dx du \\ &= -2\pi \int_{\mathbb{R}} \phi(x) \chi_{(0,\infty)}(x) \int_x^\infty g(u) du dx. \end{aligned}$$

Since $g(u)$ is integrable, this implies that

$$\chi_{(0,\infty)}(x) \int_x^\infty g(u) du$$

decays as $|x| \rightarrow \infty$ and so the function (5.1) extends to a pseudo-function on the abscissa $\sigma = 1$. \square

Hence, if J denotes the integers generated by the primes not in Q , then the condition (4.1) always holds with $A = \lim_{\sigma \rightarrow 1^+} \zeta_J^{-1}(\sigma)$. By the Euler product representation of ζ_J it is seen that $\zeta_J(1) < \infty$ if and only if

$$\sum_{p \in \mathbb{P} \setminus Q} p^{-1} < \infty. \tag{5.2}$$

Under these conditions (c) is equivalent to (5.2). This means that we get the following simpler form of Theorems 1 to 3.

Theorem 4. *Let $I \subset \mathbb{R}$ be a bounded symmetric interval, $Q \subset \mathbb{P}$ generate the integers K , and J be the integers generated by the primes not in Q . Then*

$$\mathcal{Z}_{K,I} = \zeta_J^{-1}(1) \text{Id} + \Psi_{K,I}, \tag{5.3}$$

for a compact operator $\Psi_{K,I}$. Moreover, the operator $\mathcal{Z}_{K,I}$ is bounded below on $L^2(I)$ if and only if

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{1}{p} < \infty.$$

Proof. By Lemma 3 the limit

$$\lim_{x \rightarrow \infty} \frac{\pi_K(x)}{x} = A$$

always holds with $A = \zeta_J^{-1}(\sigma)$. With this, Theorem 3 implies the formula for $\mathcal{Z}_{K,I}$.

Finally, Theorem 2 says that $\mathcal{Z}_{K,I}$ is bounded below if and only if $A > 0$. By considering the Euler product of $\zeta_J(s)$ it follows that $\zeta_J(1) < +\infty$ is exactly the condition of the theorem. \square

Since (5.2) is equivalent to (4.2) with $A > 0$, the formula (5.3) may be seen as a direct consequence of the Ikehara–Korevaar theorem. However, more can be said in relation to the formula (4.2). Note that $f \in L^p_{\text{loc}}$ if $f \in L^p(E)$ for any compact $E \subset \mathbb{R}$.

Theorem 5. *Let $Q \subset \mathbb{P}$ generate the integers K , and J be the integers generated by the primes not in Q , and assume that (5.2) holds. Then*

$$\psi_K(s) := \frac{1}{s} \zeta_K(s) - \frac{\zeta_J^{-1}(1)}{s-1}$$

extends, in the sense of distributions, to a function in L^1_{loc} on the abscissa $\sigma = 1$ if and only if

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{\log \log p}{p} < \infty.$$

For $q > 1$, the extension is in L^q_{loc} on the abscissa $\sigma = 1$ if

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{\log^{1/q'} p}{p} < \infty,$$

where $q' > 1$ is the real number satisfying $q^{-1} + q'^{-1} = 1$. Conversely, if

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{\log^{1/q'} p}{p} = \infty,$$

then the extension of ψ on $\sigma = 1$ is not in L^r_{loc} for $r > q$.

Before the proof of the theorem, which follows an argument similar to that of Lemma 3, we give a lemma which ties together summability conditions on subsets of integers and their generating prime numbers. It follows easily by using the measure calculus described by P. Malliavin in [17], however we provide a more elementary argument for the readers convenience.

Lemma 4. Let $f : \mathbb{N} \rightarrow \mathbb{R}_+$ satisfy $f(nm) \leq f(n) + f(m)$, $f(1) = 0$ and $f(n) \geq 1$ for n big enough. If the primes P generate the integers J then

$$\sum_{n \in J} \frac{f(n)}{n} < \infty \iff \sum_{p \in P} \frac{f(p)}{p} < \infty.$$

Proof. One way to prove this is, for $\sigma > 1$, to establish the inequality

$$\sum_{n \in J} f(n)n^{-\sigma} \lesssim \left(\sum_{p \in P} f(p)p^{-\sigma} \right) e^{\sum_p p^{-\sigma}},$$

and then conclude by the monotone convergence theorem. To achieve this we study the linear map $D_f : \sum a_n n^{-\sigma} \rightarrow \sum a_n f(n)n^{-\sigma}$. It is not hard to show that the abscissa of absolute convergence is invariant under D_f . Moreover, for Dirichlet series F, G with positive coefficients, it holds that $D_f(FG)(\sigma) \leq D_f(F)G(\sigma) + FD_f(G)(\sigma)$. We use this on the identity

$$\sum_{n \in J} n^{-\sigma} = e^{\sum_{p \in P} p^{-\sigma}} R(\sigma),$$

where the function

$$R(\sigma) = e^{-\sum_{p \in P} \log(1-p^{-\sigma}) - \sum_{p \in P} p^{-\sigma}}$$

is given by a Dirichlet series that converges absolutely for $\sigma > 1/2$. Here we used the Euler product formula for the function ζ_J . The desired inequality is now seen to hold since

$$D_f(e^{\sum_{p \in P} p^{-\sigma}}) \leq \left(\sum_{p \in P} f(p)p^{-\sigma} \right) e^{\sum_{p \in P} p^{-\sigma}}. \quad \square$$

Proof of Theorem 5. Both the formula

$$\mathcal{Z}_{K,I} = \zeta_J^{-1}(1) \text{Id} + \Psi_{K,I},$$

where $\Psi_{K,I}$ is a compact operator, and the statement that ψ_K extends to a pseudo-function on $\sigma = 1$, follow immediately from Theorem 3.

As in the hypothesis, assume that $\zeta_J^{-1}(1) > 0$. By the factorisation $\zeta(s) = \zeta_K(s)\zeta_J(s)$ and the formula (1.2) for the Riemann zeta function we have the identity

$$\begin{aligned} \psi_K(s) &= \frac{\zeta_K(s)}{s} - \frac{1}{\zeta_J(1)} \frac{1}{s-1} \\ &= \frac{1}{s-1} \left(\frac{1}{s\zeta_J(s)} - \frac{1}{\zeta_J(1)} \right) + \frac{\psi(s)}{s}. \end{aligned}$$

Under our assumption, it follows from the Euler product formula that $\zeta_J(1 + it)$ is a continuous function bounded away from zero. Therefore

$$\frac{1}{t} \left(\frac{1}{(1 + it)\zeta_J(1 + it)} - \frac{1}{\zeta_J(1)} \right) \in L^q_{\text{loc}}(\mathbb{R}) \iff \frac{\zeta_J(1 + it) - \zeta_J(1)}{t} \in L^q_{\text{loc}}(\mathbb{R}).$$

Let $q' \in [1, \infty]$ be such that $q^{-1} + q'^{-1} = 1$. Hence, by duality

$$\left\| \frac{\zeta_J(1 + it) - \zeta_J(1)}{t} \right\|_{L^q(I)} = \sup_{\phi \in L^{q'}(I)} \left| \int_I \phi(t) \frac{\zeta_J(1 + it) - \zeta_J(1)}{t} dt \right|.$$

A brief calculation, where we use Fubini’s theorem twice along with the properties of the counting measure $d\pi_J$, shows that

$$\begin{aligned} \int_I \phi(t) \frac{\zeta_J(1 + it) - \zeta_J(1)}{t} dt &= \int_1^\infty \int_I \phi(t) \frac{x^{-it} - 1}{t} dt \frac{d\pi_J(x)}{x} \\ &= \sqrt{2\pi} \int_1^\infty \int_0^{\log x} \hat{\phi}(\xi) d\xi \frac{\pi_J(x)}{x} \\ &= \sqrt{2\pi} \sum_{n \in J} \frac{1}{n} \underbrace{\int_0^{\log n} \hat{\phi}(\xi) d\xi}_{(*)}. \end{aligned}$$

We proceed to estimate $(*)$ for ϕ in $L^{q'}(I)$. First we let $q = 1$. Then $q' = \infty$. Recall that I is a bounded and symmetric interval, hence $I = (-T, T)$ for some $T > 0$. By Fubini’s theorem and a change of variables,

$$\begin{aligned} \left| \int_0^u \hat{\phi}(\xi) \, d\xi \right| &= \frac{1}{\sqrt{2\pi}} \left| \int_{-T}^T \phi(t) \frac{e^{-itu} - 1}{t} \, dt \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_{-uT}^{uT} \phi\left(\frac{t}{u}\right) \frac{e^{-it} - 1}{t} \, dt \right| \\ &\leq \frac{\|\phi\|_{L^\infty(I)}}{\sqrt{2\pi}} \int_{-uT}^{uT} \frac{|e^{-it} - 1|}{t} \, dt. \end{aligned}$$

The integral in the last expression is clearly $\mathcal{O}(\log u)$. It now follows that

$$\left\| \frac{\zeta_J(1 + it) - \zeta_J(1)}{t} \right\|_{L^1(I)} \lesssim \sum_{n \in J} \frac{1}{n} \log \log n.$$

By Lemma 4,

$$\sum_{n \in J} \frac{\log \log n}{n} < \infty \iff \sum_{p \in \mathbb{P} \setminus Q} \frac{\log \log p}{p} < \infty.$$

This proves the sufficiency for $q = 1$. As for the necessity, assume that

$$\sum_{p \in \mathbb{P} \setminus Q} p^{-1} \log \log p = \infty$$

and set $\phi = \chi_I(x)$. With this choice

$$\int_0^u \hat{\phi}(\xi) \, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-T}^T \frac{e^{-itu} - 1}{it} \, dt = \int_{-uT}^{uT} \frac{e^{-it} - 1}{it} \, dt.$$

The result now follows since it is clear that

$$\int_0^1 \frac{|e^{-it} - 1|}{t} \, dt < \infty,$$

while as u grows we have

$$\left| \int_1^u \frac{e^{-it}}{t} \, dt \right| < \infty \quad \text{and} \quad \int_1^u \frac{dt}{t} = \log u.$$

For $q > 1$, the necessity is proved by using the fact that for $\phi \in L^{q'}(I)$ we have

$$\left| \int_0^u \hat{\phi}(\xi) \, d\xi \right| \lesssim \|\phi\|_{q'} u^{1/q'}.$$

The sufficiency follows since for all $r > q'$ there exists $\phi \in L^{q'}(I)$ for which

$$\int_0^u \hat{\phi}(\xi) \, d\xi = x^{1/r} + \mathcal{O}(1).$$

We leave the details to the reader. \square

6. Remarks on the relation to the prime number theorem

In this section, we present our final result on the operator $\mathcal{Z}_{K,I}$. Let $R = (r_i)$ be an increasing sequence of real numbers greater than one and let N be the multiplicative semi-group it generates. We say that R is the Beurling prime numbers for the Beurling integers N . This point of view leads to a generalised type of number theory, initiated by A. Beurling in [3]. The focus of the theory is to investigate how the asymptotic structure of R relates to that of N . For more on Beurling primes see [20, Chapter 8.4] and the references therein. In our case, Q corresponds to the Beurling primes and K to the Beurling integers. We say that the prime number theorem holds for Q if

$$\pi_Q(x) \sim \frac{x}{\log x}.$$

The symbol $f(x) \sim g(x)$ is taken to mean $f(x)/g(x) \rightarrow 1$ as $x \rightarrow \infty$.

The following is now true.

Theorem 6. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$. Then the prime number theorem for Q neither implies nor is implied by the lower boundedness of the operator $\mathcal{Z}_{K,I}$.*

To prove the theorem, we need a lemma.

Lemma 5. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$ and let J denote the integers generated by the primes not in Q . Then the prime number theorem holds for the set K if and only if*

$$\sum_{p \in \mathbb{P} \setminus Q \cap (\delta x, x)} \frac{\log p}{p} = o(1), \quad \text{for all } \delta \in (0, 1). \tag{6.1}$$

Proof. Let $P = \mathbb{P} \setminus Q$. It is clear that the prime number theorem holds for K if and only if $\pi_P(x) = o(x/\log x)$. Moreover, it is readily seen that

$$\frac{\log x}{x} (\pi_P(x) - \pi_P(\delta x)) \leq \sum_{p \in (\delta x, x)} \frac{\log p}{p} \leq \frac{\log \delta x}{\delta x} (\pi_P(x) - \pi_P(\delta x)).$$

So we have to show that $\pi_P(x) = o(x/\log x)$ is equivalent to the statement that for all $\delta > 0$ it holds that $\pi_P(x) - \pi_P(\delta x) = o(x/\log x)$. One direction is immediate. For the other, assume that $\pi_P(x) - \pi_P(\delta x) = o(x/\log x)$. Rewrite this assumption in the form

$$\pi_P(x) \frac{\log x}{x} = \delta \left(\pi_P(\delta x) \frac{\log \delta x}{\delta x} \right) \frac{\log x}{\log \delta x} + o(1).$$

Hence, for all $\delta > 0$, we have

$$\limsup_{x \rightarrow \infty} \pi_P(x) \frac{\log x}{x} < \delta. \quad \square$$

Proof of Theorem 6. Recall that by Theorem 4, the lower boundedness of the operator $\mathcal{Z}_{K,I}$ is equivalent to the condition

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{1}{p} < \infty. \tag{6.2}$$

First we seek a set of primes Q for which Panejah’s condition holds but the prime number theorem does not. This part of the theorem follows by comparing the condition (6.2) to the condition (6.1) of Lemma 5 in combination with a variant of Merten’s formula (see e.g. [20, p. 50]):

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + \mathcal{O}(1). \tag{6.3}$$

One the one hand, (6.3) implies that for $\delta > 0$ small enough, then

$$\liminf_{x \rightarrow \infty} \sum_{\delta x \leq p \leq x} \frac{\log p}{p} > 0.$$

We choose a sequence $(x_n)_{n \in \mathbb{N}}$ which realises this condition and for which the intervals $(\delta x_n, x_n)$ do not overlap. On the other hand, (6.3) implies that

$$\sum_{\delta x \leq p \leq x} \frac{1}{p} \lesssim \frac{1}{\log \delta x}.$$

Choose a sub-sequence of (x_{n_k}) for which $\sum_k (\log x_{n_k})^{-1} < \infty$. Let $P = \mathbb{P} \cap (\bigcup (\delta x_{n_k}, x_{n_k}))$, and set $Q = \mathbb{P} \setminus P$. This set does the job.

Next, we seek a set Q for which the prime number theorem holds, but Panejah’s condition fails. For $M > 0$, consider the consecutive intervals $I_k = (M^k, M^{k+1})$. By Merten’s formula (6.3) it holds that

$$\sum_{M^k \leq p \leq M^{k+1}} \frac{\log p}{p} = \log M + \mathcal{O}(1).$$

So, for large enough M and $k \in \mathbb{N}$ we can choose a set of primes P_k from each of the intervals I_k in such a way that

$$\sum_{M^k \leq p \leq M^{k+1}} \frac{\log p}{p} = \frac{C}{\log k} + O\left(\frac{k}{M^k}\right).$$

Set $P = \bigcup P_k$ and let $Q = \mathbb{P} \setminus P$. It now follows that the condition (6.2) does not hold, since

$$\sum_{p \in P} \frac{1}{p} = \sum_{k \in \mathbb{N}} \sum_{p \in P \cap (M^k, M^{k+1})} \frac{1}{p} \gtrsim \sum_{k \in \mathbb{N}} \frac{1}{k \log k} = \infty.$$

Let $n \in \mathbb{N}$ be a number such that $M^{-n} \leq \delta$, and assume that $x > 0$ is large enough for the relation $M^{n+j} < x \leq M^{n+j+1}$ to hold for some integer $j \geq 2$. Then

$$\sum_{p \in P \cap (\delta x, x)} \frac{\log p}{p} \leq \sum_{p \in P \cap (M^j, M^{n+j+1})} \frac{\log p}{p} \lesssim \frac{n}{\log j}.$$

Since n is fixed and $j \rightarrow \infty$ as $x \rightarrow \infty$, the condition (6.1) holds. \square

Remark 1. In the first half of the proof of the above theorem, we actually show that the relation

$$\pi_Q(x) \sim C \frac{x}{\log x} \tag{6.4}$$

need not be true for any $C > 0$ even though (6.2) holds. This is holds since we can let the sequence x_n increase arbitrarily fast, while keeping δ fixed. This illustrates that it is not enough for ζ_J to be continuous and bounded away from zero on the abscissa $\sigma = 1$ for the usual proof of the prime number theorem using Ikehara’s theorem to work on the function $\zeta_K = \zeta/\zeta_J$ (see e.g. [31]). Indeed, if this was the case then (6.4) would hold for some $C > 0$.

7. Concluding remarks

The theory of sampling measures offer an alternative to Panejah’s theorem in proving the equivalence (a) \Leftrightarrow (c) of Theorem 2. Indeed, given a measure μ on \mathbb{R} , J. Ortega-Cerda and K. Seip [24, Proposition 1] give a necessary and sufficient condition in terms of sampling sequences on the Paley–Wiener space $PW(I)$ for $\int_I |f(t)|^2 dt \simeq \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\mu$ to hold for all $f \in L^2(I)$. In light of this, Corollary 1 connects the classical density results on sampling sequences for Paley–Wiener spaces [29, Chapter 6], which depend on the interval I , to the counting function π_N . Additional arguments, which use regularity properties of the measure μ , show that the description obtained is independent of I , both necessary and sufficient, and can be expressed as (c).

The connection between the operator $\mathcal{Z}_{K,I}$ and the frame (3.1) was essentially observed in the paper [23]. There the operator $\mathcal{Z}_{\mathbb{N},I}$ was used to study the Dirichlet–Hardy space

$$\mathcal{H}^2 = \left\{ \sum_{n \in \mathbb{N}} a_n n^{-s}; \sum_{n \in \mathbb{N}} |a_n|^2 < +\infty \right\}.$$

By the Cauchy–Schwarz inequality, the functions in this space are analytic for $\sigma > 1/2$. It has been shown [19,7] that for a bounded interval I there exists a constant $C > 0$, only depending on the length of the interval I , such that for every $F \in \mathcal{H}^2$,

$$\int_I \left| F\left(\frac{1}{2} + it\right) \right|^2 dt \leq C \|F\|_{\mathcal{H}^2}^2.$$

This implies that for $F \in \mathcal{H}^2$ then $F(s)/s$ is in the classical Hardy space H^2 on the half-plane $\sigma > 1/2$. In particular, it follows that functions in \mathcal{H}^2 have non-tangential boundary values almost everywhere on the abscissa $\sigma = 1/2$. This gives meaning to the notation $F(1/2 + it)$ for $F \in \mathcal{H}^2$. A special case of the main result of [23] is now stated as follows.

Theorem (Olsen and Saksman 2009). *Let I be some bounded interval in \mathbb{R} and $v \in L^2(\mathbb{R})$. Then there exists a function $F \in \mathcal{H}^2$ such that $\operatorname{Re} F(1/2 + it) = v(t)$ almost everywhere in $L^2(I)$.*

This result may be reformulated as saying that $\mathcal{G}_{\mathbb{N}}$, as defined in (3.1), forms a frame for $L^2(I)$. As a consequence, Corollary 1 implies that we can replace \mathcal{H}^2 by any of the subspaces

$$\mathcal{H}_K^2 = \left\{ \sum_{n \in K} a_n n^{-s} : \sum_{n \in K} |a_n|^2 < +\infty \right\}$$

for which K satisfies condition (c) of Theorem 2. For more on the emerging theory of the space \mathcal{H}^2 , see [7,6,1,14,8,2,18,9,27,28,30].

Finally we mention that the spaces \mathcal{H}^p were defined for arbitrary $p > 0$ by F. Bayart in [1]. By an idea of H. Bohr, they are defined to be the Dirichlet series for which the coefficients are Fourier coefficients of functions in the Hardy space $H^p(\mathbb{T}^\infty)$, where $\mathbb{T}^\infty = \{(z_1, z_2, \dots) : z_j \in \mathbb{T}\}$ is equipped with the product topology. For $p = 2$ this definition coincides with the definition of \mathcal{H}^2 given above. The behaviour of functions in these spaces is for the most part unknown.

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