Theoretical Computer Science 77 (1990) 321-330
North-Holland


NOTE

# ANALYSIS OF PETRI NETS BY PARTITIONING: SPLITTING TRANSITIONS

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Communicated by M. Nivat
Received October 1989

**Abstract.** In this paper, a method of analysis of large Petri nets by partitioning is proposed. This method permits a great saving of computation time and storage. Useless efforts spent in the analysis of large Petri nets are spared by a look to the partitions of interest. It is possible to study the characteristics of the required places by involving them in a partitition. It was shown that partitioning preserves the characteristics of the main Petri net. The reachability tree method or the matrix equations approach, which were untractable at the whole net level, may be used at the subnet level to get the needed analysis criteria.

## 1. Introduction

Petri nets are designed specifically to model systems with interacting concurrent components. Since the components of the systems interact, it is necessary that synchronization occurs. The transfer of information or materials from one component to another requires that the activities of the involved components be synchronized while the interaction is going on.

In what follows a method to partition a Petri net into sub-Petri nets is presented. It will be proved that studying these sub-Petri nets gives the same results as obtained from the original Petri net. Great saving of computation effort, run time, and storage area is obtained from the analysis of Petri nets by partitioning.

The available methods of analysis, such as the reachability tree [3, 4] and the matrix equations [5, 6, 7], are applicable for small size Petri nets. But when the size of the net becomes larger, both methods become practically unusable and another technique must thus be devised [2]. The proposed method partitions a net by a cutting line that goes through some transitions. Each of the subnets may be separately studied, which may be specially useful when only some places are of interest. Partitioning uses the reachability tree method or the matrix equations approach, which, impossible at the whole net level, is easy and helpful.

0304-3975/90/$03.50 

## 2. Partitioning method

### 2.1. First presentation

A Petri net structure $C$ is a four-tuple [1]:

- $C=(P, T, I, O)$, where
- $P=(p_1, p_2, p_3, \ldots, p_m)$, $m \geq 0$,
- $T=(t_1, t_2, t_3, \ldots, t_n)$, $n \geq 0$.

$P$ and $T$ are sets of places and transitions respectively such that $P \cap T = \emptyset$,

- $I$ = input function,
- $O$ = output function.

It is desired to divide a Petri net $C$ into two Petri nets $C1$ and $C2$ provided that studying $C1$, $C2$ will give results that would have been obtained if $C$ was studied.

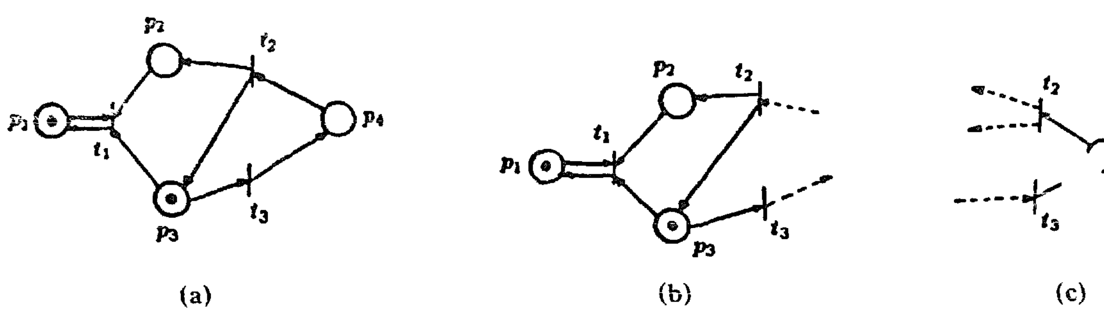



Fig. 1. (a) Main Petri net. (b) First subnet. (c) Second subnet.

Let us introduce the proposed method via the following example: $C=(P, T, I, O)$ (see Fig. 1(a)), $P=(p_1, p_2, p_3, p_4)$, $T=(t_1, t_2, t_3)$; the initial marking is $\mu = (1, 0, 1, 0)$.

$$I(t_1)=(p_1, p_2, p_3), \qquad O(t_1)=(p_1),$$
$$I(t_2)=(p_4), \qquad O(t_2)=(p_2, p_3),$$
$$I(t_3)=(p_3), \qquad O(t_3)=(p_4).$$

To partition this Petri net, we proceed as follows:

(1) Partition the transitions among the subnets according to the following restrictions:

- The *boundary* (*common*) transitions must be selected such that there are arrows going through each transition from one subnet to the other. Failing to satisfy this condition results in the dead-end or insignificant subnets obtained from the example shown in Fig. 2.
- There should be no common places between the input/output functions of the *nonboundary* transitions belonging to a subnet, and the input/output functions of the *nonboundary* transitions belonging to other subnets. The net shown in Fig. 3 does not satisfy these conditions, accordingly, it cannot be partitioned by the

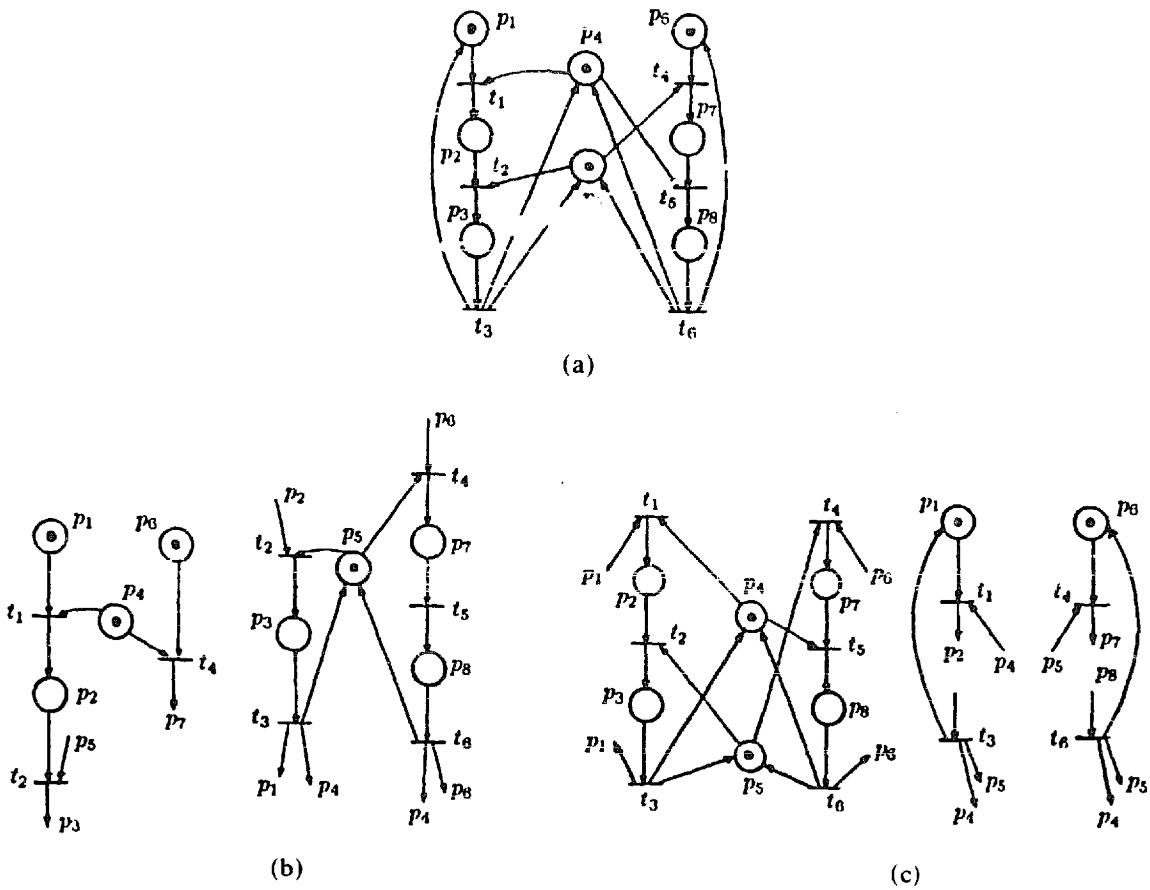



Fig. 2. (a) Main Petri net. (b) Dead-end partitioning. (c) Insignificant partitioning.

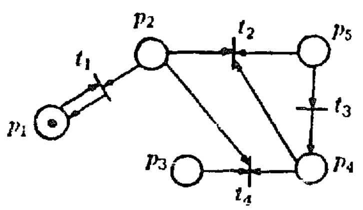



Fig. 3. Unsuitable net.

proposed method. The reachability tree method can be used to check the convenience of a partitioning, as shown throughout this paper.

(2) Thus, transitions are partitioned such that the sets $T1$ and $T2$ are formed as follows:

$$T1=(t_1, \underline{t_2}, \underline{t_3}), \quad \text{see Fig. 1(b)}, \qquad T2=(\underline{t_2}, \underline{t_3}), \quad \text{see Fig. 1(c)}.$$

$$T_{\text{com}}=(t_2, t_3), \qquad T1=T_1 \cup T_{\text{com}}, \qquad T2=T_2 \cup T_{\text{com}}, \qquad T1 \cap T2=T_{\text{com}}$$

where $T_1$, $T_2$ are the sets containing the *nonboundary* transitions in $T1$ and $T2$ respectively, $T_{\text{com}}$ is the set of transitions common to $T1$ and $T2$.

(3) The input function of $C1$ will consist of all inputs to $T_1$ and the inputs from $C1$ to $T_{\text{com}}$, while the input function of $C2$ will contain all inputs to $T_2$, and the

inputs from $C2$ to $T_{com}$. The sets $I1$ and $I2$ will then contain the following elements:

$$I1 = [I(t_1) \cup I(\underline{t_2}) \cup I(\underline{t_3})] = (p_1, p_2, p_3),$$

$$I2 = [I(\underline{t_2}) \cup I(\underline{t_3})] = (p_4).$$

An important differentiation is to be drawn between sets and bags [1]. In a bag it is possible to have more than one occurrence for any element, while in a set just one occurrence is allowed. With such distinction any bag $I(t_i)$ may contain duplicated places, but $I1$, $I2$ as sets are entitled to a single occurrence of the involved places.

(4) The output function of $C1$ will consist of all outputs from $T_1$ and the outputs from $T_{com}$ to $C1$, while the output function of $C2$ will contain all outputs from $T_2$, and the outputs from $T_{com}$ to $C2$. The sets $O1$ and $O2$ will then contain the following elements:

$$O1 = [O(t_1) \cup O(\underline{t_2}) \cup O(\underline{t_3})] = (p_1, p_2, p_3),$$

$$O2 = [O(\underline{t_2}) \cup O(\underline{t_3})] = (p_4).$$

The above differentiation between bags and sets is to be recalled again.

(5) $P1$, $P2$, the sets of places in $C1$ and $C2$ respectively, will then be a composed of the following sets:

$$P1 = I1 \cup O1 = (p_1, p_2, p_3),$$

$$P2 = I2 \cup O2 = (p_4).$$

Thus, Petri net $C$ is subdivided into subnets $C1$ and $C2$. Now, how to study the newly obtained nets, given that $C$ is of initial marking $\mu$? The answer comes out from the following steps:

(1) Choose any subnet. The marking of each place starts up with the value supplied by the initial marking of the original net.

(2) While considering the effect of the other subnets, transmitted through $T_{com}$, use the reachability tree method or the matrix equations approach to analyze the considered subnet.

(3) Repeat steps 1, 2 for each of the remaining subnets.

Assuming that the main Petri net is subdivided into two subnets is just to simplify the method presentation; in fact, this method can be applied successfully to a higher number of partitions. If the original net has $m$ places, and $n$ transitions, the analysis is of order $m \cdot n$, but by partitioning it is on the average of order $(m/l) \cdot (n/l)$ for each subnet, where $l$ is the number of subnets (partitions).

### 2.2. Formal description

Given the original Petri net $C = (P, T, I, O)$

$$P = (p_1, p_2, p_3, \ldots, p_m), \qquad T = (t_1, t_2, t_3, \ldots, t_n).$$

Required: divide the Petri net into sub-Petri nets such that

$$C = C1 \cup C2$$

where

$$C1 = (P1, T1, I1, O1) \subset C, \qquad C2 = (P2, T2, I2, O2) \subset C$$

provided that:

$$P1 \subset P, \quad T1 \subset T, \quad P2 \subset P, \quad T2 \subset T, \quad P1 \cap P2 = \emptyset,$$

$$T = T1 \cup T2, \qquad T1 \cap T2 = T_{com}$$

where

$$T1 = (T_1, T_{com}), \qquad T2 = (T_2, T_{com}).$$

$T_1$ is the set of transitions $t_i$ $(i = 1, \ldots, n)$, $T_1 \cap T_{com} = \emptyset$. $T_2$ is the set of transitions $t_j$ $(j = 1, \ldots, n, j \neq i)$, $T_2 \cap T_{com} = \emptyset$. $T_{com}$ is the set of transitions common to $C1$ and $C2$.

The input and output functions of $T1$ and $T2$ must be composed as follows:

$$I1 = [I(t_1) \cup I(T_{com})], \qquad O1 = [O(t_1) \cup O(T_{com})],$$

$$I2 = [I(t_2) \cup I(T_{com})], \qquad O2 = (O(t_2) \cup O(T_{com})]$$

such that there are no common places between the input and output functions of $T_1$ and $T_2$:

$$I(T_1) \cap I(T_2) = \emptyset, \qquad O(T_1) \cap O(T_2) = \emptyset,$$

$$I(T_1) \cap O(T_2) = \emptyset, \qquad O(T_1) \cap I(T_2) = \emptyset.$$

Also, there should be at least one arrow going from $C1$ to $C2$ through $T_{com}$ and vice versa:

$$\left.\begin{array}{l}\exists(p_r, I(t_h)) \\ \exists(p_s, O(t_h))\end{array}\right\} \quad p_r \in P1, p_s \in P_2, t_h \in T_{com}$$

and

$$\left.\begin{array}{l}\exists(p_x, I(t_l)) \\ \exists(p_y, O(t_l))\end{array}\right\} \quad p_x \in P2, p_y \in P_1, t_l \in T_{com}.$$

$t_h$ and $t_l$ may be the same transition. It should be noted that there are no limits on the number of boundary transitions or their parity.

The sets of places $P1$ and $P2$ are then found to be

$$P1 = (I1 \cup O1), \quad P1 \in P, \qquad P2 = (I2 \cup O2), \quad P2 \in P, \qquad P1 \cap P2 = \emptyset.$$

If we start arbitrarily by studying $C1$, then

$$(C1, \mu_1) \subset (C, \mu),$$

i.e., $C1 \subset C$, $\mu$ and $\mu_1$ are the sets of the markings of places contained in $C$ and $C1$ respectively. In the example of Fig. 1, $\mu = (1, 0, 1, 0)$ and $\mu_1 = (1, 0, 1)$. Also,

$$R(C1, \mu_1) \subset R(C, \mu),$$

i.e., the reachability set of $C1$ is a subset of the reachability set of $C$. For $C2$, if

$$(C2, \mu_2) \subset (C, \mu) \quad \text{then} \quad R(C2, \mu_2) \subset R(C, \mu).$$

## 3. Case studies

### 3.1. Case A

Figure 4 displays the reachability trees of the partitioned net previously introduced in Fig. 1. From the study of $C1$, the following may be noticed:

- $p_1$ and $p_3$ are safe while $p_2$ is unbounded; the whole net is thus unsafe.
- The subnet as well as the original net are not conservative since the number of tokens increases infinitely in $p_2$.
- The symbol $\omega$ in the reachability tree makes any conclusive statements regarding reachability and coverability impossible.
- The firing sequence $t_3t_2t_1$ leads to a dead-end as confirmed from the reachability tree of $C$ and that of $C1$ (Fig. 4(a), (b)).

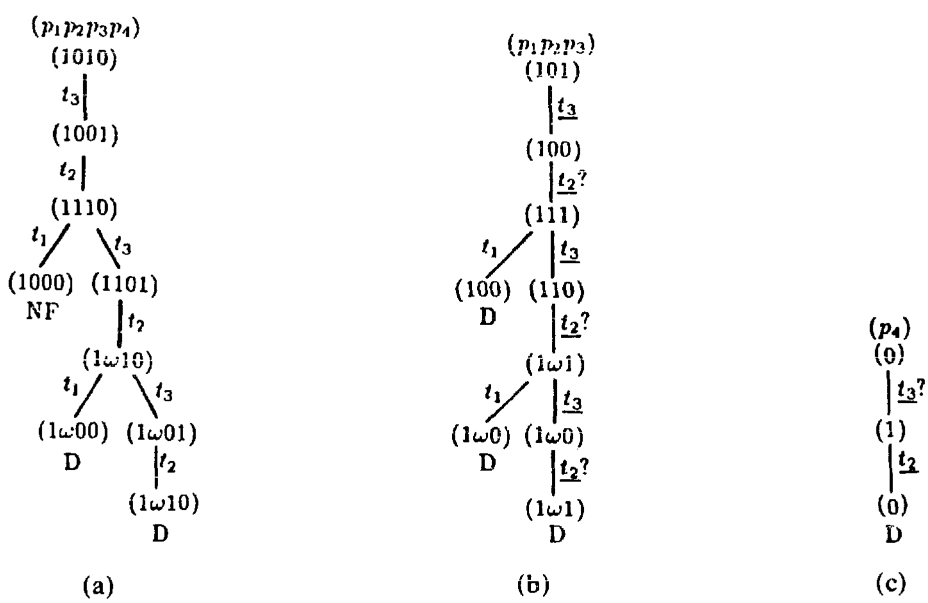



Fig. 4. Reachability trees. (a) Main Petri net. (b) First subnet. (c) Second subnet. **Legend:** D: *Duplicate*, NF: *no firing*, $\underline{t_i}$?: *may or may not be fired.*

Studying $C2$, the following is concluded from Fig. 4(c):

- If $t_3$ is fired from $C1$ (takes the effect of $C1$ on $C2$), $p_4$ will receive a token; otherwise, it remains empty.
- $p_4$ is safe.
- The absence of $\omega$ permits clear statements about reachability and coverability. But for the whole net, as seen from $C1$, any conclusive results are impossible.

- There are no dead-ends; $p_4$ has thus some degree of liveness.

These results may be equally obtained from the original net, as well as from the subnets. If there is an interest in $p_1$ or $p_3$ only, it is clear that $C1$ can provide the required characteristics without having to go through the whole net.

### *3.2. Case B*

In Fig. 5 a Petri net and its reachability tree are given, where

$$C=(P, T, I, O), \qquad P=(p_1, p_2, p_3, p_4), \qquad T=(t_1, t_2, t_3)$$

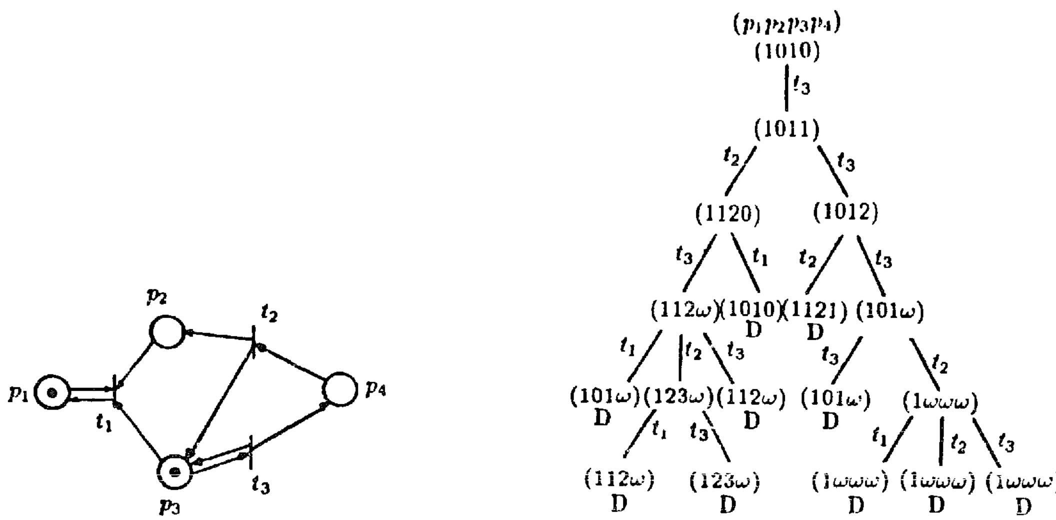
and the initial marking $\mu=(1,0,1,0)$. Partitioning this net, take

$C1=(P1, T1, I1, O1)$, see Fig. 6,

$T1=(t_1, \underline{t_2}, \underline{t_3})$, $\qquad P1=(p_1, p_2, p_3)$;

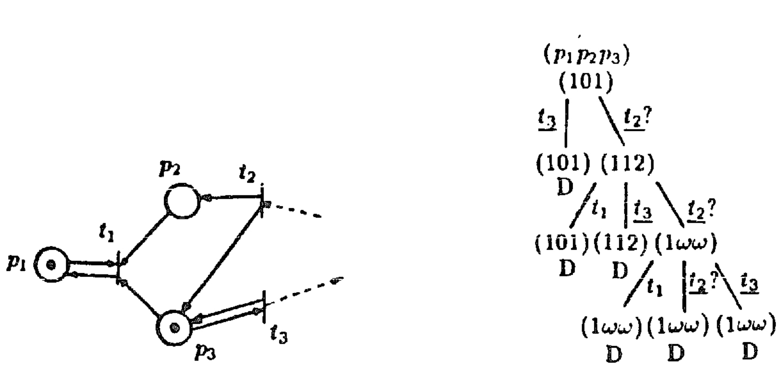
$C2=(P2, T2, I2, O2)$, see Fig. 7,

$T2=(\underline{t_2}, \underline{t_3})$, $\qquad P2=(p_4)$.




(a) (b)

Fig. 5. (a) Main net. (b) Reachability tree.




(a) (b)

Fig. 6. (a) First subnet. (b) Reachability tree.
**Legend:** The effect of $C2$ on $C1$ ($\underline{t_2}$?) can be considered at any node.

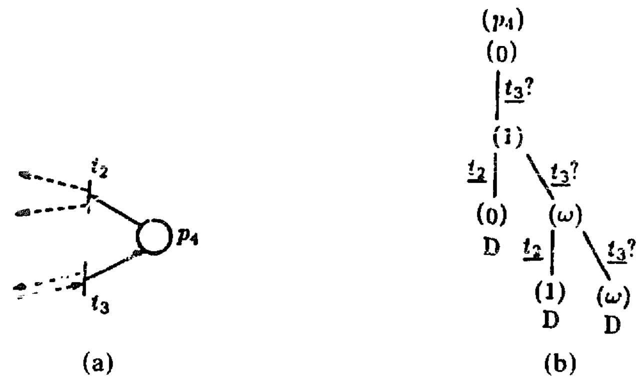



Fig. 7. (a) Second subnet. (b) Reachability tree.
**Legend:** The effect of $C1$ on $C2$ ($\underline{t_3}?$) can be con 'dered at any node.

Combining the analysis criteria obtained from the reachability trees of $C1$ and $C2$ (Figs. 6(b), 7(b)), we get the following:

- $p_1$ is safe, the other places are not. Hence, the whole net is not safe.
- The net is obviously nonconservative due to the presence of $\omega$.
- The presence of $\omega$ leads to inconclusive results about reachability and coverability.
- There are no dead-ends, all places satisfy some degree of liveness.

Comparing the sizes of the reachability trees will not favor a study of the main net.

### *3.3. Case C*

In Fig. 8 the Petri net and a portion of its extensive reachability tree are given: $C=(P, T, I, O)$, $P=(p_1, p_2, p_3, p_4, p_5, p_6)$, $T=(t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8)$, and the initial marking $\mu=(1,0,0,0,1,0)$. Partitioning this net, take

$$C1=(P1, T1, I1, O1), \quad \text{see Fig. 9},$$

$$T1=(t_1, t_2, t_3, t_4, \underline{t_5}, \underline{t_6}), \qquad P1=(p_1, p_2, p_3);$$

$$C2=(P2, T2, I2, O2), \quad \text{see Fig. 10},$$

$$T2=(\underline{t_5}, \underline{t_6}, t_7, t_8), \qquad P2=(p_4, p_5, p_6).$$

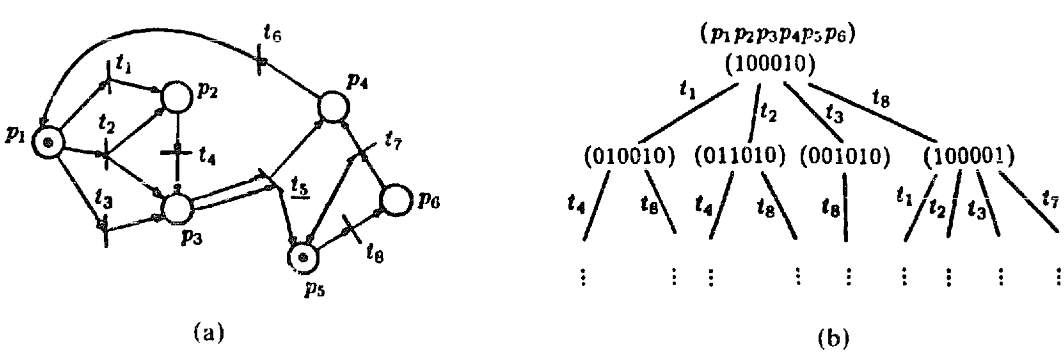



Fig. 8. (a) Main net. (b) Reachability tree.

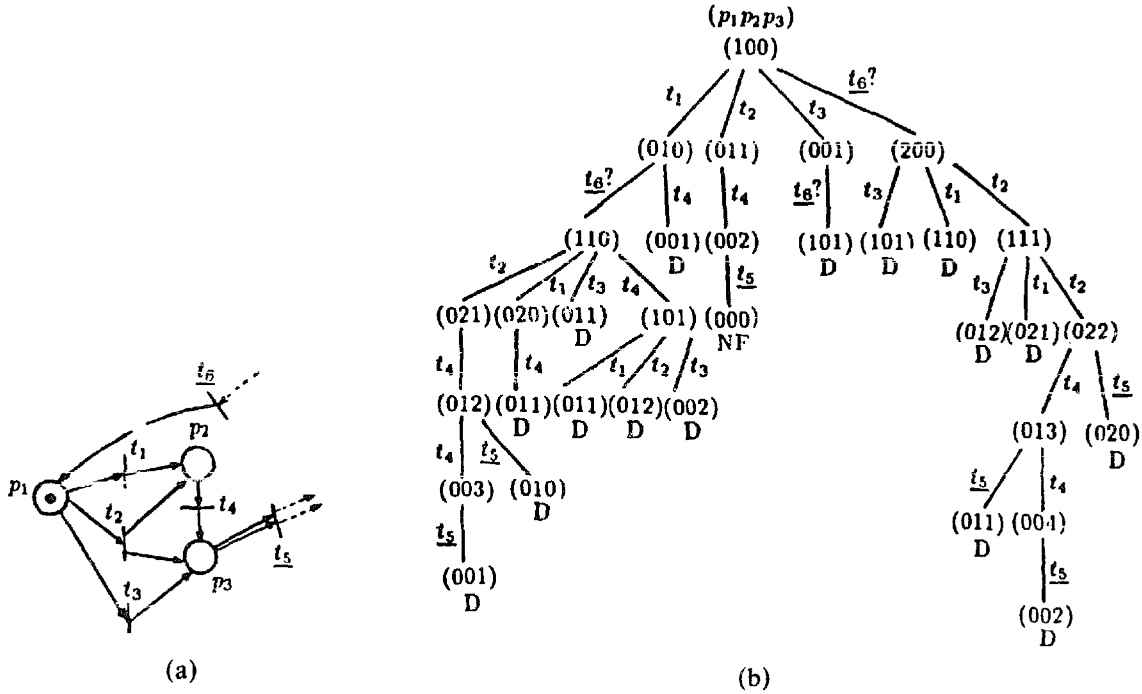



Fig. 9. (a) First subnet. (b) Reachability tree.

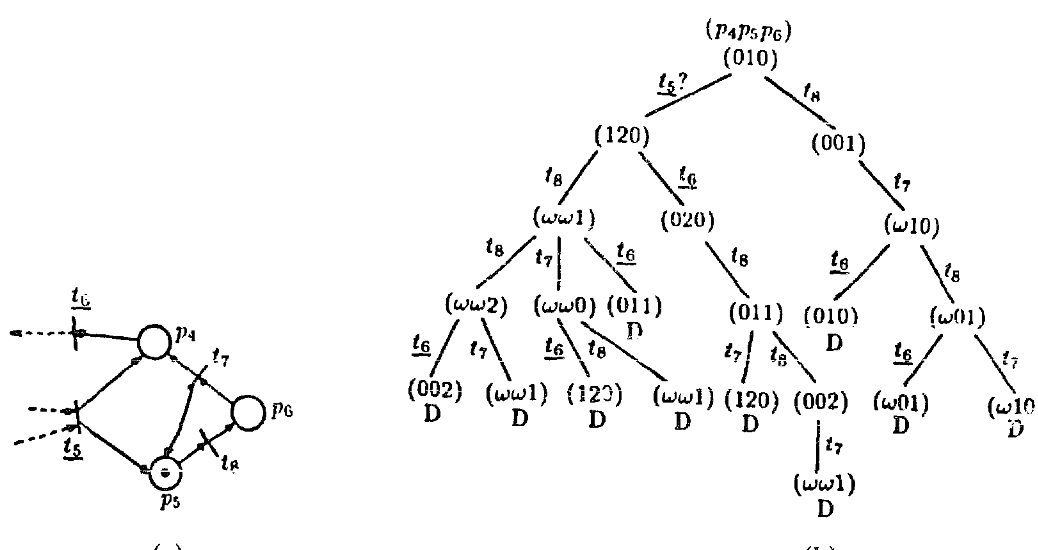



Fig. 10. (a) Second subnet. (b) Reachability tree.

Starting by a study of $C1$, we find the following:

- $p_1$ and $p_2$ are safe, $p_3$ is not, the whole net is not safe.
- The subnet is not conservative since the total number of tokens varies from one state to another. Conclusive results regarding the whole net cannot be taken before studying $C2$.
- The absence of $\omega$ permits clear statements about reachability and coverability at the subnet level.

- There are no dead-end (no firing) paths, which discloses the liveness of all subnet places.

Studying $C2$ we conclude the following:

- $p_4$ and $p_5$ are not safe, while $p_6$ is safe. The whole net is not safe as has been previously concluded.
- The subnet is not conservative as the total number of tokens in $p_4, p_5, p_6$ is not fixed. Combining this conclusion with that obtained from studying $C1$, it can be deduced that the whole net is not conservative.
- Due to the presence of $\omega$, reachability and coverability cannot be checked at the subnet level as well as at the whole net level.
- There are no dead-end (no firing) paths, which reveals the liveness of all subnet paths. From $C1$, $C2$ the whole net is live.

## 4. Conclusions

The proposed partitioning method permits a great saving of computation time and storage. Useless efforts spent in the analysis of large Petri nets are spared by a look at the partitions of interest. It is possible to study the characteristics of the required places by involving them in a partition. It was shown that partitioning preserves the characteristics of the main Petri net, namely, safeness, boundedness, conservation, coverability, reachability, and liveness. The reachability tree method or the matrix equations approach, which were untractable at the whole net level, may be used at the subnet level to get the needed analysis criteria. Different case studies were tackled to prove the efficiency of the proposed method. Independent of the number of partitions, the validity of the proposed method relies on fulfilling some net structure-related restrictions.

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