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Representation of Para-Fermi Rings and Generalised Clifford Algebra

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1. Introduction

Recently a programme was initiated to arrive at the representations of different types of algebras from the elements of the generalised Clifford algebra. As a first step [1] the generators of the Kemmer algebra $K(n)$ of n elements were synthesised from the elements of C_2^{n+1}, the generalised Clifford algebra whose two generating elements are the $(n+1)$-th roots of the unit matrix.

It is well-known that Kemmer algebra corresponds to para-Fermi statistics of order $p = 2$. We are now encouraged to carry the programme further to obtain representations of para-Fermi rings of *any order* p, relating to any number of operators v. We therefore outline the relations defining the operators occurring in para-Fermi rings. We extend the results of reference [1] to obtain the next higher representation of $K(n)$. We then obtain the representations for any number of these operators, for any order p of the statistics, and deduce the dimensions of the representations.

2. Operators of Para-Fermi Theory

The theory of generalized statistics, including the Bose and Fermi statistics as special cases has been studied by a number of authors [2], [3], [4]. We here attempt to give an explicit representation of the para-Fermi operators for any order p of the para-Fermi statistics.

Let a_α $(\alpha = 1, 2,..., v)$ and their adjoints a_α^+ be the operators of the para-Fermi rings satisfying commutation relations

$$[a_\lambda, \tfrac{1}{2}[a_\mu^+, a_\nu]_-]_- = \delta_{\lambda\mu} a_\nu \qquad (\lambda, \mu, \nu = 1,..., v) \tag{2.1}$$

$$[a_\lambda, \tfrac{1}{2}[a_\mu, a_\nu]_-]_- = 0 \tag{2.2}$$

and if the order of the para-Fermi statistics is p,

$$(a_\alpha)^{p+1} = 0; \qquad a_\alpha^{\,j} \neq 0 \qquad \text{for} \qquad j \leqslant p. \tag{2.3}$$

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The operators a_α and $a_\alpha{}^+$ are identified in physics as the creation and annihilation operators. Green [2] had noticed that

$$a_\alpha = \sum_{r=1}^{p} b_\alpha^{(r)} \tag{2.4}$$

will yield the para-Fermi ring as defined in (2.1) to (2.3) if $b_\alpha^{(r)}$ are commuting Fermi-Dirac operators. This gives a reducible representation of dimension 2^{pv}.

Let us define $2v$ hermitian operators [3, 4]

$$\beta_{2\alpha-1} = \frac{1}{2}(a_\alpha + a_\alpha{}^+); \qquad \beta_{2\alpha} = \frac{i}{2}(a_\alpha - a_\alpha{}^+) \qquad (\alpha = 1,..., v) \tag{2.5}$$

obeying the commutation relations:

$$[\beta_\lambda, [\beta_\mu, \beta_\nu]_-]_- = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu . \tag{2.6}$$

The condition (2.3) is equivalent to

$$(\beta_{2\alpha-1} - i\beta_{2\alpha})^{p+1} = 0; \qquad (\beta_{2\alpha-1} - i\beta_{2\alpha})^j \neq 0 \qquad \text{for} \qquad j \leqslant p. \tag{2.7}$$

It is known [3] that one can generate the algebra of the rotation group in $(2v + 1)$ dimensions, i.e., $O(2v + 1)$ from the β's.

For the case $p = 2$ these β's are the Kemmer elements of $K(2v)$ and representations of the lowest dimension for $K(2v)$ using Clifford elements have been constructed earlier [1].

We will now describe a method by which we obtain the next higher representation of the Kemmer algebra. The lowest representation of the β's is of dimension $(2v + 1)$. We can show that the next representation is of dimension

$$N = 2v + {}^{2v}C_2 \tag{2.8}$$

To obtain this, we take all commutators $[\beta_m, \beta_n]_- = J_{mn}$ of the generating elements, ${}^{2v}C_2$ in number. If we add the $2v$ generators, $\beta_m = J_{0m}$, to the above we get a closed set under commutation. Let us take an aggregate A of the resulting set, say,

$$A = \sum a_{mn} J_{mn}; \qquad (m \neq n;\ m, n = 0, 1,..., 2v). \tag{2.9}$$

Let us now define mappings $\hat{E}_i$'s such that

$$A \xrightarrow{\hat{E}_i} A' = [A, J_{0i}]_-; \qquad (i = 1,..., 2v). \tag{2.10}$$

It is verified that the $\hat{E}_i$'s obey

$$[\hat{E}_\lambda, [\hat{E}_\mu, \hat{E}_\nu]_-]_- = \delta_{\mu\lambda}\hat{E}_\nu - \delta_{\lambda\nu}\hat{E}_\mu$$

and (2.11)

$$\hat{E}^3 = \hat{E} \qquad \text{for all} \qquad \lambda, \mu, \text{and } \nu.$$

Thus we have the representation of the generators of the Kemmer algebra $K(2v)$ which is of dimension $2v + {}^{2v}C_2$. It is to be noted that we need not know the actual matrix representation of the β's themselves to obtain $\hat{E}_i$ matrices.

3. Representations for any Order

This section deals with a method of obtaining representations of the β's given by (2.5) and (2.6) for any order p of the para-Fermi ring. To be specific we define $\beta_\alpha^{(p)}$, (α running from 1 to $2v$) as the generators of the ring belonging to the order p of the statistics. Let $\beta_\alpha^{(p)}$ be constructed as

$$\beta_\alpha^{(p)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(p-1)}; \qquad (p = 3, 4, \text{etc.}) \tag{3.1}$$

where γ_α's are the elements of the Clifford algebra, C_{2v}^2, where the square of each of the $2v$ generators is the identity, the generators obeying the anti-commutation relations. Starting from Pauli matrices, generators of C_{2v}^2 can be obtained by the σ-operation detailed by one of the authors [5] (A.R.). The dimension of $\gamma_\alpha \in C_{2v}^2$ is 2^v. If we start with $p = 3$, we have

$$\beta_\alpha^{(3)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(2)} \tag{3.2}$$

$\beta_\alpha^{(2)} \in K(2v)$ which, for the basic representation, has the dimension $(2v + 1)$. It can be seen that $\beta_\alpha^{(3)}$ obeys the triple commutation relation (2.5). Equations (2.7) for any α are also seen to be satisfied noting that $(\gamma_\mu - i\gamma_{\mu-1})^2 = 0$ and $(\beta_\mu^{(2)} - i\beta_{\mu-1}^{(2)})^j = 0$ for $j = 3$ only; for $j < 3$ it is nonzero. Similarly defining

$$\beta_\alpha^{(4)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(3)} \tag{3.3}$$

it is verified that all the relations for para-Fermi statistics for order $p = 4$ are obeyed. In general (3.1) is found to be valid for all p. Starting with the

$(2v + 1) \times (2v + 1)$ dimensional representation of $\beta_\alpha^{(2)}$, the matrices $\beta_\alpha^{(p)}$ has the dimension

$$N \times N = [2^{v(p-2)} \times (2v + 1)] \times [2^{v(p-2)} \times (2v + 1)]; \qquad (p = 3, 4, \ldots). \tag{3.4}$$

Starting with Kemmer matrices a representation of lower dimension can be obtained by compounding the $\beta_\alpha^{(p)}$ in the following way also:

$$\beta_\alpha^{(2m)} = \beta_\alpha{}^m \otimes 1 + 1 \otimes \beta_\alpha^{(m)} \tag{3.5a}$$

and

$$\beta_\alpha^{(2m+1)} = \frac{\gamma_\alpha}{2} \otimes 1 + 1 \otimes \beta_\alpha^{(2m)} \qquad (m = 1, 2, \ldots). \tag{3.5b}$$

$\beta_\alpha^{(p)}$'s defined by (3.5) satisfy the equations (2.6) and (2.7). This representation of the para-Fermi ring of order p is of dimensions

$$N \times N = [(2v + 1)^{2n-1} \times 2^{(p-2^n)v}] \times [(2v + 1)^{2n-1} \times 2^{(p-2^n)v}] \tag{3.6}$$

where n is the maximum power of 2 such that 2^n is less than p. If we use for $\beta_\alpha^{(2)}$ higher representations [6], N will naturally be altered. However, it is to be noted that if we had begun with $\gamma_\alpha/2 \otimes 1 + 1 \otimes \gamma_\alpha/2$, and proceeded further according to Eq. (3.1) we would have obtained 2^{pv} dimensional representation for $\beta_\alpha^{(p)}$ which is the same as that of Green [2]. The representations arrived at in this section are also not irreducible.

In the above, representations of para-Fermi rings are obtained by adding γ_μ's of suitable dimension to the para-Fermi operators of smaller order. On referring to the literature of the theory of relativistic wave equations [7, 8, 9] there seems to be grounds for hoping that higher values of p may be related to higher spin. Hence successive additions of half spin fields may be thought of as a method of obtaining representations for higher p corresponding to those given here.

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