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THE CHROMATIC TOWER FOR $D(R)$

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§0. INTRODUCTION

IN [10], Hopkins states a remarkable theorem.

THEOREM. (Hopkins). *Let R be a commutative ring. Let $D^b(R)$ be the derived category of bounded complexes of finitely generated projective R-modules. Then there are maps of sets:*

$$\begin{pmatrix} \textit{Triangulated full subcategories of } D^b(R) \\ \textit{closed under direct summands} \end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \begin{pmatrix} \textit{Subsets of } \mathrm{Spec}(R) \\ \textit{closed under specialization} \end{pmatrix}$$

where $f(L) = \{p \in \mathrm{Spec}(R) \mid p \in \mathrm{supp}\, X \textit{ for some } X \in L\}$

$$\textit{and } g(P) = \begin{Bmatrix} \textit{the smallest triangulated category closed under} \\ \textit{specialization, containing } R/p \textit{ for all } p \in P \end{Bmatrix}$$

The maps f and g are inverse isomorphisms.

Following the conventions of algebraists, we will call triangulated subcategories of $D^b(R)$ *epaisse* if they are full and closed under direct summands.

Hopkins' theorem is a beautiful result. Among other things, it establishes that out of something seemingly nonsensical, like the derived category of R, one can recover a very sensible object, like *Spec*(R). However, there is a gap in the proof. Without some added hypotheses (e.g. R Noetherian) the theorem is false. A counterexample may be found in Section 4.

I should immediately add that Hopkins obtained his result by studying analogous properties in the topological setting, where [6] obtained some really remarkable and powerful results. The theorem quoted above occurs in a paper in a conference proceedings, where he explained the topological result and remarked in passing that the algebraic analogue is also correct. It should be stressed that Hopkins' result is very intriguing, and possibly very important. He discovered a parallel between stable homotopy theory and algebraic geometry, and this parallel should be explored further.

This is perhaps the appropriate point to briefly outline the topological parallel of what we do here. The starting point for us (this is historically quite wrong) is that $\Sigma^\infty S^0$ can be given the structure of an E^∞ ring spectrum. Therefore, in some sense it may be viewed as a commutative ring, and one may wish to study the algebraic geometry of this curious

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object. Following the sloppy notation of the topologists, we will freely confuse the spectrum $\Sigma^\infty S^0$ with its associated space $QS^0 = \Omega^\infty \Sigma^\infty S^0$. Following Waldhausen, such bizarre rings will be referred to as *brave new rings.*

Although QS^0 itself is very different from the rings we are used to, its derived category is far more ordinary. Any finite CW complex (or rather its suspension spectrum) may be viewed as a bounded complex of free $\Sigma^\infty S^0$ modules. Thus $D^b(QS^0)$ can be identified with the stable homotopy category: the category of finite CW complexes and stable maps (= maps defined on sufficiently high suspensions). And whatever may be wrong with QS^0, $D^b(QS^0)$ is an ordinary triangulated category. What Devinatz, Hopkins and Smith do in [6] is, among other things, classify all the epaisse subcategories of $D^b(QS^0)$, following a conjecture of Ravenel. Thus, if we believe Hopkins' theorem above, this should allow us to define $Spec(QS^0)$.

In this paper, we begin by proving that, at least when R is Noetherian, Hopkins' result is true. We tried to keep the proof as similar to Hopkins as possible, so that the gaps in the argument could be pointed out. That is Section 1.

In Sections 2 and 3 we study localizing subcategories of the unbounded derived category $D(R)$. We define

Definition 0.1 (Bousfield). A subcategory L in $D(R)$ is called *localizing* if:

(0.1.1) It is full
(0.1.2) It is triangulated
(0.1.3) It is closed with respect to the formation of arbitrary direct sums.

Definition 0.2 (Ravenel). A subcategory $L \subset D(R)$ is called *smashing* if

(0.2.1) It is localizing
(0.2.2) Localization at L commutes with direct sums.

When R is Noetherian, it is not hard to show:

THEOREM 2.8. *There are maps*

$$\begin{pmatrix} \textit{Localizing subcategories} \\ \textit{of } D(R) \end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \left(\textit{Subsets of } \operatorname{Spec}(R) \right)$$

where

$$f(L) = \{p \in \operatorname{Spec}(R) \mid \exists X \in L \textit{ with } X \otimes k(p) \neq 0\}$$

and

$$g(P) = \textit{the localizing subcategory generated by } k(p), \textit{for all } p \in P.$$

These maps are inverse isomorphisms. □

We can also show:

THEOREM 3.3. *Under the correspondence of Theorem* 2.8, *the smashing subcategories correspond to sets of primes closed under localization* (*in Grothendieck's terminology, to* systems of supports).

In particular, for $D(R)$ we can prove the "smashing conjecture": all smashing subcategories of $D(R)$ are generated by their intersection with $D^b(R)$. Theorem 1.5 (Hopkins' theorem) follows easily from Theorem 2.8 and Theorem 3.3. Thus we have really given two proofs of Hopkins' theorem in this article; one which closely parallels the nilpotence proof of Hopkins, and one which goes by way of infinite complexes.

Thus for a *Noetherian ring*, one has a complete and very satisfactory description of the spectral theory of its derived category. The whole point of this paper is that, although very nice, this is completely beside the point.

The fact that Noetherian rings are so orderly makes them very different from the topological analogue. This theory started with an attempt to understand the "chromatic tower" of Ravenel, i.e. the spectrum of QS^0. As can be shown by many pathological examples, the spectral theory of the category of spectra is anything but simple. It is quite different from Noetherian rings. The point of this article is that QS^0 should simply be considered as a fairly ordinary non-Noetherian ring. The pathological behaviour exhibited by QS^0 is closely analogous to what might happen for $D(R)$ for non-Noetherian rings R. And it is really here that the gap in [10] is very fortunate for the subject. As we discuss in Section 4, the results in [6] are in fact in perfect harmony with viewing QS^0 as a non-Noetherian discrete ring. But had Hopkins' theorem been true without restriction, there would have been a real difference. In particular, $Spec(QS^0)$ as defined from $D^b(QS^0)$ is probably the wrong space.

Finally, this paper depends on results in [1], where certain construction in $D(R)$ are made formal. At the beginning of Section 2 we list most of the results from [1] which we will rely on.

§1. A PROOF OF HOPKINS' THEOREM

Hopkins' proof of this theorem is mostly quite correct, but all the same it seems preferable to write up a complete new proof. The main reason is that some of Hopkins' reduction steps become far easier in the Noetherian case. If one accepts that the theorem is in any case only true for Noetherian rings, there seems little point in going through general, complicated arguments.

In this section R will always be a Noetherian, commutative ring, and $D^b(R)$ will stand for the derived category of bounded complexes of finitely generated projective R-modules (in the literature this is often referred to as $D^b(\mathrm{Proj}\ R)$. When X is an object of $D^b(R)$, $D(X)$ will stand for its "dual"; $D(X) = \mathrm{RHom}(X, R)$. For any two objects X, Y in $D^b(R)$, $X \overset{L}{\otimes} Y$ will simply be denoted $X \otimes Y$.

THEOREM 1.1. (*Smash Nilpotence Theorem*): *Let $f: X \to Y$ be a morphism in $D^b(R)$. Suppose that for every homomorphism $g: R \to k$, where k is a field, $f \otimes k = 0$. Then f is smash-nilpotent: There exists $n > 0$, so that $f^{\otimes n}: X^{\otimes n} \to Y^{\otimes n}$ is the zero map.*

Proof. f induces a map $f': R \to D(X) \otimes Y$, and $(f')^{\otimes n} = 0$ iff $f^{\otimes n} = 0$. Therefore we may reduce ourselves to the case $X = R$.

We define:

$$\mathrm{Ann}(f^{\otimes n}) = \{x \in R \mid xf^{\otimes n} = 0\}.$$

$\mathrm{Ann}(f^{\otimes n})$ is an ideal of R, and $\mathrm{Ann}(f^{\otimes n}) \subset \mathrm{Ann}(f^{\otimes n+1})$. Because R is Noetherian, the ideals must stabilize. There exists n such that $\mathrm{Ann}(f^{\otimes n}) = \mathrm{Ann}(f^{\otimes m})$ for all $m \geq n$. Replacing f by $f^{\otimes n}$, we may assume $n = 1$. We need to show $\mathrm{Ann}(f) = R$.

We will suppose $\mathrm{Ann}(f) \neq R$, and prove a contradiction. Because $\mathrm{Ann}(f) \neq R$, there is a minimal prime ideal $p \subset R$ containing $\mathrm{Ann}(f)$. Localizing the entire problem at p, we may assume:

(1.1.1) R is a local ring, with maximal ideal p.

(1.1.2) For every $n > 0$, $\mathrm{Ann}(f) = \mathrm{Ann}(f^n)$.

(1.1.3) For some m, $p^m \subset \mathrm{Ann}(f) \subset p$.
(1.1.4) $f \otimes R/p = 0$.

Here is where the first serious gap in [10] occurs. [10] reasons that f is a morphism $f\colon R \to X$, and it defines a class, which we will also call f, in $H^0(X)$. We know that $f \otimes R/p = 0$. Therefore $f = \sum_{i=1}^{n} x_i g_i$ where $x_i \in p$ and $g_i \in H^0(X)$. Thus if $p^m \subset \mathrm{Ann}(f)$, then $f^{m+1} = (\Sigma x_i g_i)^m f \in p^m f = 0$.

What is wrong with the argument is that it simply is not true that when $f \otimes R/p = 0$, then $f = \Sigma x_i g_i$ as above. The identity $f \otimes R/p = 0$ should be read as an identity in $H^0(X \otimes R/p)$, whereas $f = \Sigma x_i g_i$ asserts that f is already zero in $H^0(X) \otimes R/p$. In other words, we have natural maps

$$H^0(X) \xrightarrow{\alpha} H^0(X) \otimes R/p \xrightarrow{\beta} H^0(X \otimes R/p)$$

and $\beta \cdot \alpha(f) = 0$. However, this does not imply that $\alpha(f) = 0$. More concretely, one could say this as follows: $f \otimes R/p = 0$ means that, up to boundaries, $f = \Sigma x_i g_i$ where $x_i \in p$ and $g_i \in X^0$, where X is the chain complex

$$\cdots \to X^{-1} \xrightarrow{\partial} X^0 \xrightarrow{\partial} X^1 \to \cdots$$

i.e. x_i are *chains*. Of course, f is a *cycle*: $\partial(f) = 0$. In general, we have no reason to expect that g_i can be chosen to be cycles.

Naturally, our proof of Hopkins' Theorem will run somewhat differently. We will prove by induction the following assertions:

$F(n)$: Let R be a local, Noetherian, *reduced* ring of dimension $\dim(R) \leqslant n$. Suppose $f\colon R \to X$ satisfies 1.1.1, 1.1.3 and 1.1.4 above. Then f is nilpotent.

$G(n)$: Let R be a local, Noetherian ring (not necessarily reduced) of dimension $\dim(R) \leq n$. Let $f\colon R \to X$ satisfy 1.1.1, 1.1.3 and 1.1.4 above. Then f is nilpotent.

We will prove $F(n) \Rightarrow G(n) \Rightarrow F(n+1)$. Because $F(0)$ is trivial, this shows that 1.1.2 is incompatible with 1.1.1, 1.1.3 and 1.1.4; hence our required contradiction.

$F(n) \Rightarrow G(n)$: Let $f\colon R \to X$ be as in $G(n)$. Then by $F(n)$ we know that $f \otimes R/_{\mathrm{rad}(R)}$ is nilpotent. Thus for some m, $f^m \otimes R/_{\mathrm{rad}(R)} = 0$, and as above we may assume $m = 1$. By Hopkins' argument $f = \Sigma x_i g_i$ where $x_i \in \mathrm{rad}(R)$ and $g_i \in X^0$. But now the x_i's are nilpotent in R, not only in $R/_{\mathrm{Ann}\, f}$, and if $(\mathrm{rad}(R))^m = 0$, it follows that $f^{\otimes m} = (\Sigma x_i g_i)^m = 0$, even as an element in $(X^0)^{\otimes m}$.

$G(n) \Rightarrow F(n+1)$: Let R be a reduced ring of dimension $n + 1 \geq 1$. Let $f\colon R \to X$ be as above. Because R is reduced and $p^m \subset \mathrm{Ann}(f)$, where p is the maximal ideal, there must be a regular element in $\mathrm{Ann}(f)$; i.e. there is an element $x \in \mathrm{Ann}(f)$ which is not a zero divisor in R. Choose such an x.

Let f be represented by the map of chain complexes:

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
0 & \longrightarrow & X^1 \\
\uparrow & & \uparrow \\
R & \xrightarrow{f} & X^0 \\
\uparrow & & \uparrow \\
0 & \longrightarrow & X^{-1} \\
\uparrow & & \uparrow \\
\vdots & & \vdots
\end{array}$$

Because $xf = 0$, it follows that $f(x) \in X^0$ must be a boundary; i.e. we can extend to a map of complexes:

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
0 & \longrightarrow & X^1 \\
\uparrow & & \uparrow \\
R & \xrightarrow{f} & X^0 \\
x\uparrow & & \uparrow \\
R & \xrightarrow{g} & X^{-1} \\
\uparrow & & \uparrow \\
0 & \longrightarrow & X^{-2} \\
\uparrow & & \uparrow
\end{array}$$

But the fact that the sequence

$$0 \to R \xrightarrow{x} R \to R/_{xR} \to 0$$

is exact (the regularity of $x \in R$) tells us that the complex $\to 0 \to R \xrightarrow{x} R \to 0 \to$ is quasi-isomorphic to $R/_{xR}$. What we have shown is exactly that f factorizes as

$$R \xrightarrow{\alpha} R/_{xR} \xrightarrow{\beta} X.$$

Therefore $f^{\otimes n+1}$ factorizes as

$$R \xrightarrow{\alpha \otimes 1} R/_{xR} \xrightarrow{1 \otimes f^{\otimes n}} R/_{xR} \otimes X^{\otimes n} \xrightarrow{\beta \otimes 1} X^{\otimes n+1}$$

and so $(f \otimes R/_{xR})^{\otimes n} = 0$ implies $f^{\otimes n+1} = 0$. But $R/_{xR}$ is of dimension n, and by $G(n)$ we therefore know that $f \otimes R/_{xR}$ is nilpotent $\square$

LEMMA 1.2. *Let X and Y be objects in $D^b(R)$, and suppose* $\operatorname{Supp}(X) \subset \operatorname{Supp}(Y)$, *that is the support of the cohomology of X is contained in the support of the cohomology of Y. Then X is in the epaisse subcategory generated by Y.*

Proof. The proof is identical with Hopkins'. We included it only for the convenience of the reader.

There is a natural morphism $R \to Y \otimes D(Y)$, and we complete it to a triangle

$$\begin{array}{ccc}
R & \xrightarrow{f} & Y \otimes D(Y) \\
{}_h\nwarrow & & \swarrow \\
& M_f &
\end{array}$$

When $\alpha: R \to k$ is a homomorphism of R to the field k, then it is clear that if $\ker(\alpha) \in \operatorname{Supp}(Y)$, $f \otimes k: k \to Y \otimes D(Y) \otimes k$ is a split monomorphism. Therefore $h \otimes k = 0$, at least for those $\alpha: R \to k$ for which $\ker(\alpha) \in \operatorname{Supp}(Y)$. We assumed that $\operatorname{Supp}(X) \subset \operatorname{Supp}(Y)$, therefore X is quasi-isomorphic to a complex of $R/_I$ modules, where $\operatorname{Supp}(R/_I) = \operatorname{Supp}(X) \subset \operatorname{Supp}(Y)$. Then $h \otimes R/_I$ satisfies the hypotheses of Theorem 1.1, and it follows that $h^{\otimes n} \otimes_R R/_I = 0$. But then $h^{\otimes n} \otimes_R X = h^{\otimes n} \otimes_R R/_I \otimes_{R/_I} X = 0$.

For each k, we complete the morphism $M_f^{\otimes k} \xrightarrow{h^{\otimes k}} R$ to a triangle

$$\begin{array}{ccc}
M_f^{\otimes k} & \xrightarrow{h^{\otimes k}} & R \\
\nwarrow & & \swarrow \\
& N_{h^{\otimes k}} &
\end{array}$$

and the identity $h^{\otimes k} = h\circ(1\otimes h^{\otimes k-1})$ gives an octahedron:

$$\begin{tikzcd}
 & & M_f\otimes N_{h^{\otimes k-1}} \arrow[dl, "d"'] \arrow[ddll, bend right=40] & & \\
+ & M_f^{\otimes k} \arrow[rr, "1\otimes h^{\otimes k-1}"] \arrow[dr, "h^{\otimes k}"'] & & M_f \arrow[ul] \arrow[dl, "h"] & + \\
 & & R \arrow[dll] \arrow[drr] & & \\
N_{h^{\otimes k}} \arrow[uur, "d"] \arrow[rrrr, bend right=20, "d"'] & & + & & N_h = Y\otimes D(Y) \arrow[uul, "d"'] \arrow[uuull, bend right=40]
\end{tikzcd}$$

and in particular we have for each k a triangle

$$\begin{tikzcd}
Y\otimes DY = N_h \arrow[r] & M_f\otimes N_{h^{\otimes k-1}} \arrow[dl] \\
N_{h^{\otimes k}} \arrow[u] &
\end{tikzcd}$$

By induction on k, one easily proves that $N_{h^{\otimes k}}$ is in the triangulated category generated by Y for every integer $k \geq 1$. Now from the triangle

$$\begin{tikzcd}
M_f^{\otimes n}\otimes X \arrow[r, "h^{\otimes n}\otimes X"] & X \arrow[dl] \\
N_{h^{\otimes n}}\otimes X \arrow[u] &
\end{tikzcd}$$

and from the fact that $h^{\otimes n}\otimes X = 0$, we deduce that X is a direct summand of $N_{h^{\otimes n}}\otimes X$, and is therefore in the epaisse subcategory generated by Y. □

LEMMA 1.3. *Let $L \subset D^b(R)$ be epaisse. Let*

$$P(L) = \{p\in \operatorname{Spec} R \mid \exists X\in L \text{ with } p\in \operatorname{Supp}(X)\}.$$

Put $X\in D^b(R)$ with $\operatorname{Supp}(X)\subset P(L)$. Then $X\in L$.

Proof. Because R is *Noetherian*, X has only finitely many minimal prime ideals in its support. Therefore there are finitely many $Y_i\in L$ so that $\bigcup_i \operatorname{Supp}(Y_i)$ contains all the minimal prime ideals in $\operatorname{Supp}(X)$, and because $\bigcup_i \operatorname{Supp}(Y_i)$ is closed, it contains $\operatorname{Supp}(X)$. In particular, $\operatorname{Supp}(X)\subset \operatorname{Supp}(\bigoplus_i Y_i)$ and then, by Lemma 1.2, it immediately follows that $X\in L$. □

LEMMA 1.4. *Let $p\in\operatorname{Spec}(R)$. Then there is in $D^b(R)$ an object X with $\operatorname{Supp}(X) = \bar{p}$, where $\bar{p}$ is the closure of p in $\operatorname{Spec} R$.*

Proof. Because R is *Noetherian*, p is finitely generated, say by generators $a_1, \ldots, a_n$. Then the Koszul complex

$$X = \bigotimes_i (R \xrightarrow{a_i} R)$$

is an object of $D^b(R)$ with $\operatorname{Supp}(X) = \bar{p}$. □

THEOREM 1.5. *There are maps*:

$$\begin{pmatrix}\textit{Epaisse subcategories}\\ \textit{of } D^b(R)\end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \begin{pmatrix}\textit{Subsets of Spec}(R) \textit{ closed under}\\ \textit{localization}\end{pmatrix}$$

where

$$f(L) = \{p \in \mathrm{Spec}(R) \mid \exists X \in L \text{ with } p \in \mathrm{Supp}(X)\}$$

and

$$g(P) = \{X \in L \mid \mathrm{Supp}(X) \subset P\}.$$

These maps are inverse isomorphisms.

Proof. By Lemma 1.3 f is injective, whereas Lemma 1.4 is the surjectivity of f. □

Remarks 1.6. Theorem 1.1 is [10], Theorem 10(i). As we have already said, Lemma 1.2 is lifted straight out of [10]; it is the proof to Theorem 11 in [10]. Theorem 11 of [10] is actually our Theorem 1.5, and in the argument of [10], Lemmas 1.3 and 1.4, with the appeal to the Noetherian hypothesis, are both missing. Thus there are three points where we used the Noetherian hypothesis, namely the proofs of Theorem 1.1, Lemma 1.3 and Lemma 1.4.

What is true without the Noetherian hypothesis? Lemma 1.4 is definitely false; a counterexample may be found in Section 4. Theorem 1.1 (the smash nilpotence) is open. However, Lemma 1.2 can be proved in general, by reducing to the Noetherian case. Therefore, for arbitrary commutative R, one can show that for the maps f, g of Theorem 1.5, $g \circ f = 1$. For details, see [Appendix]. Lemma 1.3 is also still open, but is probably false in general.

§2. LOCALIZING SUBCATEGORIES OF $D(R)$

As in Section 1, R is a Noetherian, commutative ring. However, $D(R)$ will from now on denote the derived category of unbounded complexes of R-modules. This is in contrast with Section 1, where $D(X) = RHom(X, R)$ was the dual of X. The reason for this discrepancy is an attempt to be consistent with the literature; Section 1 is consistent with the notation in [10], whereas from now on we will be consistent with the classical algebraic literature.

We will also be making use of the following results, the proofs of which may be found in [1] or [14]:

(2.1.1) There exists a tensor product $D(R) \times D(R) \to D(R)$, which we will denote $X \otimes Y$.

(2.1.2) The tensor is bitriangulated; given a triangle $Y \to Y' \to Y'' \to \Sigma Y$ and an arbitrary $X \in D(R)$, there is an induced triangle $X \otimes Y \to X \otimes Y' \to X \otimes Y'' \to X \times \Sigma Y = \Sigma(X \otimes Y)$ and similarly for triangles in X.

(2.1.3) The tensor commutes with direct sums in either variable; the natural map $\oplus_\alpha (X \otimes Y_\alpha) \to X \otimes (\oplus_\alpha Y_\alpha)$ is an isomorphism.

(2.1.4) The tensor is associative and commutative; there are natural isomorphisms

$$4(\mathrm{a}){:}\ (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

and

$$(4\mathrm{b}){:}\ X \otimes Y = Y \otimes X.$$

(2.1.5) Let $\alpha: R \to k$ be a homomorphism of R into a field k. Let $X \in D(R)$ be arbitrary. Then $X \otimes k$ is a direct sum of suspensions of k.

(2.1.6) Every $X \in D(R)$ is isomorphic to a complex of injectives.

Remark 2.2. 2.1.2 and 2.1.3 imply that the tensor product commutes with direct limits. Note that the properties are all seemingly trivial and very reasonable. To convince himself that a little care is required, the reader should look at the introduction to [1].

Definition 2.3. A full subcategory $L \subset D(R)$ is called *localizing* if it is triangulated and closed under arbitrary direct sums.

With the aid of Definition 2.3, we are ready to quote another result from [1] which we will use:

(2.1.7) Let L be a localizing subcategory of $D(R)$. Suppose $X \in L$, $Y \in D(R)$ arbitrary. Then $X \otimes Y \in L$.

Definition 2.4 (Bousfield). Let L be a localizing subcategory of $D(R)$. The object $X \in D(R)$ is called *L-local* if $\operatorname{Hom}(L, X) = 0$ (i.e. for all $Y \in L$, $\operatorname{Hom}(Y, X) = 0$).

Definition 2.5 (Bousfield). Let L be a localizing subcategory of $D(R)$. Let X be an object in $D(R)$. The morphism $X \xrightarrow{f} X_L$ is caled a *localization* of X if X_L is local, and for any L-local $Y \in D(R)$, $\operatorname{Hom}(X_L, Y) \xrightarrow{\operatorname{Hom}(f,Y)} \operatorname{Hom}(X, Y)$ is an isomorphism.

THEOREM 2.6. (Bousfield). *Let L be a localizing subcategory of $D(R)$. Then every object $X \in D(R)$ has a localization (possibly after increasing the universe). In the triangle $X \to X_L \to Z \to \Sigma X$, the object Z is in L.*

Remark 2.7. I have attributed all these results to Bousfield, because he is the latest author to have worked on them, and he has the best, most general results. However, this matter has a long history. Localizations of spaces were first considered by Sullivan, and work on the subject was done also by Adams.

What we will do in this section is to completely describe the localizing subcategories of $D(R)$, where R is a Noetherian, commutative ring. We will prove:

THEOREM 2.8. *There are maps of sets:*

$$\begin{pmatrix} \textit{Localizing subcategories} \\ \textit{of } D(R) \end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \left(\textit{Subsets of } \operatorname{Spec}(R) \right)$$

where

$$f(L) = \{p \in \operatorname{Spec}(R) \mid \exists X \in L \textit{ with } X \otimes (R/_p)_0 \neq 0\}$$

(Here $(R/_p)_0$ is the quotient field of $R/_p$. We will often denote it by $k(p)$) and

$$g(P) = \textit{the localizing category generated by } k(p), \textit{for all } p \in P.$$

These maps are inverse isomorphisms.

The proof of Theorem 2.8 will be a succession of easy lemmas.

LEMMA 2.9. *Let $X \in D(R)$ be a complex which consists entirely of injectives, each of which is a direct sum of copies of I_p, the injective hull of $R/_p$, where $p \in \operatorname{Spec}(R)$ is given. Then X is in the localizing category generated by $k(p)$.*

Proof. Recall that the injective hull of $R/_p$ is an indecomposable injective, and is in fact an R_p module (elements outside p act invertibly). Further, every element of I_p is annihilated by p^n for some n.

Thus the complex X has a filtration:

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$$

where X_i is the subcomplex of X annihilated by p^n. Clearly, X_i/X_{i-1} is a complex of vector spaces over $k(p)$, and is therefore quasi-isomorphic to a direct sum of suspensions of $k(p)$. Thus X_i/X_{i-1} is in the localizing subcategory generated by $k(p)$. From the triangle $X_{i-1} \to X_i \to X_i/X_{i-1} \to \Sigma X_{i-1}$, we deduce by induction that so is X_i for every i. But $X = \varinjlim X_i$, and by 2.2.2 in [1] a localizing subcategory is closed under the formation of direct limits. □

Let X be a complex in $D(R)$. By 2.1.6, X is quasi-isomorphic to a complex of injectives. Every injective is a direct sum of indecomposable injectives, and these correspond to prime ideals in R. Consider the complex obtained by taking in a complex of injectives X all the terms isomorphic to direct sums of I_p. It is well-known that this is in fact a complex. One can obtain it for instance by considering in $X \otimes_R R_p$ the subcomplex supported at the closed point. In Grothendieck's notation, this is the complex $\Gamma_{Y/Y'}(X)$, where $Y = \bar{p}$ and $Y' = \bar{p} - \{p\}$ are systems of supports in $\operatorname{Spec}(R)$.

LEMMA 2.10. *Let $P \subset \operatorname{Spec}(R)$ be given. Suppose X is a complex in $D(R)$ such that for $p \in \operatorname{Spec}(R) - P$, $\Gamma_{\bar{p}/\bar{p}-\{p\}}(X) = 0$. Then X is in the localizing subcategory $L = g(P)$ generated by all the $k(p)$, $p \in P$.*

Proof. Let $\wedge \subset \mathcal{P}\mathcal{P}(\operatorname{Spec}(R))$ be the set of all systems of support $Y \subset \mathcal{P}(\operatorname{Spec}(R))$ such that $\Gamma_Y(X)$ is in our localizing subcategory. As localizing subcategories are closed under the formation of direct limits, $\wedge$ must be closed under the formation of increasing unions. Hence, by Zorn's Lemma, $\wedge$ contains a maximal element Y. We assert $Y = \operatorname{Spec}(R)$.

Suppose $Y \neq \operatorname{Spec}(R)$. Because R is Noetherian, $\operatorname{Spec}(R) - Y$ contains a maximal element p. But now

$$\Gamma_{Y \cup \{p\}/Y}(X) = \Gamma_{\bar{p}/\bar{p}-\{p\}}(X)$$

and by our hypothesis, either $k(p) \in L$ or $\Gamma_{\bar{p}/\bar{p}-\{p\}}(X) = 0$. By Lemma 2.9, in either case we have $\Gamma_{\bar{p}/\bar{p}-\{p\}}(X) \in L$. Thus we deduce easily that $\Gamma_{Y \cup \{p\}}(X) \in L$, and this is a contradiction to the maximality of Y. □

LEMMA 2.11. *Let $P \subset \operatorname{Spec}(R)$, and let $X \in D(R)$ be such that for all $p \in P$, $X \otimes k(p) = 0$. Then if $L = g(P)$ is the localizing subcategory generated by all $k(p)$, $p \in P$, then $X \otimes L = 0$.*

Proof. A trivial consequence of properties 2.1.2 and 2.1.3 of the tensor product. □

LEMMA 2.12. *If $X \in D(R)$ has the property that for every $p \in \operatorname{Spec}(R)$ $X \otimes \mathrm{k(p)} = 0$, then $X = 0$.*

Proof. By Lemma 2.11, it would follow that for every Y in $L = g(\operatorname{Spec}(R))$, $X \otimes Y = 0$. But by Lemma 2.10 we know that $g(\operatorname{Spec}(R)) = D(R)$ (every complex is in L). In particular, we deduce that $X = X \otimes R = 0$. □

LEMMA 2.13. *Let X be a complex of injectives, all of which are direct sums of I_p for one particular $p \in \operatorname{Spec}(R)$. Then for all $p' \neq p$, $X \otimes k(p') = 0$.*

Proof. Trivial. □

LEMMA 2.14. *If $X \in D(R)$ is a complex of injectives as in Lemma 2.13, then $X \otimes k(p) = 0$ iff $X = 0$.*

Proof. Lemma 2.12 and Lemma 2.13. □

Proof of Theorem 2.8. We have the maps

$$\begin{pmatrix}\text{Localizing subcategories}\\ \text{of } D(R)\end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \begin{pmatrix}\text{Subsets of } Spec(R)\end{pmatrix}$$

and we have to show that $f \circ g$ and $g \circ f$ are identities. For $f \circ g$ this is just Lemma 2.11. For $g \circ f$, we observe the following. Let L be a localizing category. Then we need to show

(1) $g \circ f(L) \subset L$.

To prove this, observe that $p \in f(L)$ iff there exists an X with $X \otimes k(p) \neq 0$. Now $X \otimes k(p) \in L$ because X is (2.1.7), and by 2.1.5, $X \otimes k(p)$ is a direct sum of suspensions of $k(p)$; i.e. $k(p)$ must be in L. (L is epaisse; see [1]). Thus $g \circ f(L)$, the localizing category generated by all $k(p)$, $p \in f(L)$ is contained in L. □

(2) $L \subset g \circ f(L)$.

Let $X \in L$. We know that

$$X \otimes k(p) = \Gamma\bar{p}/_{\bar{p}-\{p\}}(X) \otimes k(p)$$

and by Lemma 2.14, this is zero if and only if $\Gamma\bar{p}/_{\bar{p}-\{p\}}(X) = 0$. By Lemma 2.10, X is in the localizing subcategory generated by the $k(p)$'s for which $\Gamma\bar{p}/_{\bar{p}-\{p\}}(X) \neq 0$, or equivalently $X \otimes k(p) \neq 0$. Thus $X \in g \circ f(L)$. □

§3. SMASHING SUBCATEGORIES OF $D(R)$

Definition 3.1 (Ravenel). A localizing subcategory $L \subset D(R)$ is called *smashing* if localization commutes with direct sums; i.e. if the natural map $\oplus(X_\alpha)_L \to (\oplus X_\alpha)_L$ is an isomorphism.

Remark 3.2. If L is smashing, then localizing at L must commute with arbitrary homotopy colimits.

In this section we will classify all the smashing subcategories of $D(R)$, where R is, as always, a Noetherian, commutative ring. We will prove:

THEOREM 3.3. *Under the isomorphism of Theorem* 2.8,

$$\begin{pmatrix}\textit{Localizing subcategories}\\ \textit{of } D(R)\end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \begin{pmatrix}\textit{Subsets of } \mathrm{Spec}(R)\end{pmatrix}$$

the smashing subcategories correspond to systems of support; i.e. to subsets closed under specialization.

COROLLARY 3.4. (*The smashing conjecture*): *The smashing subcategories of $D(R)$ are precisely the localizing subcategories generated by epaisse subcategories of $D^b(R)$.*

Again, the proof of Theorem 3.3 will be by a sequence of easy lemmas. We begin with:

LEMMA 3.5. *Let L be a localizing category, and let $p \in \mathrm{Spec}(R)$ be a prime. Then either $k(p)$ is L-local, or $k(p) \in L$.*

Proof. Suppose $k(p)$ is not L-local. Then there exists $X \in L$ and a non-trivial morphism $X \to k(p)$. We then have a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & k(p) \arrow[d] \arrow[dd, bend left=60, "\mathrm{id}"] \\
X \otimes k(p) \arrow[r] & k(p) \otimes k(p) \arrow[d] \\
 & k(p)
\end{tikzcd}$$

which establishes that our morphism $X \to k(p)$ factors through $X \otimes k(p)$. But the morphism was assumed non-zero, hence $X \otimes k(p) \neq 0$. But $X \otimes k(p)$ is contained in L because X is, and as $X \otimes k(p)$ is a direct sum of suspensions of $k(p)$, it follows that $k(p) \in L$. □

LEMMA 3.6. *Suppose L is smashing. Then for any indecomposable injective I_p, either $I_p \in L$ or I_p is L-local.*

Proof. As in Lemma 2.9, I_p is a direct limit of objects that can be obtained as extensions of $k(p)$. Because L is smashing, direct limits commute with L-localization, and of course localization always respects triangles. Because $k(p)_L = k(p)$ or zero, it follows that $(I_p)_L = I_p$ or zero.

LEMMA 3.7. *Let $p \in q$ be primes in R. If L is a smashing subcategory and $k(p) \in L$, then $k(q) \in L$.*

Proof. There is a non-zero map $R/_p \to R/_q$, which induces a non-zero map of the injective hulls $I_p \to I_q$. If $k(p) \in L$, then $I_p \in L$, and $(I_p)_L = 0$. Therefore, I_q cannot be L-local; for if I_q were L-local, then we would have an isomorphism

$$\begin{aligned} \mathrm{Hom}(I_p, I_q) &\cong \mathrm{Hom}((I^p)_L, I_q) \\ &= \mathrm{Hom}(0, I_q) = 0 \end{aligned}$$

which contradicts the existence of our non-zero map $I_p \to I_q$. Thus $(I_q)_L \neq I_q$, and by Lemma 3.5 we deduce that $(I_q)_L = 0$ and $k(q)_L = 0$, i.e. $k(q) \in L$. □

Proof of Theorem 3.3. It follows immediately from Lemma 3.7 that, under the correspondence of Theorem 2.8, smashing subcategories correspond to families of prime ideals closed under localization (= support systems). It remains to show that every support system corresponds to a smashing subcategory. One way to do this is to observe that every support system is generated by objects in $D^b(R)$, and is therefore trivially smashing. □

Remark 3.8. The reader can reflect that Hopkins' Theorem (= Theorem 1.5) follows easily from Theorem 2.8 and Theorem 3.3. Thus we have really given two different proofs of Hopkins' Theorem.

§4. REMARKS ABOUT THE NON-NOETHERIAN CASE

In the non-Noetherian case the situation is far more complicated. Let us consider:

Example 4.1. Let $R = k[X_1, X_2, \ldots]$ be the polynomial ring in infinitely many variables. There is an obvious homomorphism $R \to k$ which sends all X_i to zero. Its kernel m is maximal. We can consider the support system $\{m\}$, consisting of the singleton closed point.

It is very clear that there is no finite complex of projectives whose support is only m. Thus in the Hopkins correspondence, there is no epaisse subcategory of $D^b(R)$ corresponding to $\{m\}$.

This example is more significant that meets the eye at first glance. Following contemporary fads, one wants to view the stable homotopy category as the category of modules over the "brave new ring" QS^0. We now know the structure of the epaisse subcategories of this triangulated category. Thanks to [6] and [13], there is a "chromatic tower" of "prime" ideals. It can be defined as follows. Consider the fields $K(Q, 0)$ and $K_i(p)$ where $K(Q, 0)$ is the Eilenberg–McLane spectrum, whereas $K_i(p)$ is the ith Morava K-theory at the prime p. To each of these corresponds a "prime" epaisse subcategory; it can be defined as the smallest epaisse L with $L \otimes K \neq 0$. If we denote these by $L(K)$, then every epaisse subcategory is a join of $L(K)$'s, and the $L(K)$'s are not joins of smaller localizing subcategories (in this sense, the $L(K)$'s are very analogous to the sets $\bar{p}$, where p is a prime in R and the correspodence is as in Theorem 1.5). But, of course, the sad thing about the analogy is that for any prime $p \in \mathbb{Z}$ we have inclusions $L(K_i(p)) \subsetneq L(K_{i+1}(p))$, and there are infinitely many Morava K-theories. This was used by topologists to conclude that QS^0 is really different from discrete rings; it is possible in QS^0 to construct an unbounded, ascending chain of prime ideals; in discrete rings, such a chain must be contained in a maximal ideal.

What Example 4.1 shows is that what happens for QS^0 is not at all unlike discrete rings. The same will happen to the ring R of Example 4.1; it is possible to construct a chain of prime ideals in R such that the corresponding localizing subcategories form an unbounded chain; in $\mathrm{Spec}(R)$ there is a natural bound, namely the maximal ideal m, but there is no localizing subcategory corresponding to it. Moreover, in QS^0 there is a natural analogue to m, namely the Eilenberg–McLane spectrum $K(\mathbb{Z}/_{p\mathbb{Z}}, 0)$. This is a field which does not under the correspodence, give rise to any epaisse subcategory.

What I tried to show in this paper is that for discrete *Noetherian* rings, $D(R)$ behaves very nicely. Far too nicely to be at all analogous to the categories of spectra. Thus to study QS^0, one should begin with the assumption that it behaves like a non-Noetherian ring. It is possible to reproduce many, if not all, of the pathological examples of very bad spectra due to Bousfield and Ravenel in the category $D(R)$. And my belief is that a thorough understanding of $D(R)$ for non-Noetherian R will help a great deal in elucidating some of the remaining problems about the chromatic tower.

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APPENDIX BY M. BÖKSTEDT

We will show here that even in the non-Noetherian case, there is a construction which, out of the epaisse subcategory of $D^b(R)$, reconstructs $\operatorname{Spec}(R)$. We begin with some straightforward observations:

LEMMA A.1. *Let $X \in D^b(R)$. Then there exists a Noetherian subring $S \subset R$, an object $X' \in D^b(S)$ and an isomorphism $X' \otimes_R S = X$.*

Proof. Trivial.

LEMMA A.2. *The set* $\operatorname{Spec}(R) - \operatorname{supp}(X)$ *is a quasi-compact open subset of Spec* (R).

Proof. Let $\operatorname{Spec}(R) \to \operatorname{Spec}(S)$ be the natural map. Then $\operatorname{supp}(X) = \varphi^{-1}(\operatorname{supp}(X'))$ corresponds to a finitely generated ideal $(f_1, \ldots, f_n) \subset R$. Thus

$$\operatorname{Spec}(R) - \operatorname{supp}(X) = \bigcup_{i=1}^{n} \operatorname{Spec}\left(R\left[\frac{1}{f_i}\right]\right)$$

is a finite union of quasi-compacts. □

LEMMA A.3. *Let X and Y be objects of $D^b(R)$ with* $\operatorname{supp}(X) \subset \operatorname{supp}(Y)$. *Then X is in the epaisse category generated by Y.*

Proof. Choose a ring $S \subset R$ and objects X', Y' in $D^b(S)$ with $X = X' \otimes_S R$, $Y = Y' \otimes_S R$ as in the proof of Lemma A.2. Suppose $\operatorname{supp}(X') = (f_1, \ldots, f_n)$ whereas $\operatorname{supp}(Y') = (g_1, \ldots, g_n)$. Then because $\varphi^{-1}(\operatorname{supp}(X')) \subset \varphi^{-1}(\operatorname{supp}(Y'))$, we have relations:

$$f_i^N = \sum r_{ij} g_j$$

for some $N \in \mathbb{N}$, $r_{ij} \in R$. Replacing S by $S[r_{ij}] \subset R$, we may assume $\operatorname{supp}(X') \subset \operatorname{supp}(Y')$. Then by the Noetherian case of Hopkins' theorem, X' is in the epaisse subcategory generated by Y'. Hence the Lemma. □

Definition A.4. An epaisse subcategory L of $D^b(R)$ is called *principal* if it is generated by one element $X \in D^b(R)$; $\langle X \rangle$ will stand for the smallest epaisse subcategory containing X.

Definition A.5. Let $PS(D^b(R))$ be the set of all principal epaisse subcategories of $D^b(R)$. $PS(L)$ is clearly a partially ordered set, ordered by inclusion.

Definition A.6. A subset $S \subset PS(D^b(R))$ is called *filtering* if it satisfies the following two hypotheses

A.6.1: If $\langle X \rangle \in PS(D^b(R))$, $\langle Y \rangle \in S$ and $\langle Y \rangle \subset \langle X \rangle$, then $\langle X \rangle \in S$.
A.6.2: If $\langle X \rangle$ and $\langle Y \rangle$ are in S, there is a $\langle Z \rangle \in S$ with $\langle Z \rangle \subset \langle X \rangle$, $\langle Z \rangle \subset \langle Y \rangle$.

PROPOSITION A.7. *There are inverse isomorphisms*

$$\begin{pmatrix} \text{Filtering subsets} \\ \text{of } PS(D^b(R)) \end{pmatrix} \underset{g}{\overset{f}{\rightleftarrows}} \begin{pmatrix} \text{Closed subsets of} \\ \operatorname{Spec}(R) \end{pmatrix}$$

where

$$f(S) = \bigcap_{\langle X \rangle \in S} \operatorname{supp}\langle X \rangle$$

and

$$g(P) = \{\langle X \rangle \mid P \subset \operatorname{supp}(X)\}.$$

Proof. First observe that g is well defined; given $\langle X \rangle$ and $\langle Y \rangle$ in $g(P)$, then $\langle X \otimes Y \rangle$ satisfies $\langle X \otimes Y \rangle \subset \langle X \rangle$, $\langle X \otimes Y \rangle \subset \langle Y \rangle$, and hence A.6.2 is satisfied.

Step 1. $g \circ f = 1$. Suppose S is a filtering subset of $PS(D^b(R))$ and suppose $\operatorname{supp}\langle X \rangle \supset f(S)$. We need to show $\langle X \rangle \in S$.

By Lemma A.2, $\operatorname{Spec}(R) - \operatorname{supp}(X)$ is compact. But $\bigcap_{\langle Y \rangle \in S} \operatorname{supp}(Y) \subset \operatorname{supp}(X)$ by hypothesis, thus $\operatorname{Spec}(R) - \operatorname{supp}(Y)$ form an open cover of the quasi-compact $\operatorname{Spec}(R) - \operatorname{supp}(X)$. Therefore there is a finite subcover;

$$\bigcap_{i=1}^{n} \operatorname{supp}\langle Y_i \rangle \subset \operatorname{supp}\langle X \rangle.$$

By A.6.2, there is a $\langle Z \rangle \in S$ with $\operatorname{supp}(Z) \subset \operatorname{supp}(X)$. Thus, by Lemma A.3, $\langle Z \rangle \subset \langle X \rangle$ and hence by A.6.1, $\langle X \rangle \in S$. □

Step 2. $f \circ g = 1$. This is the trivial observation that, given $p \in \operatorname{Spec}(R) - P$, there is $X \in D^b(R)$ with $P \subset \operatorname{supp}(X)$ but $p \notin \operatorname{supp}(X)$. □

The reader is invited to see what this gives for QS^0. At least one of the authors feels there should be a simpler way to describe Spec (R) in terms of localizing subcategories of $D(R)$.