Convex Polyhedra of Doubly Stochastic Matrices—IV

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ABSTRACT

Basic geometrical properties of general convex polyhedra of doubly stochastic matrices are investigated. The faces of such polyhedra are characterized, and their dimensions and facets are determined. A connection between bounded faces of doubly stochastic polyhedra and faces of transportation polytopes is established, and it is shown that there exists an absolute bound for the number of extreme points of d-dimensional bounded faces of these polyhedra.

1. INTRODUCTION

A real matrix is doubly stochastic provided all row and column sums equal 1. Let \( N_n = \{1, \ldots, n\} \), and let \( S \subseteq N_n \times N_n \). The set of all \( n \times n \) doubly stochastic matrices \( X = [x_{ij}] \) such that \( x_{ij} \geq 0 \) for all \( (i, j) \in S \) is denoted by \( \Omega(S) \). It is clear that \( \Omega(S) \) is a convex polyhedron in Euclidean \( n^2 \)-space whose dimension equals \( (n-1)^2 \). In particular, \( \Omega(\emptyset) \) is the hyperplane of all \( n \times n \) doubly stochastic matrices, and \( \Omega(N_n \times N_n) \) is the polytope \( \Omega_n \) of all \( n \times n \) non-negative doubly stochastic matrices. In [3], [4], and [5] geometrical properties of \( \Omega_n \) were presented. We continue our study by investigating

*Research supported by National Science Foundation Grant No. GP-37978X.
†Research conducted while on sabbatical leave at the University of Wisconsin.
basic geometrical properties of $\Omega(S)$ for arbitrary $S$. We characterize the faces of $\Omega(S)$ and determine their dimensions and facets. The extreme points of $\Omega(S)$ are characterized. It is shown that if $\Omega(S)$ is a polytope, then a $d$-dimensional face of $\Omega(S)$ has at most $2^d$ extreme points. We conclude by pointing out a connection between the bounded faces of $\Omega(S)$ and the faces of transportation polytopes, and show that there is an absolute bound for the number of extreme points of $d$-dimensional bounded faces of these polyhedra.

2. FACES OF $\Omega(S)$

Let $S \subseteq \mathbb{N}_n \times \mathbb{N}_n$. The polyhedron $\Omega(S)$ consists of all $n \times n$ real matrices $X = [x_{ij}]$ which satisfy the following constraints:

\begin{align*}
    x_{ij} & \geq 0 \quad (i,j) \in S, \\
    \sum_{k=1}^{n} x_{ik} & = 1 = \sum_{k=1}^{n} x_{kj} \quad (i \in \mathbb{N}_n). \tag{2.2}
\end{align*}

Thus $\Omega(S)$ is a polyhedron in the $(n - 1)^2$-dimensional linear manifold defined by the equations (2.2).

**Theorem 2.1.** Let $n > 2$, and let $S_1, S_2 \subseteq \mathbb{N}_n \times \mathbb{N}_n$ with $S_1 \neq S_2$. Then $\Omega(S_1) \neq \Omega(S_2)$.

**Proof.** There is no loss in generality in assuming $S_1 - S_2 \neq \emptyset$. Let $(r,s) \in S_1 - S_2$. Since $n > 2$, there exists an $n \times n$ doubly stochastic matrix $X = [x_{ij}]$ such that $x_{rs} = 0$ and $x_{ij} > 0$ whenever $i \neq r$ or $j \neq s$. Let $u, v \in \mathbb{N}_n$ with $u \neq r$ and $v \neq s$. For $\varepsilon > 0$ let $X(\varepsilon)$ be the doubly stochastic matrix obtained from $X$ by adding $-\varepsilon$ to $x_{ru}$ and $x_{uv}$, and $\varepsilon$ to $x_{ru}$ and $x_{uv}$. Then for $\varepsilon$ sufficiently small, $X(\varepsilon) \in \Omega(S_2) - \Omega(S_1)$. Hence $\Omega(S_1) \neq \Omega(S_2)$. \ \ 

Let $S \subseteq \mathbb{N}_n \times \mathbb{N}_n$. We obtain the faces of $\Omega(S)$ by replacing some of the inequalities of (2.1) with equalities. Let $K \subseteq S$ and define an $n \times n$ $(0,1,*)$-matrix $A = [a_{ij}]$ by

\[
a_{ij} = \begin{cases} 
1 & \text{if } (i,j) \in K, \\
0 & \text{if } (i,j) \in S - K, \\
* & \text{if } (i,j) \notin S.
\end{cases}
\]
We denote by $\mathcal{F}(A)$ the face of $\Omega(S)$ obtained when (2.1) is replaced with

\begin{align*}
x_{ij} & > 0 \quad ((i,j) \in K), \\
x_{ij} & = 0 \quad ((i,j) \in S - K).
\end{align*}

Note that if $P$ and $Q$ are $n \times n$ permutation matrices and $A^T$ is the transpose of $A$, then $\mathcal{F}(A)$, $\mathcal{F}(PAQ)$, and $\mathcal{F}(A^T)$ are all congruent. We define $\sigma_l(A)$ to be the number of 1's of $A$, $\sigma_*(A)$ the number of *'s of $A$, and $\sigma(A) = \sigma_l(A) + \sigma_*(A)$.

It is clear that we may have $\mathcal{F}(A) = \mathcal{F}(B)$ for two distinct $(0,1,\star)$-matrices $A$ and $B$. To establish a one-to-one correspondence between the non-empty faces of $\Omega(S)$ and certain $(0,1,\star)$-matrices we make the following definition. Let $A = [a_{ij}]$ be an $n \times n$ $(0,1,\star)$-matrix. Then $A$ has total 1-support provided $\mathcal{F}(A) \neq \emptyset$ and $a_{ij} = 1$ implies there exists a matrix $X = [x_{ij}] \in \mathcal{F}(A)$ such that $x_{rs} > 0$. Let $M(S)$ be the set of all $n \times n$ $(0,1,\star)$-matrices $A = [a_{ij}]$ with $a_{ij} = \star$ if and only if $(i,j) \in S$. Then it follows that there is a one-to-one correspondence between the non-empty faces of $\Omega(S)$ and the matrices in $M(S)$ with total 1-support.

Using the theory of network flows [7], we now characterize matrices with total 1-support.

**Theorem 2.2.** Let $A = [a_{ij}]$ be an $n \times n$ $(0,1,\star)$-matrix. Then $A$ has total 1-support if and only if the following condition is satisfied:

(†) Whenever $e$ and $f$ are non-negative integers and $P$ and $Q$ are permutation matrices such that

\begin{align*}
PAQ = \begin{bmatrix}
A_1 & 0 \\
Z & A_2
\end{bmatrix},
\end{align*}

(2.3)

where $Z$ is an $e \times f$ $(0,1)$-matrix, then $e + f \geq n$, with equality if and only if $Z = 0$.

**Proof.** If $A$ has total 1-support it easily follows that (†) holds. Now suppose that (†) holds. Consider the network $N$ which has vertices $u_1, \ldots, u_n, v_1, \ldots, v_n, s, t$, where there is an arc $(u_i, v_j)$ from $u_i$ to $v_j$ if and only if $a_{ij} \neq 0$, and arcs $(s, u_i)$ and $(v_i, t)$ for $i = 1, \ldots, n$. We establish lower and upper
bounds on arc flows as follows:

\[ l(u_i, v_j) = \epsilon, \quad c(u_i, v_j) = +\infty \quad \text{if} \quad a_{ij} = 1, \]

\[ l(u_i, v_j) = -\infty, \quad c(u_i, v_j) = +\infty \quad \text{if} \quad a_{ij} = *, \]

(2.4)

\[ l(s, u_i) = l(v_i, t) = 1 = c(s, u_i) = c(v_i, t) \quad \text{for} \quad i = 1, \ldots, n. \]

The number \( \epsilon \) above is an arbitrary positive real number. By applying [7, p. 51] we see that there exists a flow in \( N \) from the source \( s \) to the sink \( t \) satisfying the lower and upper bounds on arc flows if and only if

\[ f > (n - \epsilon) + \sigma(Z)\epsilon \]

(2.5)

whenever \( e \) and \( f \) are non-negative integers and \( P \) and \( Q \) are permutation matrices such that (2.3) holds, where \( Z \) is an \( e \times f (0,1) \)-matrix. It follows from (f) that \( \epsilon \) can be chosen so that (2.5) is always satisfied. Hence there exists a flow \( f \) satisfying the constraints (2.4). Let \( X = [x_{ij}] \) be the \( n \times n \) matrix where \( x_{ij} = f(u_i, v_j) \) for \( i, j = 1, \ldots, n \). Then \( X \in \mathcal{F}(A) \) and \( a_{ij} = 1 \) implies \( x_{ij} > \epsilon > 0 \). Hence \( A \) has total 1-support.

The preceding proof is similar to the proof of Theorem 2.1 in [2]. If in the proof of Theorem 2.2 we replace \( \epsilon \) by 0, we obtain the following. Let \( A \) be an \( n \times n (0, 1, *) \)-matrix. Then \( \mathcal{F}(A) \neq \emptyset \) if and only if \( e + f > n \) whenever \( e \) and \( f \) are non-negative integers and \( P \) and \( Q \) are permutation matrices such that (2.3) holds where \( Z \) is an \( e \times f (0,1) \)-matrix.

Let \( A = [a_{ij}] \) be an \( n \times n (0, 1, *) \)-matrix. Let the rows of \( A \) be \( u_1, \ldots, u_n \) and the columns of \( A \) be \( v_1, \ldots, v_n \). We define the graph [respectively \( 1 \)-graph, \( * \)-graph] of \( A \) to be the bipartite graph \( G(A) \) [respectively \( G_1(A), G_*(A) \)] whose vertices are \( u_1, \ldots, u_n \) and \( v_1, \ldots, v_n \), where there is an edge \{ \( u_i, v_j \) \} joining \( u_i \) and \( v_j \) if and only if \( a_{ij} \neq 0 \) [respectively, \( a_{ij} = 1, a_{ij} = * \)]. The matrix \( A \) is said to have total support provided \( A \neq 0 \) and \( a_{rs} \neq 0 \) implies there exists a matrix \( X = [x_{ij}] \in \mathcal{F}(A) \) such that \( x_{rs} \neq 0 \).

**Theorem 2.3.** Let \( A = [a_{ij}] \) be an \( n \times n (0, 1, *) \)-matrix. Then \( A \) has total support if and only if \( A \) has total 1-support and the following condition is satisfied:
(‡) If \( P \) and \( Q \) are permutation matrices such that (2.3) holds, where \( A_1 \) is a square matrix and \( Z \) is not a \((0,1)\)-matrix, then \( Z \) has at least two nonzero entries.

**Proof.** If \( A \) has total support, then clearly \( A \) has total 1-support and (‡) is satisfied. Now suppose that \( A \) has total 1-support and (‡) is satisfied. Since \( A \) has total 1-support, it follows that there exists a matrix \( X = [x_{ij}] \in \bar{\mathcal{F}}(A) \) such that \( x_{ij} > 0 \) whenever \( a_{ij} = 1 \). Suppose that \( a_{rs} = *, \) while \( x_{rs} = 0 \). First suppose that the edge \( \{u_r, v_s\} \) is an isthmus of \( G(A) \). Then there exist permutation matrices \( P \) and \( Q \) such that (2.3) holds, where \( a_{rs} \) is the only nonzero entry of \( Z \). Since \( x_{rs} = 0 \), we see that the matrix \( A_1 \) of (2.3) is square. This contradicts (‡), and hence \( \{u_r, v_s\} \) is not an isthmus of \( G(A) \). Therefore there is a cycle \( \gamma \) of \( G(A) \) containing the edge \( \{u_r, v_s\} \). Let \( \varepsilon > 0 \). By alternately adding \( \varepsilon \) and \( -\varepsilon \) to the entries of \( X \) corresponding to the edges of the cycle \( \gamma \), we obtain a doubly stochastic matrix \( X(\varepsilon) \) whose \((r,s)\)-entry is not zero. For \( \varepsilon \) sufficiently small, \( X(\tau) \in \bar{\mathcal{F}}(A) \). Therefore \( A \) has total support.

An \( n \times n \) \((0,1,*)\)-matrix is called **connected** if its graph \( G(A) \) is connected. It is easy to see that if \( A \) is a connected \((0,1,*)\)-matrix with total support and \( B \) is obtained from \( A \) by replacing a zero entry with a 1 or \(*\), then \( B \) is connected and has total support.

**Theorem 2.4.** Let \( A = [a_{ij}] \) be an \( n \times n \) \((0,1,*)\)-matrix such that \( \bar{\mathcal{F}}(A) \neq \emptyset \). Then there exist permutation matrices \( P \) and \( Q \) such that \( PAQ \) has the form

\[
\begin{bmatrix}
A_1 & 0 & \cdots & 0 & 0 \\
X_{21} & A_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_{k-1,1} & X_{k-1,2} & \cdots & A_{k-1} & 0 \\
X_{k,1} & X_{k,2} & \cdots & X_{k,k-1} & A_k
\end{bmatrix}
\]

(2.6)

where \( A_1, \ldots, A_k \) are connected matrices with total support and \( \bar{\mathcal{F}}(PAQ) = \bar{\mathcal{F}}(A_1 \oplus \cdots \oplus A_k) \).

**Proof.** Let \( B = [b_{ij}] \) be the \((0,1,*)\)-matrix obtained from \( A \) by replacing with 0 each \( a_{rs} \) for which \( Y = [y_{ij}] \subset \bar{\mathcal{F}}(A) \) implies \( y_{rs} = 0 \). Clearly \( \bar{\mathcal{F}}(A) = \bar{\mathcal{F}}(B) \), and \( B \) has total support. There exist permutation matrices \( R \) and \( S \) such that \( RBS = B_1 \oplus \cdots \oplus B_k \), where each \( B_i \) is a connected matrix with
total support. It follows that

$$\begin{bmatrix}
B_1 & Z_{12} & \cdots & Z_{1,k-1} & Z_{1k} \\
Z_{21} & B_2 & \cdots & Z_{2,k-1} & Z_{2k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Z_{k-1,1} & Z_{k-1,2} & \cdots & B_{k-1} & Z_{k-1,k} \\
Z_{k1} & Z_{k2} & \cdots & Z_{k,k-1} & B_k
\end{bmatrix}$$

Let $\Gamma$ be the directed graph with vertices $B_1, \ldots, B_k$ such that there is an arc from $B_i$ to $B_j$ if and only if $Z_{ij} \neq 0$. Suppose $\Gamma$ has a directed cycle. Using the connectivity of the bipartite graphs $G(B_1), \ldots, G(B_k)$, we see that there is a cycle $\gamma$ in $G(A)$, where $\gamma$ is not a cycle of $G(B)$, such that the edges of $\gamma$ which are in $G(A)$ but not in $G(B)$ have the same parity in a consecutive numbering of the edges of $\gamma$. There exists a matrix $Y = [y_{ij}] \in \mathbb{F}(B)$ such that $y_{ij} > 0$ whenever $b_{ij} = 1$. Let $\epsilon > 0$ be sufficiently small. By alternately adding $\epsilon$ and $-\epsilon$ to the entries of $Y$ corresponding to the edges of $\gamma$, we obtain a matrix $Y(\epsilon) \in \mathbb{F}(A) - \mathbb{F}(B)$. This contradicts $\mathbb{F}(A) = \mathbb{F}(B)$. Therefore $\Gamma$ has no directed cycles. Hence by [9, pp. 268-269] it is possible to order $B_1, \ldots, B_k$ so that the adjacency matrix of $\Gamma$ is triangular. Hence there exist permutation matrices $P$ and $Q$ such that (2.6) holds, where $A_i = B_{\tau(i)}$ for some permutation of $\tau$ of $\{1, \ldots, k\}$.

Let $A$ be an $n \times n$ $(0, 1, *)$-matrix with $\mathbb{F}(A) \neq \emptyset$. The matrices $A_1, \ldots, A_k$ in Theorem 2.4 are called the principal components of $A$. The principal components are unique except for permutations of their rows and columns. Clearly the following holds.

**Corollary 2.5.** If $A_1, \ldots, A_k$ are the principal components of an $n \times n$ $(0, 1, *)$-matrix $A$ with $\mathbb{F}(A) \neq \emptyset$, then $\mathbb{F}(A)$ is congruent to the orthogonal vector sum of $\mathbb{F}(A_1), \ldots, \mathbb{F}(A_k)$.

**Theorem 2.6.** Let $A = [a_{ij}]$ be an $n \times n$ $(0, 1, *)$-matrix with total 1-support. Then there exist permutation matrices $P$ and $Q$ such that $PAQ$ has the form (2.6), where $A_1, \ldots, A_k$ are connected matrices with total support, $\mathbb{F}(PAQ) = \mathbb{F}(A_1 \oplus \cdots \oplus A_k)$, the $X_{ij}$ are $(0, *)$-matrices, and the multigraph $H$ with vertices $A_1, \ldots, A_k$ and $\sigma(X_{ij})$ edges joining $A_i$ and $A_j$ has no cycles.

**Proof.** By Theorem 2.4 there exist permutation matrices $P$ and $Q$ such that (2.6) holds where $A_1, \ldots, A_k$ are connected matrices with total support and $\mathbb{F}(PAQ) = \mathbb{F}(B)$, where $B = [b_{ij}] = A_1 \oplus \cdots \oplus A_k$. Since $A$ has total
1-support, it follows that the $X_{ij}$ are $(0, \ast)$-matrices. Suppose $H$ has a cycle. Using the connectivity of the bipartite graphs $G(A_1), \ldots, G(A_k)$ we see that there is a cycle $\gamma$ in $G(PAQ)$ that is not a cycle of $G(B)$. There exists a matrix $Y = [y_{ij}] \in \mathcal{F}(B)$ such that $y_{ij} > 0$ whenever $b_{ij} = 1$. Let $\varepsilon > 0$ be sufficiently small. Since each edge of $\gamma$ which is not an edge of $G(B)$ is an edge of $G^*(PAQ)$, by alternately adding $\varepsilon$ and $-\varepsilon$ to the entries of $Y$ corresponding to the edges of $\gamma$ we obtain a matrix $Y(\varepsilon) \in \mathcal{F}(PAQ) - \mathcal{F}(B)$. This contradicts $\mathcal{F}(PAQ) = \mathcal{F}(B)$. Therefore $H$ has no cycles.

Let $A$ be an $n \times n$ $(0,1,\ast)$-matrix with total 1-support. It follows from Theorem 2.5 that a cycle of $G(A)$ is a cycle of $G(A_i)$ for some principal component $A_i$ of $A$. Observe that the number of principal components of $A$ is at least as great as the number of connected components of $G(A)$.

We now determine the dimension of faces of $\Omega(S)$. By Corollary 2.5 it suffices to consider $\mathcal{F}(A)$ such that $A$ is a connected $(0,1,\ast)$-matrix with total support.

**Theorem 2.7.** Let $A = [a_{ij}]$ be a connected $n \times n$ $(0,1,\ast)$-matrix with total support. Then

$$\dim \mathcal{F}(A) = \sigma(A) - 2n + 1.$$  

**Proof.** It is clear there exists a matrix $X = [x_{ij}] \in \mathcal{F}(A)$ such that $x_{ij} \neq 0$ if and only if $a_{ij} \neq 0$. Let $H$ be a spanning tree of $G(A)$, and let $d = \sigma(A) - 2n + 1$. Then there are exactly $d$ edges, $e_1, e_2, \ldots, e_d$, which are edges of $G(A)$ but not edges of $H$. For each $i = 1, \ldots, d$ there exists a unique elementary cycle $\gamma_i$ such that $e_i$ is an edge of $\gamma_i$ while all other edges of $\gamma_i$ are edges of $H$. Let $C_i$ be an $n \times n$ $(0,1,1)$-matrix with row and column sums equal to 0 having non-zero entries precisely in those positions corresponding to the edges of the cycle $\gamma_i$. Let $m$ equal the minimum absolute value of the non-zero entries of $X$. Let

$$\mathcal{S} = \{X + t_1 C_1 + \cdots + t_d C_d : 0 \leq |t_i| \leq m/d \ (1 \leq i \leq d)\}.$$ 

It follows that $\mathcal{S} \subseteq \mathcal{F}(A)$. Moreover, the projection of $\mathcal{S}$ onto the coordinate subspace determined by the edges $e_1, \ldots, e_d$ is a $d$-dimensional cube. Therefore

$$\dim \mathcal{F}(A) \geq d = \sigma(A) - 2n + 1.$$ 

Let $k = n^2 - \sigma(A)$. Let $A_0 = A$. And for $i = 1, \ldots, k$ let $A_i$ be obtained from $A_{i-1}$ by replacing a 0 of $A_{i-1}$ with a 1. Since $A$ is connected with total
support, $\mathcal{F}(A_{i-1})$ is a proper subface of $\mathcal{F}(A_i)$ for $i = 1, \ldots, k$. Therefore
\[
\dim \mathcal{F}(A) = \dim \mathcal{F}(A_0) < \dim \mathcal{F}(A_1) < \cdots < \dim \mathcal{F}(A_k) = (n - 1)^2,
\]
and hence
\[
\dim \mathcal{F}(A) \leq (n - 1)^2 - k = \sigma(A) - 2n + 1.
\]
The theorem now follows.

It follows from Corollary 2.5 and Theorem 2.7 that if $A_1, \ldots, A_k$ are the principal components of $A$, then
\[
\dim \mathcal{F}(A) = \left( \sum_{i=1}^{k} \sigma(A_i) \right) - 2n + k.
\]
Moreover, from Theorems 2.6 and 2.7 we obtain the following.

**Theorem 2.8.** Let $A$ be an $n \times n$ $(0,1,*)$-matrix with total 1-support. If $G(A)$ has $m$ connected components, then
\[
\dim \mathcal{F}(A) = \sigma(A) - 2n + m.
\]

Given a polyhedron $\mathcal{P}$ of dimension $k$, we define a *facet* of $\mathcal{P}$ to be a face of $\mathcal{P}$ with dimension $k - 1$. Let $A$ be an $n \times n$ $(0,1,*)$-matrix with total 1-support. The non-empty faces of $\mathcal{F}(A)$ are the polyhedra $\mathcal{F}(B)$, where $B$ is obtained from $A$ by replacing certain 1's with 0's. Since $A$ has total 1-support, if $B$ is obtained from $A$ by replacing a 1 with a 0, then $\dim \mathcal{F}(B) < \dim \mathcal{F}(A)$. Hence, if $\dim \mathcal{F}(A) > 1$, all of the facets of $\mathcal{F}(A)$ can be found among the polyhedra $\mathcal{F}(B)$ where $B$ is obtained from $A$ by replacing a 1 by a 0.

**Theorem 2.9.** Let $A = [a_{ij}]$ be an $n \times n$ $(0,1,*)$-matrix with total 1-support such that $\dim \mathcal{F}(A) \geq 1$. Then $\mathcal{F}(A)$ has a facet if and only if there is a cycle of $G(A)$ which is not a cycle of $G_*(A)$.

**Proof.** Suppose $\mathcal{F}(A)$ has a facet $\mathcal{F}(B)$ where $B$ is obtained from $A$ by replacing a 1 with a 0. Since $\dim \mathcal{F}(A) \geq 1$ and $A$ has total 1-support, it follows that the edge of $G(A)$ corresponding to this 1 is an edge of a cycle $\gamma$ of $G(A)$ which is not a cycle of $G_*(A)$. Now suppose there is a cycle $\gamma$ of
$G(A)$ which is not a cycle of $G_t(A)$. Since $A$ has total $1$-support, there exists a matrix $X = [x_{ij}] \in \mathcal{F}(A)$ such that $x_{ij} \neq 0$ whenever $a_{ij} = 1$. Let $m = x_{rs}$ be the smallest of those entries of $X$ which correspond to edges of $\gamma$ which are edges of $G_t(A)$. By alternately adding $m$ and $-m$ to the entries of $X$ corresponding to the edges of $\gamma$, we can obtain a matrix $Y = [y_{ij}] \in \mathcal{F}(A)$ such that $y_{rs} = 0$. Let $C$ be the $(0,1,\ast)$-matrix obtained from $A$ by replacing $a_{rs} = 1$ with $0$. Since $Y \in \mathcal{F}(C)$, $\mathcal{F}(C)$ is a non-empty face of $\mathcal{F}(A)$. Therefore $\mathcal{F}(A)$ has a facet.

In view of Corollary 2.5, we now restrict our consideration to connected matrices $A$ with total support.

**Theorem 2.10.** Let $A$ be an $n \times n$ connected $(0,1,\ast)$-matrix with total support such that $\dim \mathcal{F}(A) > 0$. Let $B$ be a matrix obtained from $A$ by replacing a $1$ with a $0$. Then $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$ if and only if one of the following holds.

(i) $B$ is connected with total support.

(ii) There exist permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} B_1 & C_2 \\ C_1 & B_2 \end{bmatrix}, \quad PBQ = \begin{bmatrix} B_1 & C_2' \\ C_1 & B_2' \end{bmatrix},$$

where $B_1$ and $B_2$ are the principal components of $B$ and $\sigma(C_1) + \sigma(C_2) = 2$.

(iii) There exist an integer $k \geq 3$ and permutation matrices $P$ and $Q$ such that

$$PAQ = \begin{bmatrix} B_1 & F_1 & \cdots & 0 & E_k \\ E_1 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{k-1} & F_{k-1} \\ 0 & 0 & \cdots & E_{k-1} & B_k \end{bmatrix},$$

and

$$PBQ = \begin{bmatrix} B_1 & F_1 & \cdots & 0 & 0 \\ E_1 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{k-1} & F_{k-1} \\ 0 & 0 & \cdots & E_{k-1} & B_k \end{bmatrix}.$$
where $B_1, \ldots, B_k$ are the principal components of $B$, and $\sigma(E_i) + \sigma(F_i) = 1$ for $i = 1, \ldots, k - 1$.

**Proof.** If (i) holds, it follows from Theorem 2.7 that $\dim \mathcal{F}(B) = \dim \mathcal{F}(A) - 1$. Now suppose (iii) holds. Let the order of $B_i$ be $n_i$ for $i = 1, \ldots, k$. From Corollary 2.5 and Theorem 2.7 we see that

$$\dim \mathcal{F}(B) = \sum_{i=1}^{k} \dim \mathcal{F}(B_i)$$

$$= \sum_{i=1}^{k} \left[ \sigma(B_i) - 2n_i + 1 \right]$$

$$= \sigma(A) - 2n$$

$$= \dim \mathcal{F}(A) - 1.$$

If (ii) holds, it similarly follows from Corollary 2.5 and Theorem 2.7 that $\dim \mathcal{F}(B) = \dim \mathcal{F}(A) - 1$. Therefore if (i), (ii), or (iii) holds, then $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$.

Now suppose $\mathcal{F}(B)$ is a facet of $\mathcal{F}(A)$. Since $\dim \mathcal{F}(A) > 0$, $\mathcal{F}(B) \neq \emptyset$. Hence since $A$ is connected with total support, $B$ is connected. Suppose (i) does not hold. It follows from Theorem 2.4 that there exist permutation matrices $U$ and $V$ such that $UBV$ has the form

$$\begin{bmatrix}
    M_1 & 0 & \cdots & 0 & 0 \\
    X_{21} & M_2 & \cdots & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    X_{k-1,1} & X_{k-1,2} & \cdots & M_{k-1} & 0 \\
    X_{k,1} & X_{k,2} & \cdots & X_{k,k-1} & M_k
\end{bmatrix}, \quad (2.7)$$

where $k \geq 2$ and $M_1, \ldots, M_k$ are the principal components of $B$. Since $\dim \mathcal{F}(B) = \dim \mathcal{F}(A) - 1$, it follows from Corollary 2.5 and Theorem 2.7 that

$$\sum_{k > i > j > 1} \sigma(X_{ij}) = k - 1.$$

Let $H$ be the multigraph with vertices $M_1, \ldots, M_k$ and $\sigma(X_{ij})$ edges joining $M_i$ and $M_j$. Since $B$ is connected, it follows that $H$ is a connected graph with $k$
vertices and $k - 1$ edges. Therefore $H$ is a tree. Suppose $UAV$ has the form (2.7) with $M_1, \ldots, M_k$ replaced by $M'_1, \ldots, M'_k$. Since $H$ is a tree, it follows that $A$ does not have total support. Therefore $UAV$ has the form

$$
\begin{bmatrix}
M_1 & Y_{12} & \cdots & Y_{1,k-1} & Y_{1,k} \\
Y_{21} & M_2 & \cdots & Y_{2,k-1} & Y_{2,k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
Y_{k-1,1} & Y_{k-1,2} & \cdots & M_{k-1} & Y_{k-1,k} \\
Y_{k,1} & Y_{k,2} & \cdots & Y_{k,k-1} & M_k
\end{bmatrix}
$$

Let $L$ be the multigraph with vertices $M_1, \ldots, M_k$ and $\sigma(Y_{ij}) + \sigma(Y_{ji})$ edges joining $M_i$ and $M_j$. Then $L$ is a connected graph with $k$ vertices and $k$ edges. Moreover, since $A$ has total support, $L$ has no pendant edges. Therefore $L$ is a cycle of length $k$. It now follows that (ii) or (iii) holds according to whether $k = 2$ or $k > 3$.

It is clear that a $(0,1,\ast)$-matrix $A$ satisfying (iii) of Theorem 2.10 with $\sigma(E_k) = \sigma_1(E_k) = 1$ is connected. We now determine when such a matrix $A$ has total support.

**Theorem 2.11.** Let $A$ be an $n \times n$ $(0,1,\ast)$-matrix such that

$$
A =
\begin{bmatrix}
B_1 & F_1 & \cdots & 0 & E_k \\
E_1 & B_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B_{k-1} & F_{k-1} \\
0 & 0 & \cdots & E_{k-1} & B_k
\end{bmatrix}
$$

where $k \geq 3$, $B_1, \ldots, B_k$ are connected matrices with total support, $\sigma(E_i) + \sigma(F_i) = 1$ for $i = 1, \ldots, k-1$, and $\sigma(E_k) = \sigma_1(E_k) = 1$. Then $A$ has total support if and only if $\sigma_i(F_i) = \sigma(F_i)$ for $i = 1, \ldots, k-1$.

**Proof.** Suppose for some $i, \sigma_1(F_i) = 1$. Then $E_i = 0$, and it follows that

$$
A =
\begin{bmatrix}
A_1 & Z \\
0 & A_2
\end{bmatrix}
$$
where \( A_1 \) and \( A_2 \) are square matrices and \( Z \) is a non-zero \((0,1)\)-matrix. It is now a consequence of Theorem 2.2 that \( A \) does not have total support. Therefore if \( A \) has total support, then \( \sigma_i(F_i) = \sigma(F_i) \) for \( i = 1, \ldots, k - 1 \).

Now suppose that \( \sigma_i(F_i) = \sigma(F_i) \) for \( i = 1, \ldots, k - 1 \). Since \( B_1, \ldots, B_k \) are connected, we see that there is a cycle \( \gamma \) in \( G(A) \) which contains all the edges of \( G(A) \) corresponding to the non-zero entries of \( E_1, \ldots, E_k, F_1, \ldots, F_{k-1} \). There exists a consecutive numbering of the edges of \( \gamma \) such that the edges corresponding to the non-zero entries of \( E_1, \ldots, E_k \) have even parity while the edges corresponding to the non-zero entries of \( F_1, \ldots, F_{k-1} \) have odd parity. Let \( B = [b_{ij}] = B_1 \oplus \cdots \oplus B_k \). Since \( B \) has total support, there exists \( X = [x_{ij}] \in \mathcal{F}(B) \) such that \( x_{ij} \neq 0 \) whenever \( b_{ij} \neq 0 \). Let \( \epsilon > 0 \) be sufficiently small. By adding \( \epsilon \) to the entries of \( X \) corresponding to the even numbered edges of \( \gamma \) and \(-\epsilon\) to the entries of \( X \) corresponding to the odd numbered edges of \( \gamma \), we obtain a matrix \( Y = [y_{ij}] \in \mathcal{F}(A) \) such that \( y_{ij} \neq 0 \) whenever \( a_{ij} \neq 0 \). Hence \( A \) has total support.

It is not difficult to modify the proof of Theorem 2.1 to show that if \( A \) is a \((0,1,*\))-matrix satisfying (ii) of Theorem 2.10 with \( \sigma_1(C_2) > 1 \), then \( A \) has total support if and only if \( \sigma_1(C_2) = 1 \).

3. EXTREME POINTS

Let \( A \) be an \( n \times n \) \((0,1,*\))-matrix with rows \( u_1, \ldots, u_n \) and columns \( v_1, \ldots, v_n \). Let \( X = [x_{ij}] \in \mathcal{F}(A) \). Then the graph of \( X \) is the bipartite graph \( G(X) \) whose vertices are \( u_1, \ldots, u_n \) and \( v_1, \ldots, v_n \), where there is an edge \( \{u_i, v_i\} \) joining \( u_i \) and \( v_i \) if and only if \( x_{ij} \neq 0 \). Let \( G \) and \( H \) be graphs with the same vertex set \( V \) and edge sets \( E \) and \( F \), respectively. Then their union \( G \cup H \) is the graph with vertex set \( V \) and edge set \( E \cup F \).

**Theorem 3.1.** Let \( A = [a_{ij}] \) be an \( n \times n \) \((0,1,*\))-matrix with total \( 1 \)-support, and let \( X = [x_{ij}] \in \mathcal{F}(A) \). Then \( X \) is an extreme point of \( \mathcal{F}(A) \) if and only if the graph \( G(X) \cup G_*(A) \) has no cycles.

**Proof.** Let \( B \) be the \((0,1,*\))-matrix obtained from \( A \) by replacing \( a_{ij} \) with \( 0 \) whenever \( a_{ij} = 1 \) and \( x_{ij} = 0 \). Then \( \mathcal{F}(B) \) is the smallest face of \( \mathcal{F}(A) \) which contains \( X \), and \( G(B) = G(X) \cup G_*(A) \). We prove the theorem by showing \( \mathcal{F}(B) \neq \{X\} \) if and only if \( G(B) \) has a cycle. Suppose there exists a cycle \( \gamma \) in \( G(B) \). Let \( \epsilon > 0 \) be sufficiently small. By alternately adding \( \epsilon \) and \(-\epsilon\) to the entries of \( X \) corresponding to the edges of \( \gamma \), we obtain a matrix \( Y \in \mathcal{F}(B) \) with \( Y \neq X \).
Now suppose there exists \( Y \in \mathcal{F}(B) \) with \( Y \neq X \). Then \( X - Y \) is a non-zero matrix with all row and column sums equal to 0. It is easy to see that this implies there is a cycle in \( G(B) \).

**Corollary 3.2.** Let \( A \) be an \( n \times n \) \((0,1,*)\)-matrix with total 1-support, and let \( X \) be an extreme point of \( \mathcal{F}(A) \). Then \( X \) is an integral matrix.

**Proof.** It follows from Theorem 3.1 that \( G(X) \) has no cycles. Thus the row and column sum constraints determine a triangular system of linear equations with all non-zero scalars equal to 1. It follows that the unique solution of this system is integral.

**Theorem 3.3.** Let \( A \) be an \( n \times n \) \((0,1,*)\)-matrix with total 1-support. Then \( \mathcal{F}(A) \) has an extreme point if and only if \( G(A) \) has no cycles.

**Proof.** First suppose \( G \) has a cycle. It follows from Theorem 3.1 that \( \mathcal{F}(A) \) has no extreme points. Now suppose \( G(A) \) has no cycles. We prove by induction on \( \dim \mathcal{F}(A) \) that \( \mathcal{F}(A) \) has an extreme point. This trivially holds if \( \dim \mathcal{F}(A) = 0 \). Let \( \dim \mathcal{F}(A) \geq 1 \). It follows from Theorem 2.10 that \( G(A) \) has a cycle \( \gamma \), which by our assumption is not a cycle of \( G(A) \). Therefore by Theorem 2.9 \( \mathcal{F}(A) \) has a facet \( \mathcal{F}(B) \). Since \( \mathcal{F}(B) = \mathcal{F}(A) \), the inductive assumption applied to \( B \) implies that \( \mathcal{F}(A) \) has an extreme point.

Let \( A \) be an \( n \times n \) \((0,1,*)\)-matrix with total 1-support. It follows from Caratheodory's theorem [8] that \( \mathcal{F}(A) \) is bounded [that is, \( \mathcal{F}(A) \) is a polytope] if and only if \( \mathcal{F}(A) \) is the convex hull of its extreme points. We now obtain a characterization of those matrices \( A \) for which \( \mathcal{F}(A) \) is bounded.

**Theorem 3.4.** Let \( A \) be an \( n \times n \) \((0,1,*)\)-matrix with total 1-support. Then \( \mathcal{F}(A) \) is bounded if and only if \( G(A) \) has no elementary cycle \( u_{ij}, v_{ij}, u_{ij}, v_{ij}, \ldots, u_{ik}, v_{ik}, u_{ik} \) such that \( \{u_i, v_i\} \) is an edge of \( G(A) \) for \( t = 1, \ldots, k \).

**Proof.** We say that a coordinate pair \( (r,s) \) is positively (respectively, negatively) unbounded in \( \mathcal{F}(A) \), provided for all \( m > 0 \) there exists an \( X = [x_{ij}] \in \mathcal{F}(A) \) such that \( x_{rs} > m \) (respectively, \( x_{rs} < -m \)). Since \( \mathcal{F}(A) \) consists of doubly stochastic matrices, for each \( r = 1, \ldots, n \) there exists \( s_1 \) such that the coordinate pair \( (r,s_1) \) is negatively unbounded in \( \mathcal{F}(A) \) if and only if there exists \( s_2 \) such that the coordinate pair \( (r,s_2) \) is positively unbounded in \( \mathcal{F}(A) \) where \( s_1 \neq s_2 \). Similarly, for each \( s = 1, \ldots, n \), there exists \( r_1 \) such that
the coordinate pair \((r_1, s)\) is negatively unbounded in \(\mathcal{F}(A)\) if and only if there exists \(r_2\) such that the coordinate pair \((r_2, s)\) is positively unbounded in \(\mathcal{F}(A)\) where \(r_1 \neq r_2\). Now suppose \(\mathcal{F}(A)\) is unbounded. Then there exists a coordinate pair which is either positively or negatively unbounded in \(\mathcal{F}(A)\). It follows that there exist an integer \(k > 2\) and distinct \(i_1, \ldots, i_k\) and distinct \(j_1, \ldots, j_k\) such that the coordinate pairs \((i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\) are negatively unbounded in \(\mathcal{F}(A)\) and the coordinate pairs \((i_2, j_1), (i_3, j_2), \ldots, (i_1, j_k)\) are positively unbounded in \(\mathcal{F}(A)\). Hence \(G(A)\) has an elementary cycle \(u_{i_1}, u_{i_2}, \ldots, u_{i_k}, u_{j_1}, \ldots, u_{j_k}\), where \(\{u_{i_m}, v_{i_m}\}\) is an edge of \(G, A\) for \(m = 1, \ldots, k\).

Conversely, suppose \(G(A)\) has such an elementary cycle. Let \(X = [x_{ij}] \in \mathcal{F}(A)\), and let \(m > 0\). Let \(X(m)\) be the matrix obtained from \(X\) by adding \(-m\) to \(x_{i_1, j_1}, \ldots, x_{i_k, j_k}\) and \(m\) to \(x_{i_2, j_2}, \ldots, x_{i_1, j_k}\). Then for each \(m > 0\), \(X(m) \in F(A)\), and hence \(\mathcal{F}(A)\) is unbounded.

**Theorem 3.5.** Let \(S \subseteq \mathbb{N} \times \mathbb{N}\), and let \(A = [a_{ij}]\) be the \(n \times n\) \((1, *)\)-matrix such that \(a_{ij} = 1\) if and only if \((i, j) \in S\). Then the following are equivalent.

(i) \(\Omega(S)\) is bounded.

(ii) \(A\) or \(A^T\) has an \((n - 1) \times n\) submatrix of all 1’s.

(iii) Every \(n \times n\) permutation matrix is an extreme point of \(\Omega(S)\).

**Proof.** It follows from Theorem 3.1 that (i) and (ii) are equivalent. Moreover it follows from Theorem 3.1 that (ii) and (iii) are equivalent.

In [3] it was shown that a \(k\)-dimensional face of the convex polytope \(\Omega_n\) of non-negative doubly stochastic matrices has at most \(2^k\) extreme points. More generally we have the following.

**Theorem 3.6.** Let \(S \subseteq \mathbb{N} \times \mathbb{N}\), and suppose that \(\Omega(S)\) is bounded. Let \(\mathcal{F}\) be a non-empty face of \(\Omega(S)\) of dimension \(k\). Then \(\mathcal{F}\) has at most \(2^k\) extreme points.

**Proof.** Let \(B\) be an \(n \times n\) \((0, 1, *)\)-matrix with total 1-support such that \(\mathcal{F}(B)\) is a non-empty face of \(\Omega(S)\) of dimension \(k\). We prove by induction on \(k\) that \(\mathcal{F}(B)\) has at most \(2^k\) extreme points. This is clearly true if \(k = 0\). Now suppose \(k > 1\). By Theorem 3.5 all of the entries of \(B\) which are *’s occur in the same row or the same column. There is no loss in generality in assuming that these entries all occur in row 1 of \(B\). Since \(k > 1\), it follows from Theorem 3.1 that there exists an integer \(s \geq 2\) such that row \(s\) of \(B\) contains at least two 1’s. Suppose that \(b_{sn} = 1\). Let \(B_1\) be the matrix obtained from \(B\) by replacing \(b_{st}\) by 0, and let \(B_2\) be the matrix obtained from \(B\) by replacing
bsi with 0 for all i ≠ t. Since row s of B contains no *'s, it follows from Corollary 3.2 that if X is an extreme point of \( \mathcal{F}(B) \), then X is an extreme point of \( \mathcal{F}(B_1) \) or \( \mathcal{F}(B_2) \). Hence, since B has total 1-support, \( \mathcal{F}(B_1) \) and \( \mathcal{F}(B_2) \) are non-empty faces of \( \Omega(S) \) with dimension at most \( k - 1 \). Therefore it follows from the inductive assumption that \( \mathcal{F}(B) \) has at most \( 2^k \) extreme points.

Denote the graph of a polytope \( \mathcal{P} \) by \( G(\mathcal{P}) \). The vertices of \( G(\mathcal{P}) \) are the extreme points of \( \mathcal{P} \), and two vertices are joined by an edge in \( G(\mathcal{P}) \) if and only if they are the extreme points of a 1-dimensional face of \( \mathcal{P} \). Balinski and Russakoff [1] conjectured and Imrich [10] proved that the connectivity of \( G(\Omega_n) \) equals the (minimum) degree of the vertices of \( G(\Omega_n) \). More generally we have the following.

**Theorem 3.7.** Let \( S \subseteq N_n \times N_n \), and suppose that \( \Omega(S) \) is bounded. Then the connectivity of \( G(\Omega(S)) \) equals the minimum degree of the vertices of \( G(\Omega(S)) \).

**Proof.** Let \( A = [a_{ij}] \) be the \( n \times n \) \((1,*)\)-matrix such that \( a_{ij} = 1 \) if and only if \((i, j) \in S\). If \( A \) contains no *'s or \( n = 2 \), then \( \Omega(S) = \Omega_n \) and the result follows from Imrich's result [10]. Now suppose \( n > 2 \) and \( A \) contains at least one *.

Since \( \Omega(S) \) is bounded, by Theorem 3.5 there is no loss in generality in assuming \( a_{11} = * \) and every * occurs in the first row of \( A \). Let \( X = [x_{ij}] \) be the unique matrix in \( \Omega(S) \) with \( x_{ij} = 0 \) for all \( i, j \geq 2 \). It follows from Theorem 3.1 that \( X \) is an extreme point of \( \Omega(S) \). Suppose that \( X \) is joined by an edge in \( G(\Omega(S)) \) to each of \( Y_1, \ldots, Y_m \). Let \( Y_t = [y_{ij}] \), where \( 1 < t < m \). It follows from Theorem 2.7 that there exist integers \( r_t, s_t > 2 \) such that if \( i, j \geq 2 \), then \( y_{ij} \neq 0 \) if and only if \((i, j) = (r_t, s_t)\). Since \( Y_1, \ldots, Y_m \) are distinct, we see that \((r_1, s_1), \ldots, (r_m, s_m)\) are distinct. Hence \( m \leq (n - 1)^2 \), and the connectivity of \( G(\Omega(S)) \) is at most \( (n - 1)^2 \). Hence, since \( \dim \Omega(S) = (n - 1)^2 \), it follows from a theorem of Balinski [8, p. 213] that the connectivity of \( G(\Omega(S)) \) is at least \((n - 1)^2 \). The theorem now follows.

4. BOUNDED FACES AND TRANSPORTATION POLYTOPES

Let \( R = (r_1, \ldots, r_m) \) and \( C = (c_1, \ldots, c_n) \) be vectors of positive real numbers such that \( r_1 + \cdots + r_m = c_1 + \cdots + c_n \). The transportation polytope \([6, 7]\) \( \Gamma(R, C) \) is the set of all \( m \times n \) non-negative matrices with row sum vector \( R \) and column sum vector \( C \). Let \( \mathcal{F} \) be a non-empty face of \( \Gamma(R, C) \). Then there exists an \( m \times n \) \((0,1)\)-matrix \( A = [a_{ij}] \) such that \( \mathcal{F} \) consists of all
matrices $X = [x_{ij}] \in \Gamma(R, C)$ such that $x_{ij} = 0$ whenever $a_{ij} = 0$. We write $\mathcal{F} = \mathcal{F}_{R,C}(A)$. An $m \times n$ $(0,1)$-matrix $A = [a_{ij}]$ is said to have total support relative to $\Gamma(R, C)$ provided $A \neq 0$ and $a_{ij} = 1$ implies there exists $X = [x_{ij}] \in \mathcal{F}_{R,C}(A)$ with $x_{ij} \neq 0$. Matrices with total support relative to $\Gamma(R, C)$ are characterized in [2]. We shall be primarily interested in integral transportation polytopes, that is, transportation polytopes $\Gamma(R, C)$ for which $R$ and $C$ are integral vectors.

**Theorem 4.1.** Let $S \subseteq N_n \times N_n$, and let $\mathcal{F}$ be a bounded non-empty face of $\Omega(S)$. Then there exists an integral transportation polytope $\Gamma(R, C)$ such that $\mathcal{F}$ is congruent to a face of $\Gamma(R, C)$.

**Proof.** Let $A$ be an $n \times n$ $(0,1,\ast)$-matrix such that $\mathcal{F} = \mathcal{F}(A)$. Let $B$ be an $n \times n$ $(0,1)$-matrix such that $G(B) = G_\ast(A)$. Since $\mathcal{F}$ is bounded, there exists a positive integer $k$ such that $X + kB$ is a positive matrix for all $X \in \mathcal{F}$. For $i = 1, \ldots, n$ let $r_i$ equal 1 plus the $i$th row sum of $kB$, and for $j = 1, \ldots, n$ let $c_j$ equal 1 plus the $j$th column sum of $kB$. Let $R = (r_1, \ldots, r_n)$ and $C = (c_1, \ldots, c_n)$. Then the mapping $g : \mathcal{F}(A) \to \Gamma(R, C)$ defined by $g(X) = X + kB$ is a translation. Let $A'$ be the $(0,1)$-matrix obtained from $A$ by replacing each $\ast$ with a 1. It is clear that $g(\mathcal{F}(A)) \subseteq \mathcal{F}_{R,C}(A')$. Let $Y \in \mathcal{F}_{R,C}(A')$. It follows that $X = Y - kB \in \mathcal{F}(A)$ with $g(X) = Y$. Hence $\mathcal{F}(A)$ is congruent to $\mathcal{F}_{R,C}(A')$. $\blacksquare$

The following converse also holds.

**Theorem 4.2.** Let $\mathcal{F}$ be a non-empty face of an integral transportation polytope $\Gamma(R, C)$. Then there exists an integer $k$ and an $S \subseteq N_k \times N_k$ such that $\mathcal{F}$ is congruent to a face of $\Omega(S)$.

**Proof.** Suppose that $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$, and let $\tau = r_1 + \cdots + r_m = c_1 + \cdots + c_n$. Let $A$ be an $m \times n$ $(0,1)$-matrix such that $\mathcal{F} = \mathcal{F}_{R,C}(A)$. For $i = 1, \ldots, m$ let $A_i$ be the $r_i \times 1$ matrix of all 1's, and for $j = 1, \ldots, n$ let $B_j$ be the $1 \times c_j$ matrix of all 1's. Define the $\tau \times m$ block diagonal matrix $H_1$ and the $n \times \tau$ block diagonal matrix $H_2$ by

$$H_1 = \text{diag}(A_1, \ldots, A_m), \quad H_2 = \text{diag}(B_1, \ldots, B_n).$$

Let $D_1$ and $D_2$, respectively, be the $m \times m$ and $n \times n$ diagonal matrices with all diagonal entries equal to $\ast$. Let $k = m + n + \tau$. Define a $k \times k$ $(0,1,\ast)$-
Let $X \in \mathcal{F}_{R,C}(A)$, and define a $k \times k$ doubly stochastic matrix $Y$ by

$$
Y = \begin{bmatrix}
H_1 & 0 & 0 \\
E_1 & X & 0 \\
0 & E_2 & H_2
\end{bmatrix},
$$

where $E_1 = \text{diag}(1 - r_1, \ldots, 1 - r_m)$ and $E_2 = \text{diag}(1 - c_1, \ldots, 1 - c_n)$. Then the mapping $h : \mathcal{F}_{R,C}(A) \to \mathcal{F}(B)$ establishes a congruence between $\mathcal{F}_{R,C}(A)$ and $\mathcal{F}(B)$. The theorem now follows.

Let $A$ be an $m \times n$ $(0, 1)$-matrix such that $A$ has total support relative to a transportation polytope $\Gamma(R, C)$. Direct proofs of the following can be given.

(i) A matrix $X \in \mathcal{F}_{R,C}(A)$ is an extreme point of $\mathcal{F}_{R,C}(A)$ if and only if $G(X)$ has no cycles [6, 7].

(ii) If $A$ is connected, then $\dim \mathcal{F}_{R,C}(A) = \sigma(A) - (m + n) + 1$.

These two properties can also be proved for integral transportation polytopes by using the congruence established in the proof of Theorem 4.2, and Theorems 3.1 and 2.8.

We now prove two lemmas which will be used to show that there exists an absolute bound for the number of extreme points of $k$-dimensional faces of transportation polytopes. It will follow from Theorem 4.1 that this bound holds also for bounded $k$-dimensional faces of doubly stochastic polyhedra.

**Lemma 4.3.** Let $G$ be a multigraph with cyclomatic number equal to $k \geq 2$ such that each vertex has degree at least 3. Then $G$ has at most $2(k - 1)$ vertices.

**Proof.** We prove the lemma by induction on $k$. It suffices to assume $G$ is connected. Suppose first that $k = 2$. It then follows that $G$ is a multigraph with 2 vertices and 3 edges or one with 1 vertex and 2 loops. Hence, the lemma holds in this case. Now let $k > 2$. Let $e$ be an edge of some cycle of $G$, and let $G'$ be the multigraph obtained from $G$ by removing the edge $e$. Then
G' has at most two vertices of degree less than 3. First suppose that e joins two distinct vertices or that e is a loop at a vertex x, where the degree of x in G is at least 4. Then every vertex of G has degree at least 2. Let H be the multigraph all of whose vertices have degree at least 3 which is homeomorphic to G'. Then the cyclomatic number of H is k - 1, and by the inductive assumption H has at most 2(k - 2) vertices. Hence G has at most 2(k - 1) vertices. Now suppose that e is a loop at vertex x, where the degree of x in G is 3. Let G'' be the multigraph obtained from G' by removing x and the pendant edge joining x to a vertex y. Then the degree of y in G'' is at least 2, while every other vertex of G'' has degree at least 3 in G''. Let K be the multigraph all of whose vertices have degree at least 3 which is homeomorphic to G''. By the inductive hypothesis K has at most 2(k - 2) vertices. Hence G has at most 2(k - 1) vertices.

A branch of a graph G is a chain $u_1, \ldots, u_r$, where $r \geq 4$, such that the degree of $u_i$ in G is 2 for $i = 2, \ldots, r - 1$.

**Lemma 4.4.** Let B be an $m \times n$ $(0, 1)$-matrix such that $\mathcal{F} = \mathcal{F}_{R, C}(B)$ is a face of $\Gamma(R, C)$, where $R = (r_1, \ldots, r_m)$ and $C = (c_1, \ldots, c_n)$. Let $u_1, \ldots, u_r$ be a branch of $G(B)$, let $s = 2\lceil r/2 \rceil$, $t = 2\lceil (r - 1)/2 \rceil$, and let

$$S_1 = \{ \{ u_1, u_2 \}, \{ u_3, u_4 \}, \ldots, \{ u_{s-1}, u_s \} \},$$

$$S_2 = \{ \{ u_2, u_3 \}, \{ u_4, u_5 \}, \ldots, \{ u_t, u_{t+1} \} \}.$$

Then there exists $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$ such that if $X$ is any extreme point of $\mathcal{F}$ and $E$ is the set of edges of $G(X)$, then $E \cap S_k = S'_k$ or $E \cap S_k = S'_k$, $k = 1, 2$. Indeed, for $k = 1, 2$, an edge of $S_k$ belongs to $S'_k$ if and only if it is an edge of the graph of every extreme point of $\mathcal{F}$.

The above lemma can be proved by showing that for edges $e_1$ and $e_2$ in $S_k$, if $X_i$ is an extreme point of $\mathcal{F}$ such that $e_i$ is not an edge of $G(X_i)$, $i = 1, 2$, then $e_1$ is not an edge of $G(X_2)$ and $e_2$ is not an edge of $G(X_1)$.

**Theorem 4.5.** Let $k$ be a non-negative integer. Then there exists a positive integer $b_k$ such that a $k$-dimensional face of any transportation polytope has at most $b_k$ extreme points.

**Proof.** If $k = 0$ or 1, the theorem is trivial. Suppose $k \geq 2$. Let A be a connected $(0, 1)$-matrix with total support such that $\mathcal{F}_{R, C}(A)$ is a $k$-dimensional face of a transportation polytope. It suffices to prove that there exists
an integer $b'_k$ such that $\mathcal{F}_{R,C}(A)$ has at most $b'_k$ extreme points. It is clear that there exists a connected $(0, 1)$-matrix $B$ with total support such that $\mathcal{F}_{R,C}(B)$ is a $k$-dimensional face of a transportation polytope congruent to $\mathcal{F}_{R,C}(A)$ and $G(B)$ has no pendant edges. Since $G(B)$ is connected, it follows from property (ii) which was stated for transportation polytopes that $k$ equals the cyclomatic number of $G(B)$. If $T$ is any spanning forest of $G(B)$, it follows that there exists at most one extreme point of $\mathcal{F}_{R,C}(B)$ such that $G(X) = T$. Suppose $G(B)$ has a branch $u_1, \ldots, u_r$ ($r \geq 4$). It follows from Lemma 4.4 that the number of spanning forests of $G(B)$ which are graphs of extreme points of $\mathcal{F}_{R,C}(B)$ is no greater than the number of spanning forests of the graph obtained from $G$ by deleting the vertices $u_3, \ldots, u_{r-1}$ and connecting $u_2$ and $u_r$ by an edge. From this fact and Lemma 4.3 it follows that there exists a number $\alpha$ depending only on $k$ such that the number of extreme points of $\mathcal{F}_{R,C}(B)$ is no greater than the number of spanning forests of a graph with $\alpha$ vertices. Hence the number of these extreme points is no greater than the number of spanning forests of the complete graph with $\alpha$ vertices. This proves the theorem.

It follows from Theorems 4.1 and 4.5 that the following holds.

**Theorem 4.6.** Let $k$ be a non-negative integer. Then there exists a positive integer $b_k$ such that a bounded $k$-dimensional face of a doubly stochastic polyhedron has at most $b_k$ extreme points.

To conclude we sketch a proof that a 2-dimensional face of a transportation polytope (and therefore by Theorem 4.1 a bounded 2-dimensional face of a doubly stochastic polyhedron) can have at most 6 vertices. Let $A$ be an $m \times n$ $(0, 1)$-matrix having total support relative to a transportation polytope $\Gamma(R, C)$ such that $\mathcal{F}_{R,C}(A)$ is 2-dimensional. First of all, it is not difficult to reduce the problem to the case where $G(A)$ is 2-connected. Since $\dim \mathcal{F}_{R,C}(A) = 2$, it then follows from the dimension formula (ii) that the cyclomatic number of $G(A)$ is 2. It now follows by repeated application of Lemma 4.4 that we can assume $A$ is one of the three matrices

$$A_1 = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}.$$  

If $A = A_1$ or $A_2$, it is not difficult to show $\mathcal{F}_{R,C}(A)$ has at most 6 vertices. If $A = A_3$, it is clear that $\mathcal{F}_{R,C}(A)$ has at most 6 vertices. It follows from [3] that there are 2-dimensional faces of transportation polytopes with 3 and 4 vertices. If $R = C = (2, 4, 3)$ then $A_1$ is a 2-dimensional face of $\Gamma(R, C)$ with 5
vertices. If \( R = (7,7) \) and \( C = (6,4,4) \), then \( A_3 \) is a 2-dimensional face of \( \Gamma(R, C) \) with 6 vertices.

REFERENCES

4. R. A. Brualdi and P. M. Gibson, Convex polyhedra of doubly stochastic matrices: II. The graph of \( \Omega_n \), submitted for publication.
5. R. A. Brualdi and P. M. Gibson, Convex polyhedra of doubly stochastic matrices: III. Affine and combinational properties of \( \Omega_n \), submitted for publication.

Received 10 October 1975