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Tighter bound for MULTIFIT scheduling on uniform processors

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Abstract

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We examine one of the basic, well studied problem of scheduling theory, that of nonpreemptive assignment of independent tasks on m parallel processors with the objective of minimizing the makespan. Because this problem is NP-complete and apparently intractable in general, much effort has been directed toward devising fast algorithms which find near optimal schedules. Two well-known heuristic algorithms LPT (largest processing time first) and MULTIFIT, shortly MF, find schedules having makespans within $\frac{4}{3}$, $\frac{13}{11}$, respectively, of the minimum possible makespan, when the m parallel processors are identical. If they are uniform, then the best worst-case performance ratio bounds we know are 1.583, 1.40, respectively. In this paper we tighten the bound to 1.382 for MF algorithm for the uniform-processor system. On the basis of some of our general results and other investigations, we conjecture that the bound could be tightened further to 1.366.

Keywords. Bin packing, multiprocessor scheduling, heuristic algorithms, uniform processors, worst-case analysis, performance ratio.

1. Introduction

A well-known deterministic scheduling problem concerns the nonpreemtive assignment of independent tasks to a set of processors in an effort to minimize the makespan (the total elapsed time from the start of execution until all tasks are completed). Formally, we are given a list $\mathscr{L}=\{a_1,\dots,a_n\}$ of independent tasks, each task a_i having processing time $s(a_i)$ and a set $\mathscr{P}=\{P_1,\dots,P_m\}$ of $m\geq 2$ uniform processors. With each P_i associated a relative speed α_i. The objective is to find a schedule, i.e., an assignment of $\mathscr{L}$ to $\mathscr{P}$, which minimizes the maximum finishing time:

$$z_m^*(\mathscr{L})\equiv \min \max_{1\leq i\leq m} \frac{s(P_i)}{\alpha_i},$$

where $s(P_i)=\sum_{a\in P_i} s(a)$ and the minimization is over all assignments of $\mathscr{L}$.

This problem can be readily demonstrated to be NP-complete [5] and is therefore intractable in general. Hence practical heuristic algorithms, which provide near optimal solutions, have been enjoying great favor among our schedulers. Two of them, called LPT and MULTIFIT, shortly MF, are well known. When all α_i's are equal we know that, [6, 7],

$$R(\mathrm{LPT}) = \tfrac{4}{3}, \quad \text{and} \quad R(\mathrm{MF}) = \tfrac{13}{11},$$

where, $R(\cdot)$ is defined as follows:

$$R(A) = \sup\left\{ \max_{1\le i\le m} \frac{s(P_i)}{\alpha_i} \Big/ z_m^*(\mathscr{L}) \colon A \text{ constructs an assignment } \mathscr{P} \text{ of } \mathscr{L} \right\},$$

where the supremum is over all $\mathscr{L}$, m and α_i's. If α_i's are not equal, then the best results we know are [3,2,4]

$$1.52 < R(\mathrm{LPT}) < 1.583 \quad \text{and} \quad 1.341 < R(\mathrm{MF}) < 1.4.$$

In this paper we show that

$$R(\mathrm{MF}) < r,$$

where r is the positive root of equation $2r^3+4r^2-5r-6=0$, i.e., $r=1.381501643\ldots$.

After briefly describing the MF algorithm in the next section, in Section 3 we assume the existence of a counterexample to a more general bound 1.366, and hence the existence of a minimal counterexample whose properties we analyze. In Section 4 we analyze more specifically a minimal counterexample to the bound 1.382. From our assumption contradictions are deduced. Basing the general results obtained in Section 3 and some other investigations made, we conjecture that $R(\mathrm{MF}) \le 1.366$.

2. Description of MF and notations

The scheduling algorithm MF we considered is based on the bin-packing algorithm first-fit decreasing (FFD) first. We consider each processor P_i as a bin and its speed α_i as its capacity, and consider each task a_i as item with size $s(a_i)$. When all bin capacities are multiplied by a constant, or expansion factor, a deadline is specified and hence a successful packing given by FFD is actually a schedule meeting this deadline. We would like to find the smallest expansion factor r such that any list that can be packed in a set of bins of capacities $\alpha_1, \ldots, \alpha_m$ will be successfully packed by the FFD algorithm with the bin capacities multiplied by the expansion factor r. To achieve this goal the MF algorithm first arranges the bins in nondecreasing order of capacities, and arranges the list of items in nonincreasing order of sizes. Then a lower bound and an upper bound are initiated for the expansion factors. At each step we apply FFD, i.e., each item is considered in turn to be placed in the first bin (the P_i with the smallest subscript) in which it will fit, for an expansion factor value of C midway between the current upper and lower bounds. If it succeeds, C becomes the new upper bound, otherwise the new lower bound.

Our main result is that: when the expansion factor is set to be 1.382, then FFD will succeed.

In the following sections we assume to be given a list of items $\mathscr{L}=\{a_1,\dots,a_n\}$ such that $s(a_1)\geq\cdots\geq s(a_n)$, and a set of m bins with capacities $\alpha_1\leq\cdots\leq\alpha_m$. For $a,b\in\mathscr{L}$, by $a\prec b$ we mean a precedes b in $\mathscr{L}$ (hence $s(a)\geq s(b)$). By $P_i=(b_1,\dots,b_k)$ we mean that the ith bin is packed with items $b_1\prec\cdots\prec b_k$. $|P_i|$ denotes the number of items packed in P_i. $P_i[k]$ represents the kth item packed in P_i.

3. General properties of a minimal counterexample

Let r_0 be the positive root of equation $2r^2-2r-1=0$, i.e., $r_0=(\sqrt{3}+1)/2\simeq 1.366$. In this section we suppose that there exists a counterexample for expansion factor r_0, or we will call r_0-counterexample, that is, a list $\mathscr{L}$ of items and a set of bins of capacities $\alpha_1,\dots,\alpha_m$ such that $\mathscr{L}$ can be packed into these bins but FFD fails to pack $\mathscr{L}$ into the bins even of capacities $\beta_1=r_0\alpha_1,\dots,\beta_m=r_0\alpha_m$. To simplify our argument we assume that $\mathscr{L}$ and m are minimal—that no set of fewer than m bins can be used to provide a counterexample and that, given m, no list with fewer than $|\mathscr{L}|$ items will fail to be packed by FFD. *All properties we deduced in this section also apply to any minimal r-counterexample with routine changes of r_0 to r for any r: $r_0\leq r\leq 1.4$.*

We assume by the minimality that the FFD packed all items but the last. Let $\mathscr{P}=\{P_1,\dots,P_m\}$ be this packing of $\{\mathscr{L}-\textit{the last}\}$. For convenience we normalize all bin capacities and item sizes so that the final has size 1. Let $\mathscr{P}^*=\{P_1^*,\dots,P_m^*\}$ be some fixed optimal packing of $\mathscr{L}$ into bins of capacities $\alpha_1,\dots,\alpha_m$. Without loss of generality, we may assume $s(P_1^*)\leq\cdots\leq s(P_m^*)$, where $s(P_i^*)=\sum_{a\in P_i^*}s(a)$. Let $d_i=s(P_i)-s(P_i^*)$ and $\bar{d}_i=-d_i$.

First of all, we use the concept of domination from [1] and [4] to give

Definition 3.1. A union of bins $\bigcup_{i\in I}P_i\subseteq\mathscr{P}$ is said to *dominate* $\bigcup_{j\in J}P_j^*\subseteq\mathscr{P}^*$ if

(i) There is a bijection $\sigma: I\leftrightarrow J$, such that $\alpha_i\leq\alpha_{\sigma(i)}$ for every $i\in I$, and

(ii) $\bigcup_{i\in I}P_i=\{a_1,\dots,a_s\}$ and $\bigcup_{j\in J}P_j^*=\{b_1,\dots,b_t\}$. There is a function $f:\{b_1,\dots,b_t\}\rightarrow\{a_1,\dots,a_s\}$ such that for each a_k,

$$\sum\{s(b): f(b)=a_k\}\leq s(a_k).$$

Lemma 3.2 [4]. *For any $I,J\subseteq\{1,\dots,m\}$, $|I|=|J|$, $\bigcup_{i\in I}P_i$ cannot dominate $\bigcup_{j\in J}P_j^*$.*

Sketch of proof. If we had such a domination, removal of bins $\bigcup_{j\in I}P_i$ and the items packed in them would result in a smaller counterexample. □

Lemma 3.3. *For* $i=1,\dots,m$, $s(P_i)>r_0\alpha_i-1$ *and* $d_i>(r_0-1)\alpha_i-1$. *In particular, if* $\alpha_i\geq 1/(r_0-1)$ *then* $d_i>0$.

Proof. The fact that the last item has size 1 and it cannot fit in any bin gives that $s(P_i)+1>\beta_i$. Since $s(P_i^*)\leq\alpha_i$ we have $s(P_i)-s(P_i^*)>(\beta_i-1)-\alpha_i=(r_0-1)\alpha_i-1$. □

Lemma 3.4. *For any* $P_i^*\in\mathscr{P}^*$, $|P_i^*|\geq 2$. *Hence* $\alpha_j\geq 2$ *for all* j.

Proof. Suppose $P_i^*=\{x\}$. If P_i was empty when x was to be packed, then x was packed in a bin no later than P_i since it was fit in P_i. Thus the bin dominates P_i^*. If P_i was not empty, then the first item in P_i must precede x and P_i dominates P_i^*. In either case Lemma 3.2 is contradicted.

Since $|P_j^*|\geq 2$ and $s(a)\geq 1$ for any $a\in\mathscr{L}$, $\alpha_j\geq 2$ follows. □

Lemma 3.5. *If* $\alpha_i<2r_0$ *then* $|P_i^*|=2$. *Let* $P_i^*=(a,b)$. *Then both* a *and* b *are packed by FFD after* P_i.

Proof. $|P_i^*|=2$ is trivial since $2r_0<3$. Suppose $a\in P_j$ $(j\leq i)$. Since $\beta_j=r_0\alpha_j\geq 2r_0>\alpha_i\geq s(a)+s(b)$, b would fit in P_j. Thus $b\in P_k$ $(k<j)$ else P_j would dominate P_i^*. But then P_k must also contain an item at least as large as a since $\beta_k\geq 2r_0>s(a)$. Hence P_k dominates P_i^*. In either case we contradict Lemma 3.2. Suppose now $b\in P_j$ $(j\leq i)$ and $a\in P_k$ $(k>i)$. Since $\beta_j\geq 2r_0>s(a)$ we know that P_j contains another item at least as large as a. Hence P_j dominates P_i^*, causing another contradiction. □

Lemma 3.6. $|P_i|=1$ *iff* $s(P_i)<s(P_i^*)$.

Proof. If $|P_i|=1$ then $s(P_i^*)$ must be greater than $s(P_i)$ else P_i dominates P_i^*. Suppose $s(P_i)<s(P_i^*)$. By Lemma 3.3 we then have $\alpha_i<1/(r_0-1)<3$. If $|P_i|\geq 2$ then $|P_i|=|P_i^*|=2$. Let $P_i^*=(a,b)$ and $P_i=(u,v)$. Since a was after P_i by Lemma 3.5 we have $s(u)\geq s(a)$. Hence $s(b)>s(v)$ by Lemma 3.2. Since b cannot be packed before P_i by Lemma 3.5, it was not fit in P_i. Hence $s(u)+s(b)>\beta_i$. But since $s(a)+s(b)\leq\alpha_i$ we then have

$$s(u)-s(a)>\beta_i-\alpha_i=(r_0-1)\alpha_i\geq 2(r_0-1).$$

Hence $s(u)>2r_0-1$, and

$$\alpha_i\geq s(P_i^*)>s(P_i)=s(u)+s(v)\geq 2r_0\geq\frac{1}{r_0-1},$$

contradicting our assumption. □

For convenience, we give the following

Definition 3.7. An item $x \in \mathscr{L}$ is said to be *normal* if $s(x) \leq \beta_1$, otherwise *abnormal.* A bin $P_i \in \mathscr{P}$, or $P_i^* \in \mathscr{P}^*$ is *normal* if all items in it are normal, otherwise *abnormal.*

As in [1] and [4], we classify the bins of $\mathscr{P}$ and items of $\mathscr{L}$ by type according to the following scheme. If, after P_i receives its first item, there is a total of k items in P_i when the next item is placed in a bin that follows P_i, then P_i is called a k-bin. The k items are called *regular*. If no additional items are placed in P_i it is called *regular*, otherwise *fallback* and the subsequent item(s) are called *fallback item(s)*. Items in a regular k-bin will be called of type X_k, the first k items in a fallback k-bin will be of type Y_k and fallback items of type F_k. Let $b(P_i) = \sum \{a \in P_i\colon a \text{ is regular}\}$.

Lemma 3.8. *If $\alpha_i < 2r_0$, then $|P_i| = 1,2$. If $|P_i| = 2$, then P_i is a fallback 1-bin, $s(P_i[1]) > 2r_0 - 1$ and $d_i > 1 - r_0(3 - 2r_0)$.*

Proof. Suppose $P_i = (b_0, b_1, \ldots, b_k), k \geq 2$. By Lemma 3.5 we can assume $P_i^* = (b, c)$. Since b was placed after P_i by Lemma 3.5, $s(b) \leq s(b_0)$ and hence $s(c) > s(b_1)$ by Lemma 3.2. Since c was placed after P_i, c could not fit in P_i. Hence $s(c) > s(b_1) + \cdots + s(b_k) \geq k$. Moreover $2r_0 > \alpha_i \geq s(b) + s(c) \geq 2s(c) > 2k \geq 4$, a contradiction. Hence $k \leq 1$, and P_i is a fallback 1-bin if $|P_i| \geq 2$.

Suppose $|P_i| = 2$ and $d_i \leq \mu$, where $\mu = 1 - r_0(3 - 2r_0)$. Then

$$s(P_i) = s(b_0) + s(b_1) \leq \alpha_i + \mu.$$

Since c could not fit in P_i according to the above discussion,

$$s(b_0) + s(c) > \beta_i = r_0 \alpha_i. \tag{3.1}$$

Hence on one hand,

$$s(c) - s(b_1) > (r_0 - 1)\alpha_i - \mu,$$

or

$$s(c) > 1 + (r_0 - 1)\alpha_i - \mu.$$

Thus

$$\tfrac{1}{2}\alpha_i \geq \tfrac{1}{2}(s(b) + s(c)) \geq s(c) > 1 + (r_0 - 1)\alpha_i - \mu.$$

Hence

$$(\tfrac{3}{2} - r_0)2r_0 > (\tfrac{3}{2} - r_0)\alpha_i > 1 - \mu,$$

a contradiction, which shows that $d_i > \mu$.

On the other hand, since $s(b) + s(c) \leq \alpha_i$, combining (3.1), we get

$$s(b_0) > s(b) + (r_0 - 1)\alpha_i \geq 2r_0 - 1. \qquad \square$$

Lemma 3.9. *If P_i is a normal and regular k-bin, then each item in it exceeds $(1/k)(\beta_i - 1)$ in size unless P_i is the last such bin. If P_i is a fallback k-bin, then $b(P_i) > (k/(k+1))\beta_i$. If in addition P_i is normal then each regular item in it exceeds $(1/(k+1))\beta_i$ in size unless P_i is the last such bin.*

Proof. Let P_i and P_j be normal and regular k-bins ($i<j$). If there is an item of P_i, say $a \in P_i$, such that $s(a) \leq (1/k)(\beta_i - 1)$, then $s(P_j[1]) \leq s(a) \leq (1/k)(\beta_i - 1)$. Hence

$$s(P_j) = \sum_{t=1}^{k} s(P_j[t]) \leq ks(P_j[1]) \leq \beta_i - 1 \leq \beta_j - 1,$$

which contradicts Lemma 3.3.

Suppose P_i is a fallback k-bin. If $b(P_i) \leq (k/(k+1))\beta_i$ then $s(P_i[k]) \leq (1/k) \times b(P_i) \leq (1/(k+1))\beta_i$. Hence any item succeeding $P_i[k]$ can fit in P_i, contradicting the type of P_i. Thus $b(P_i) > (k/(k+1))\beta_i$. Let P_i and P_j be normal fallback k-bins ($i<j$). Then $s(P_j[1]) \geq (1/k)b(P_j) > (1/(k+1))\beta_j$ since $b(P_j) > (k/(k+1))\beta_j$. For any regular item $a \in P_i$, $s(a) \geq s(P_j[1]) > (1/(k+1))\beta_j \geq (1/(k+1))\beta_i$ since a precedes $P_j[1]$. □

Lemma 3.10. *If $P_i = \{x\}$ then $|P_j| = 1$ for all $j<i$.*

Proof. By Lemmas 3.3 and 3.6,

$$s(x) = s(P_i) < s(P_i^*) \leq \alpha_i < \frac{1}{r_0 - 1} \leq 2r_0 \leq \beta_1 \leq \beta_j.$$

Hence x is normal and P_j was not empty when x was to be packed. Thus $s(P_j[1]) \geq s(x)$. Since $\beta_j \leq \beta_i$, P_i could be packed with more than one item if so could P_j.

Set $l = \max\{i\colon |P_i| = 1,\ 1 \leq i \leq m\}$. Such an l should exist. Otherwise $s(P_i) \geq s(P_i^*)$ for all $1 \leq i \leq m$ by Lemma 3.6, and we would have

$$s(\mathscr{L}) = 1 + \sum_{i=1}^{m} s(P_i) \geq 1 + \sum_{i=1}^{m} s(P_i^*) = 1 + s(\mathscr{L}).$$

Let $T_i \in P_i$ ($i = 1, \ldots, l$). Then $T_1 \prec \cdots \prec T_l$ ($\alpha_l < 1/(r_0 - 1)$). They are the only items of type X_1. □

Lemma 3.11. *For $1 \leq i \leq l$, $s(T_i) > 2r_0 - 1$. If $T_i \in P_j^*$ then $\alpha_j \geq 2r_0$.*

Proof. $s(T_i) = s(P_i) > \beta_i - 1 = r_0\alpha_i - 1 \geq 2r_0 - 1$. If $T_i \in P_j^*$ then $\alpha_j \geq s(P_j^*) \geq 1 + s(T_i) > 2r_0$ since $|P_j^*| \geq 2$. □

Lemma 3.12. *Let P_i be a normal k-bin ($k \geq 2$) and $|P_i^*| = 2$. If P_i^* is also normal then $P_i^*[1]$ is the regular item of a normal 1-bin.*

Proof. Let $P_i^* = (a, b)$ and $P_i = (u, v, \ldots)$. If $s(a) \leq s(u)$ then by Lemma 3.2, $s(b) > s(v)$, and hence b is packed before P_i since v is regular. Let $b \in P_j$. Then $P_j[1] \prec u$ since P_i is normal. Hence P_j dominates P_i^*, which contradicts Lemma 3.2. Therefore we must have $s(a) > s(u)$. If $a \in P_j$ then $a = P_j[1]$ otherwise P_j would dominate P_i^*. If $|P_j| = 1$ then we are done. Suppose $|P_j| \geq 2$. Then $P_j[2]$ cannot be regular. Otherwise $b \prec P_j[2]$ and hence $b \in P_{j'}$, where $P_{j'}$ is before P_j, which implies that $P_{j'}$ dominates P_i^* since $P_{j'}[1] \prec a$. □

Lemma 3.13. *Let $r_0 \le r \le 1.4$. Then in any minimal r-counterexample, $s(T_l) < 2$.*

Immediate results of Lemma 3.13 are:

Corollary 3.14. *If P_i is a fallback* 1*-bin and $\alpha_i \ge 2r_0$, then $P_i[1]$ is abnormal.*

Proof. Suppose to the contrary that $P_i[1]$ is normal. We show that $s(P_i[1]) > 2$ and thus we have our contradiction by the fact that $s(P_i[1]) > s(T_l)$ and $\beta_l < \beta_i$. If $\alpha_i \ge 3$ then by Lemma 3.9,

$$s(P_i[1]) > \tfrac{1}{2}\beta_i = \tfrac{1}{2}r_0\alpha_i \ge \tfrac{3}{2}r_0 > 2.$$

So we assume $2r_0 \le \alpha_i < 3$. Then $|P_i^*| = 2$ and let $P_i^* = (a, b)$. Since

$$s(a) \le \alpha_i - s(b) \le \alpha_i - 1 < \tfrac{1}{2}r_0\alpha_i \quad \text{and} \quad \tfrac{1}{2}\beta_i < s(P_i[1]),$$

we have $s(P_i[1]) > s(a)$. Hence $s(b) > P_i[2]$ by Lemma 3.2. Since $P_i[1]$ is normal, any bin before P_i was not empty when $P_i[1]$ was to be packed. Hence if $b \in P_j$ then $j > i$ or else P_j would dominate P_i^* since it contains another item as large as $P_i[1]$. This means that b could not fit in P_i:

$$s(P_i[1]) + s(b) > \beta_i.$$

Noting that $s(a) + s(b) \le \alpha_i$, we then have

$$s(P_i[1]) - s(a) > (r_0 - 1)\alpha_i \ge (r_0 - 1)2r_0 = 1.$$

Hence $s(P_i[1]) > s(a) + 1 \ge 2$. □

Corollary 3.15. $\alpha_1 \le \cdots \le \alpha_l < 2 + 2\varepsilon_0/r_0$, *where* $\varepsilon_0 = \tfrac{3}{2} - r_0$.

Proof. Since $\beta_l < s(T_l) + 1$ by Lemma 3.3, $\beta_l < 3$, or $\alpha_l < 3/r_0 = 2 + (3 - 2r_0)/r_0$. □

As for the proof of Lemma 3.13, we leave it until finishing the proof of our main result. Then a sketch of proof is enough to make things clear.

4. Proof of the main result

In this section r is exclusively used to denote the positive root of equation $2r^3 + 4r^2 - 5r - 6 = 0$ (i.e., $r = 1.381501643\ldots$). By using a little sophisticated weight function w, we prove that the MF algorithm for scheduling uniform processors produces a schedule whose length is at most r times the minimal schedule length.

As preparations for the proof, we let

$$\alpha_l = 2+2\lambda \quad (\lambda \geq 0),$$

$$\varepsilon = \tfrac{3}{2} - r - (r-1)\lambda = 0.118498\ldots - (r-1)\lambda,$$

$$\delta = 4\varepsilon - \left(4 - \frac{5}{r}\right) = 2 - 4r + \frac{5}{r} - 4(r-1)\lambda = 0.093243\ldots - 4(r-1)\lambda,$$

$$\Delta = 3(\varepsilon+\delta) - \left(3 - \frac{4}{r}\right)$$

$$= \tfrac{15}{2} - 15r + \frac{19}{r} - 15(r-1)\lambda = 0.530625\ldots - 15(r-1)\lambda.$$

Lemma 4.1. *For* $1 \leq i \leq l$, $s(T_i) > 2r - 1 + 2r\lambda$ *and* $\bar{d}_1 \leq \cdots \leq \bar{d}_l < 2\varepsilon$.

Proof. By Lemma 3.3 we have

$$s(T_l) = s(P_l) > \beta_l - 1 = r\alpha_l - 1 = (2+2\lambda)r - 1 = 2r - 1 + 2r\lambda.$$

Hence, for any $i \leq l$,

$$s(T_i) \geq s(T_l) > 2r - 1 + 2r\lambda.$$

On the other hand,

$$\bar{d}_l = s(P_l^*) - s(P_l) \leq \alpha_l - s(T_l) < \alpha_l - (\beta_l - 1)$$

$$= 1 - (r-1)\alpha_l = 1 - (r-1)(2+2\lambda) = (3-2r) - 2(r-1)\lambda.$$

Since $s(P_1^*) \leq \cdots \leq s(P_l^*)$ and $s(P_1) \geq \cdots \geq s(P_l)$, it is then immediate that $\bar{d}_1 \leq \cdots \leq \bar{d}_l$. □

Let

$$\lambda_0 = \frac{2r+1}{4(r^2-1)} - 1 = 0.035445\ldots,$$

$$\mu_0 = \left(\frac{6}{r} - 2r\right)\lambda_0 + \left(\frac{4}{r} - 2r + 1\right) = 1.188404\ldots,$$

$$\alpha' = (2r-1) + \mu_0 + 2r\lambda = 2.951407\ldots + 2r\lambda,$$

$$\alpha'' = \frac{1}{r-1}(1+4\varepsilon) = \frac{3}{r-1} - 4 - 4\lambda = 3.863662\ldots - 4\lambda.$$

We use also X_1 to represent the set of all items of type X_1 if no confusion is caused.

Lemma 4.2. $\lambda < \lambda_0$.

For the same reason as for Lemma 3.13, we leave the proof to the next section, where a sketch is enough.

Now we can define the following weight function w:

Table 1
$s_1=(\delta+\frac{1}{2})r/(r-1)-\frac{1}{2}$, $s_2=(\frac{13}{10}\delta+\frac{1}{2})r/(r-1)-\frac{1}{2}$

Item type	$\alpha_i<2r$	$2r\le\alpha_i<\alpha''$
X_1	s	-
Y_1	$s-3\varepsilon$	$s-(\varepsilon+\delta)$
F_1	$s-\Delta$	$s-(\varepsilon+\delta)$
X_2[a]	-	$s-(r-1)^2$, if $s\le s_1$, P_i normal, $\lvert P_i^*\cap X_1\rvert=1$, $s-(\varepsilon+\frac{7}{10}\delta)$, if $s_1<s\le s_2$, $s-(\varepsilon+\delta)$, otherwise.
Y_2[b]	-	$s-(\varepsilon+\delta)$
F_2	-	$s-\Delta$, if $\alpha_i<\alpha'$, $s-2\varepsilon$, otherwise.
X_3[c]	-	$s-(\varepsilon+\delta)$, if $\alpha_i<\alpha'$, or $\beta_i-s(P_i)<2\lambda_0$, or P_i abnormal, $s-\varepsilon$, otherwise.
Y_3, F_3	-	$s-(\varepsilon+\delta)$
X_4	-	$s-(\varepsilon+\delta)$ if P_i abnormal, $s-\varepsilon$, otherwise.
others	-	$s-\varepsilon$

[a] If a is the last item in the last normal regular 2-bin, then $w(a)=s(a)-\Delta$.
[b] If a is the last regular item in the last normal fallback 2-bin and $\alpha_i<\alpha'$, $\beta_i-s(P_i)<\mu_0-1$, then $w(a)=s(a)-\Delta$.
[c] If a is one of the last two items in the last normal regular 3-bit, then $w(a)=s(a)-(\varepsilon+\delta)$.

If a is the last item in $\mathscr{L}$ then $w(a)=s(a)-\varepsilon$. Let $a\in P_i$. If $\alpha_i\ge\alpha''$ then $w(a)=s(a)-\varepsilon$. Other details are in Table 1.

We use $w(d_i)$ to denote $w(P_i)-w(P_i^*)$ and $w(\bar{d}_i)=-w(d_i)$.

The remainder of the proof consists mainly of a weight argument. It will be proved that:

The FFD bins $\mathscr{P}=\{P_1,\dots,P_m\}$ can be grouped so that the total weights of items in each group are often greater than that of items in the group of optimal bins corresponding those FFD bins. In case they are not, the loss of weights can be compensated for by a gain from other groups.

Formally, by a sequence of case analyses, we show that:

The set $\{1,\dots,m\}$ of indices can be partitioned into $I_1,\dots,I_t$ (i.e., $I_i\cap I_j=\emptyset$, $i\ne j$, $\bigcup_{i=1}^t I_i=\{1,\dots,m\}$) such that for any $I\in\mathscr{I}=\{I_1,\dots,I_t\}$, the following conditions, which will be called $D(I)$, will be satisfied:

$$\sum_{i\in I} w(P_i)\ge\sum_{i\in I} w(P_i^*)+k_I\varepsilon,$$

where $k_I=\max\{0,\sum_{i\in I}(\lvert P_i\rvert-\lvert P_i^*\rvert)\}$; or

$$\bar{k}_I = \sum_{i \in I} (|P_i^*| - |P_i|) > 0,$$

and

$$\sum_{i \in I} w(P_i^*) < \sum_{i \in I} w(P_i) + \bar{k}_I \varepsilon.$$

An argument about the conservation of total weights and numbers of items in $\mathscr{L}$ will then allow us to contradict the assumption that we had a counterexample.

We use $\mathscr{I}$ to record our appropriate partition of $\{1, \ldots, m\}$. Initially we let $\mathscr{I} \Leftarrow \emptyset$.

4.1. P_i is a regular 1-bin

The fact from Lemma 3.11 that any item of type X_1 cannot be packed in P_i^* shows that

$$w(P_i^*) \le s(P_i^*) - 2\varepsilon.$$

By Lemma 4.1 we then have

$$w(P_i) - w(P_i^*) \ge s(P_i) - (s(P_i^*) - 2\varepsilon) = 2\varepsilon - \bar{d}_i > 0.$$

Hence we set $\mathscr{I} \Leftarrow \mathscr{I} \cup \{i\}$.

Before analyzing further, we give the following:

Lemma 4.3. *Suppose $|P_i| = k \ge 2$ and $w(P_i) = s(P_i) - \mu$ ($\mu \ge 0$). If $d_i > \mu + (k-4)\varepsilon$ then an appropriate set I of indices containing i can be decided so that condition $D(I)$ is satisfied.*

Proof. Suppose first that $P_i^* \cap X_1 = \emptyset$. Then $w(P_i^*) \le s(P_i^*) - 2\varepsilon$. Hence

$$w(d_i) \ge (s(P_i) - \mu) - (s(P_i^*) - 2\varepsilon) = d_i + 2\varepsilon - \mu \ge (k-2)\varepsilon.$$

Therefore we can let $I = \{i\}$, and set $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Suppose now $|P_i^* \cap X_1| = 1$, and $T_j \in P_i^*$ $(1 \le j \le l)$. Then $w(P_i^*) \le s(P_i^*) - \varepsilon$. Hence

$$w(d_i) \ge (s(P_i) - \mu) - (s(P_i^*) - \varepsilon) = d_i + \varepsilon - \mu.$$

Thus

$$w(d_i) > (k-3)\varepsilon, \quad \text{if } k \ge 3,$$

and

$$w(\bar{d}_i) < \varepsilon, \quad \text{if } k = 2.$$

In either case we can let $I \Leftarrow \{i, j\}$ and set $\mathscr{I} \Leftarrow (\mathscr{I} - \{j\}) \cup I$.

If $|P_i^* \cap X_1| \ge 2$ then let $I' = \{j\colon T_j \in P_i^*,\ 1 \le j \le l\}$. Since

$$w(d_i) \ge d_i - \mu > (k-4)\varepsilon, \quad \text{if } k \ge 4,$$

$$w(\bar{d}_i) < \varepsilon, \quad \text{if } k = 3,$$

and

$$w(\bar{d}_i) < 2\varepsilon, \quad \text{if } k = 2,$$

we can let $I \Leftarrow I' \cup \{i\}$ and set $\mathscr{I} \Leftarrow (\mathscr{I} - \bigcup_{j\in I}\{j\}) \cup I'$. □.

4.2. P_i is a fallback 1-bin

Case 1: $\alpha_i < 2r$. By Lemma 3.8 we have $|P_i| = 2$ and $d_i > 1 - 2r\varepsilon$. Since $w(P_i) = s(P_i) - (\Delta + 3\varepsilon)$ and $w(P_i^*) \le s(P_i^*) - 2\varepsilon$, we obtain

$$w(d_i) = w(P_i) - w(P_i^*) \ge d_i + 2\varepsilon - (\Delta + 3\varepsilon) = 1 - 2r\varepsilon - \Delta - \varepsilon > 0.$$

Let $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Case 2: $\alpha_i \ge 2r$.

Case 2.1: $|P_i| = k \ge 4$. Since $w(P_i) = s(P_i) - k(\varepsilon + \delta)$, if we can show that $d_i > k(\varepsilon + \delta) + (k-4)\varepsilon$, then by Lemma 4.3 we are done. So we suppose to the contrary that $d_i \le k(2\varepsilon + \delta) - 4\varepsilon$. Noting that the first item of P_i must be larger than half the bin size by Lemma 3.9, we then have

$$s(P_i) > \max\{r\alpha_i - 1, 2(k-1)\}.$$

Hence

$$d_i \ge s(P_i) - \alpha_i > \max\{(r-1)\alpha_i - 1, 2(k-1) - \alpha_i\}.$$

Therefore

$$(r-1)\alpha_i - 1 < k(2\varepsilon + \delta) - 4\varepsilon,$$

and

$$2(k-1) - \alpha_i < k(2\varepsilon + \delta) - 4\varepsilon.$$

Combining the two inequalities, we get

$$(r-1)(2 - 2\varepsilon - \delta)k - 4(r-1)^2 < (r-1)\alpha_i < (2\varepsilon + \delta)k + (1 - 4\varepsilon),$$

or

$$k < \frac{4(r-1)^2 + (1-4\varepsilon)}{(r-1)(2-2\varepsilon-\delta) - (2\varepsilon+\delta)} \le \frac{4r^2 - 4r - 1}{6r^2 - 3r - 7} < 4,$$

a contradiction.

Case 2.2: $|P_i| = 3$. Since $P_i[1]$ is abnormal by Corollary 3.14, we have $s(P_i) > \beta_1 + 2 \ge 2r + 2$. Hence

$$d_i \ge \max\{2r + 2 - \alpha_i, (r-1)\alpha_i - 1\} \ge 2r - \frac{3}{r}.$$

But

$$\left(2r - \frac{3}{r}\right) > 2\varepsilon + 3\delta,$$

hence

$$d_i > 3(\varepsilon + \delta) - \varepsilon.$$

Noting that $w(P_i) = s(P_i) - 3(\varepsilon + \delta)$, we are then done by Lemma 4.3.

Case 2.3: $|P_i| = 2$. The fact that $P_i[1]$ is abnormal implies that

$$d_i \ge \max\{2r + 1 - \alpha_i, (r-1)\alpha_i - 1\} \ge 2r - 1 - \frac{2}{r} > 2\delta.$$

Since $w(P_i)=s(P_i)-2(\varepsilon+\delta)$ we are also done by Lemma 4.3.

4.3. P_i is a regular 2-bin

(At most $(\Delta-(r-1)^2+\frac{9}{100})$ of weight is needed for compensation to this class of bins.)

If P_i is abnormal, then $d_i \ge \max\{2r+1-\alpha_i, (r-1)\alpha_i-1\} \ge 2r-1-2/r$. If P_i^* is abnormal, then $d_i \ge (r-1)\alpha_i - 1 \ge (r-1)(2r+1)-1$. Hence

$$d_i \ge \min\left(2r-1-\frac{2}{r}, (r-1)(2r+1)-1\right) = 2r-1-\frac{2}{r} > 2\delta$$

if at least one of P_i and P_i^* is abnormal. Therefore, considering that $w(P_i) \ge s(P_i)-2(\varepsilon+\delta)$, we are done by Lemma 4.3.

In the following we then suppose that both P_i and P_i^* are normal. (If P_i is the last normal regular 2-bin, then we add $(\Delta-(r-1)^2+\frac{9}{100})$ of weight to its second item.)

Case 1: $P_i^* \cap X_1 = \emptyset$. If $|P_i^*| \ge 3$ then $w(P_i^*)-3\varepsilon$ and $d_i > (r-1)\alpha_i - 1 \ge 3(r-1)-1 = 3r-4 > 2\delta-\varepsilon$. Hence

$$w(d_i) \ge d_i + 3\varepsilon - 2(\varepsilon+\delta) = d_i + \varepsilon - 2\delta > 0.$$

Set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

If $|P_i^*|=2$ then, by Lemma 3.12 $P_i^*[1]$ is a normal, regular item in a fallback 1-bin since $P_i^* \cap X_1 = \emptyset$. Hence $w(P_i^*[1]) = s(P_i^*[1]) - 3\varepsilon$ by Corollary 3.14. Thus $w(P_i^*) \le s(P_i^*) - 4\varepsilon$ and

$$w(d_i) \ge d_i + 4\varepsilon - 2(\varepsilon+\delta) = d_i + 2\varepsilon - 2\delta > 0$$

since $d_i > 0$ by Lemma 3.6.

Set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Case 2: $|P_i^* \cap X_1| = 1$. Then $w(P_i^*) \le s(P_i^*) - \varepsilon$. Let $T_j \in P_i^*$ $(1 \le j \le l)$ and $P_i = (u_1, u_2)$. Then, by Lemma 4.1, $\alpha_i > 2r(1+\lambda)$. We are to show that $w(\bar{d}_i) < \varepsilon$ so as to set $I \Leftarrow \{i,j\}$ and $\mathscr{I} \Leftarrow (\mathscr{I} - \{j\}) \cup I$.

Let

$$s_1 = \frac{r}{r-1}(\delta + \tfrac{1}{2}) - \tfrac{1}{2}; \qquad s_2 = \frac{r}{r-1}(\tfrac{13}{10}\delta + \tfrac{1}{2}) - \tfrac{1}{2};$$

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \le 0; \end{cases}$$

$$\theta_1 = \theta(s(u_1) - s_2); \qquad \theta_2 = \theta(s(u_2) - s_2).$$

Case 2.1: $s(u_2) > s_1$. Then

$$w(P_i) = s(P_i) - (2\varepsilon + (\tfrac{7}{5} + \tfrac{3}{10}(\theta_1 + \theta_2))\delta),$$

$$d_i \ge \max\{s(u_1) + s(u_2) - \alpha_i, (r-1)\alpha_i - 1\}$$

$$\geq (s(u_1)+s(u_2)-2s_2)\frac{r-1}{r}+(2s_2+1)\frac{r-1}{r}-1$$

$$=\tfrac{13}{5}\delta+(s(u_1)-s_2)\frac{r-1}{r}+(s(u_2)-s_2)\frac{r-1}{r}.$$

Noting that $\theta_t=0$ iff

$$-\tfrac{3}{10}\delta<(s(u_t)-s_2)\frac{r-1}{r}\leq 0 \quad (t=1,2),$$

we then have

$$d_i>(\tfrac{7}{5}+\tfrac{3}{10}(\theta_1+\theta_2))\delta.$$

Hence

$$w(d_i)\geq d_i+\varepsilon-(2\varepsilon+(\tfrac{7}{5}+\tfrac{3}{10}(\theta_1+\theta_2))\delta)>-\varepsilon,$$

or

$$w(\bar{d}_i)<\varepsilon.$$

Case 2.2: $s(u_2)\leq s_1<s(u_1)$. If P_i is not the last normal regular 2-bin, then $w(P_i)=s(P_i)-(r-1)^2-(\varepsilon+(\frac{7}{10}+\frac{3}{10}\theta_1)\delta)$. Since, by Lemma 3.9, $s(u_2)>\frac{1}{2}(\beta_i-1)\geq\frac{1}{2}(2r^2(1+\lambda)-1)\geq r^2(1+\lambda)-\frac{1}{2}$, we have

$$d_i\geq\max\{(r^2-\tfrac{1}{2})+s(u_1)-\alpha_i,(r-1)\alpha_i-1\}\geq(s(u_1)+r^2(1+\lambda)+\tfrac{1}{2})\frac{r-1}{r}-1$$

$$\geq(s(u_1)-s_2)\frac{r-1}{r}+\left((s_2+\tfrac{1}{2})\frac{r-1}{r}-\tfrac{1}{2}\right)+(r(r-1)(1+\lambda)-\tfrac{1}{2})$$

$$\geq(s(u_1)-s_2)\frac{r-1}{r}+\tfrac{13}{10}\delta+((r-1)^2-\varepsilon).$$

Since $\theta_1=0$ iff

$$-\tfrac{3}{10}\delta<(s(u_1)-s_2)\frac{r-1}{r}\leq 0,$$

we obtain $d_i>(r-1)^2-\varepsilon+(1+\frac{3}{10}\theta_1)\delta$. Hence

$$w(d_i)\geq d_i+\varepsilon-(\varepsilon+(\tfrac{7}{10}+\tfrac{3}{10}\theta_1)\delta+(r-1)^2)>-\varepsilon,$$

or

$$w(\bar{d}_i)<\varepsilon.$$

If P_i is the last normal regular bin, then $w'(P_i) = s(P_i)-((r-1)^2+\varepsilon+(\frac{7}{10}+\frac{3}{10}\theta_1)\delta-\frac{9}{100})$, where w' stands for the new weight after compensation. Since

$$d_i=(s(u_1)-s(T_j))+s(u_2)-(s(P_i^*)-s(T_j))\leq s(u_2)-1,$$

and

$$d_i>(r-1)\alpha_i-1\geq(s(T_j)+1)(r-1)-1,$$

we have

$$s(u_2)\geq(s(T_j)+1)(r-1)\geq(2r+2r\lambda)(r-1).$$

Hence

$$d_i \geq \max\{s(u_2)+s(u_1)-\alpha_i, (r-1)\alpha_i-1\} \geq (s(u_1)+s(u_2)+1)\frac{r-1}{r}-1$$

$$\geq (s(u_1)-s_2)\frac{r-1}{r}+\left((s_2+\tfrac{1}{2})\frac{r-1}{r}-\tfrac{1}{2}\right)+(2r(r-1)(1+\lambda)+\tfrac{1}{2})\frac{r-1}{r}-\tfrac{1}{2}$$

$$=(s(u_1)-s_2)\frac{r-1}{r}+\tfrac{13}{10}\delta+\left(2(r-1)^2(1+\lambda)-\frac{1}{2r}\right)$$

$$>2(r-1)^2(1+\lambda)-\frac{1}{2r}+(1+\tfrac{3}{10}\theta_1)\delta.$$

By simple calculation and noting that $\lambda<\lambda_0$, we have

$$2(r-1)^2(1+\lambda)-\frac{1}{2r}+(1+\tfrac{3}{10}\theta_1)\delta>(r-1)^2-\varepsilon+(\tfrac{7}{10}+\tfrac{3}{10}\theta_1)\delta-\tfrac{9}{100}.$$

Hence

$$w'(d_i)=w'(P_i)-w(P_i^*)\geq d_i+\varepsilon-((r-1)^2+\varepsilon+(\tfrac{7}{10}+\tfrac{3}{10}\theta_1)\delta-\tfrac{9}{100})>-\varepsilon.$$

or

$$w'(d_i)<\varepsilon.$$

Case 2.3: $s(u_1)\leq s_1$. Then $w(P_i)=s(P_i)-2(r-1)^2$. Noting that $\alpha_i>2r(1+\lambda)$, we have

$$d_i>(r-1)\alpha_i-1\geq 2r(r-1)(1+\lambda)-1\geq 2(r-1)^2-2\varepsilon,$$

and

$$w(d_i)\geq d_i+\varepsilon-2(r-1)^2>-\varepsilon,$$

or

$$w(\bar{d}_i)<\varepsilon.$$

Case 3: $|P_i^*\cap X_1|\geq 2$. By Lemma 4.1 we have $\alpha_i\geq 2(2r-1)$ and hence $d_i\geq 2(2r-1)(r-1)-1>2\delta$. Since $w(P_i)=s(P_i)-2(\varepsilon+\delta)$ we are then done by Lemma 4.3.

4.4. P_i *is a fallback* 2-*bin*

(At most $\Delta-(2\varepsilon+\delta+\tfrac{99}{1000})$ of weight is needed for compensation to this class.)

At first we suppose P_i is not the last normal, regular fallback 2-bin.

Case 1: $\alpha_i<\alpha'$. If $|P_i|\geq 4$ then $s(P_i)>3\cdot 2=6$ since each regular item exceeds the sum of all fallback items in size, and hence $\beta_i\geq s(P_i)>6$, contradicting that $\alpha_i<\alpha'$. Therefore $|P_i|=3$. We have $w(P_i)=s(P_i)-(\Delta+2\varepsilon+2\delta)$. By Lemma 3.9 we also have $s(P_i)\geq 1+b(P_i)>1+\tfrac{2}{3}\beta_i$, and hence

$$d_i>1+\tfrac{2}{3}\beta_i-\alpha_i=1-(1-\tfrac{2}{3}r)\alpha_i>1-(1-\tfrac{2}{3}r)\alpha'>\Delta-\varepsilon+2\delta.$$

If $|P_i^*|\geq 3$ then $P_i^*\cap X_1=\emptyset$ or else $\alpha_i\geq(2r-1+2r\lambda)+2>\alpha'$. Hence $w(P_i^*)\leq s(P_i^*)-3\varepsilon$, and

$$w(d_i) \geq d_i + 3\varepsilon - (\Delta + 2\varepsilon + 2\delta) > 0.$$

Set $I \Leftarrow \{i\}$ and $\mathcal{I} \Leftarrow \mathcal{I} \cup I$.

Suppose $|P_i^*| = 2$ and let $P_i^* = (x, c)$. Since both P_i^* and P_i have to be normal by the restriction $\alpha_i < \alpha'$, we know that $x \in P_j$ is regular and $\alpha_j < 2r$ from Lemma 3.12 and Corollary 3.14.

If x is of type Y_1, then $w(x) = s(x) - 3\varepsilon$ and we obtain $w(P_i^*) \leq s(P_i^*) - 4\varepsilon$, since c cannot be of type X_1. Hence

$$w(d_i) \geq d_i + 4\varepsilon - (\Delta + 2\varepsilon + 2\delta) > \varepsilon.$$

Set $I \Leftarrow \{i\}$ and $\mathcal{I} \Leftarrow \mathcal{I} \cup I$. Suppose x is of type X_1. Let $P_j^* = (a, b)$ and let a' denote the largest item among a, b and c. Then $s(a') \leq \max\{1 + 2\lambda, \mu_0\} = \mu_0$ since $s(x) > 2r - 1 + 2r\lambda$ by Lemma 4.1 and $s(x) + s(c) < 2r - 1 + 2r\lambda + \mu_0$. But $\mu_0 < \frac{2}{3}r^2 \leq \frac{1}{3}\beta_i < s(P_i[2])$ by Lemma 3.9, hence $s(a') < s(P_i[2])$. In addition we have $s(a') > s(P_i[3])$ since otherwise $P_j \cup P_i$ would dominate $P_j^* \cup P_i^*$.

Case 1.1: *a' was packed before P_i.* Then $a' \in P_k$ is a fallback item since $s(a') < s(P_i[2])$. P_k is a normal 1-bin or else $P_j \cup P_k$ would dominate $P_j^* \cup P_i^*$. Hence $w(a') = s(a') - \Delta$, and

$$w(d_i) + w(d_j) \geq d_i + \Delta - (\Delta + 2\varepsilon + 2\delta) \geq d_i - 2(\varepsilon + \delta) > 0.$$

Case 1.2: *a' was packed after P_i.* Then a' could not fit in P_i, which implies that

$$\beta_i - s(P_i) < s(a') - 1 \leq \mu_0 - 1,$$

or

$$s(P_i) \geq \beta_i + 1 - \mu_0 \geq s(P_i^*) + (r-1)\alpha_i + 1 - \mu_0,$$

or

$$d_i \geq (r-1)\alpha_i + 1 - \mu_0 \geq 2r(r-1) + 1 - \mu_0 > \Delta + 2\varepsilon + \delta.$$

Hence

$$w(d_i) + w(d_j) \geq d_i + \varepsilon - (\Delta + 2\varepsilon + 2\delta) > 0.$$

Set $I \Leftarrow \{i, j\}$ and $(\mathcal{I} \Leftarrow \mathcal{I} - \{j\}) \cup I$.

Case 2: $\alpha' \leq \alpha_i < \alpha''$. For the same reason as in Case 1, we have $|P_i| = 3$. Hence $w(P_i) = s(P_i) - (4\varepsilon + 2\delta)$. As in Case 1 we have $d_i > 1 - (1 - \frac{2}{3}r)\alpha_i \geq 1 - (1 - \frac{2}{3}r)\alpha'' > 4\varepsilon + 2\delta$. Then by Lemma 4.3 we are done.

Case 3: $\alpha_i \geq \alpha''$. Let $|P_i| = 2 + k$ $(k \geq 1)$. Then $w(P_i) = s(P_i) - (k+2)\varepsilon$ and $d_i \geq \max\{k - (1 - \frac{2}{3}r)\alpha_i, (r-1)\alpha_i - 1\} \geq 3(k+1)(r-1)/r - 1 > 2(k+1)\varepsilon$. The last inequality holds since $k + 1 > r/(2r^2 - 3)$. Hence

$$w(d_i) \geq d_i - (k+2)\varepsilon \geq k\varepsilon.$$

Set $I \Leftarrow \{i\}$and $\mathcal{I} \Leftarrow \mathcal{I} \cup I$.

Now we show that if P_i is the last normal, regular fallback 2-bin, then $\Delta - (2\varepsilon + \delta + \frac{99}{1000})$ of weight compensation can make up the loss.

We only need to check when $\alpha_i < \alpha'$ and $\beta_i - s(P_i) < \mu_0 - 1$ since otherwise no compensation is needed as was proved above (note: $\beta_i - s(P_i) \geq \mu_0 - 1$ implies $\mu_0 <$

$s(P_i[2])$). The following proof is almost the same as that of Case 1. We have $|P_i|=3$ and

$$\begin{aligned} w'(P_i) &= w(P_i) + \Delta - (2\varepsilon + \delta + \tfrac{99}{1000}) \\ &= s(P_i) - (2\Delta + \varepsilon + \delta) - (-\Delta + 2\varepsilon + \delta + \tfrac{99}{1000}) \\ &= s(P_i) - (\Delta + 3\varepsilon + 2\delta + \tfrac{99}{1000}), \end{aligned}$$

$$d_i \geq (r-1)\alpha_i + 1 - \mu_0 \geq (r-1)2r + 1 - \mu_0 > \Delta + 2\delta + \tfrac{99}{1000},$$

where w' is the new weight after compensation.

If $|P_i^*| \geq 3$ then $w(P_i^*) \leq s(P_i^*) - 3\varepsilon$ and hence

$$w'(d_i) = w'(P_i) - w(P_i^*) \geq d_i + 3\varepsilon - (\Delta + 3\varepsilon + 2\delta + \tfrac{99}{1000}) > 0.$$

Set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Suppose $|P_i^*| = 2$ and let $P_i^* = (x, c)$. Then $x \in P_j$ is regular and $\alpha_j < 2r$. If x is of type Y_1 then $w(x) = s(x) - 3\varepsilon$ and hence $w(P_i^*) \leq s(P_i^*) - 4\varepsilon$. We then have

$$w'(d_i) \geq d_i + 4\varepsilon - (\Delta + 3\varepsilon + 2\delta + \tfrac{99}{1000}) > \varepsilon,$$

allowing us to set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$. Suppose x is of type X_1. Let $P_j^* = (a, b)$ and a', a'' be the largest and second largest, respectively, among a, b and c. We have $s(a') \leq \mu_0 < 2r^2/3 < s(P_i[1])$ and $s(a'') \leq 1 + 2\lambda$. Let $a'' \in P_{i'}$. If $i' < i$ then a'' is a fall-back item and $P_{i'}$ has to be a normal 1-bin since otherwise $P_j \cup P_{i'}$ would dominate $P_j^* \cup P_i^*$ (noting that $\mu_0 + 1 + 2\lambda < 2r$). Hence $w(a'') = s(a'') - \Delta$, implying that

$$w'(d_i) + w(d_j) \geq d_i + \Delta - (\Delta + 3\varepsilon + 2\delta + \tfrac{99}{1000}) > \Delta - 3\varepsilon > 0.$$

If $i' = i$ then $a'' \prec P_i[3]$ by Lemma 3.2 and thus $a'' = P_i[2]$, which implies that $w(a'') = s(a'') - \Delta$ and that $w'(d_i) + w(d_j) > 0$. If $i < i'$ then since $P_i[2] \prec a'' \prec P_i[3]$ we have $\beta_i - s(P_i) < 2\lambda$. Hence

$$d_i \geq (r-1)\alpha_i - 2\lambda \geq (r-1)2r - 2\lambda > \Delta + 2\varepsilon + 2\delta + \tfrac{99}{1000},$$

making that

$$w'(d_i) \geq d_i + \varepsilon - (\Delta + 3\varepsilon + 2\delta + \tfrac{99}{1000}) > 0.$$

Therefore, in the case where x is of type X_1, we are allowed to set $I \Leftarrow \{i, j\}$ and $\mathscr{I} \Leftarrow (\mathscr{I} - \{j\}) \cup I$.

4.5. P_i is a regular 3-bin

(At most 2δ of weight compensation is needed in this class.)

Case 1: $2r \leq \alpha_i < \alpha'$, *or* $\beta_i - s(P_i) < 2\lambda_0$, *or* P_i *is abnormal.* If $\beta_i - s(P_i) < 2\lambda_0$ then $s(P_i) > \beta_i - 2\lambda_0 \geq \alpha_i + (r-1)\alpha_i - 2\lambda_0 \geq s(P_i^*) + (r-1)\alpha_i - 2\lambda_0$. Hence $d_i \geq (r-1)2r - 2\lambda_0 \geq \varepsilon + 3(\varepsilon + \delta)$ and $w(d_i) \geq d_i - 3(\varepsilon + \delta) \geq \varepsilon$, allowing us to set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

If P_i is abnormal then $s(P_i) > 2r + 2$. Hence $d_i \geq \max\{2r + 2 - \alpha_i, (r-1)\alpha_i - 1\} \geq 2r - 3/r > 3(\varepsilon + \delta) - \varepsilon$. Noting that $w(P_i) = s(P_i) - 3(\varepsilon + \delta)$, we are then finished by Lemma 4.3.

In the following we then suppose that P_i is normal and $2r \le \alpha_i < \alpha'$ (thus P_i^* is also normal else $\alpha_i > 2r+1 > \alpha'$).

Case 1.1: $|P_i^*| \ge 3$. Then $|P_i^*| = 3$ since $\alpha' < 4$ and $P_i^* \cap X_1 = \emptyset$ since $(2r-1)+2 > \alpha'$. Let $P_i^* = (a, b, c)$, and let $P_k = (u_1, u_2, u_3)$ be the first normal regular 3-bin. We show that $w(d_i) \ge 0$ so as to set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$. Suppose to the contrary that $w(d_i) < 0$.

Then none of a, b and c satisfies $w(\cdot) = s(\cdot) - \Delta$ since otherwise $w(P_i^*) \le s(P_i^*) - (\Delta + 2\varepsilon)$, implying that

$$w(d_i) \ge d_i + \Delta + 2\varepsilon - 3(\varepsilon + \delta) = d_i + 2\varepsilon - \left(3 - \frac{4}{r}\right) > 2\varepsilon > 0,$$

a contradiction.

If it is not true that $a < u_1$ then $s(a) \le s(u_1)$, which implies that $s(b) > s(u_3)$ else P_k would dominate P_i^*. Hence either b or c, say b, was packed before P_k and thus is of type F_1 or F_2. Since $(b) > s(b) - \Delta$ we then have $w(b) = s(b) - (\varepsilon + \delta)$, which implies that b is a fallback item of an abnormal fallback 1-bin. But then this abnormal fallback 1-bin dominates P_i^* since $s(a) + s(c) \le \alpha_i s(b) < \alpha' - 1 < 2r$.

Therefore we have $a < u_1$. Let $a \in P_j$, where P_j is a 1- or 2-bin before P_k. It is apparent that P_j is normal since otherwise P_j would dominate P_i^*. a has to be regular since otherwise, as in the proof of that $a < u_1$, a would be a fallback item of an abnormal fallback 1-bin, which then dominates P_i^*. If $s(a) \ge \frac{2}{3}r^2$ then $\alpha_i \ge s(a) + s(b) + s(c) \ge \frac{2}{3}r^2 + 2 > \alpha'$. Hence $s(a) < \frac{2}{3}r^2$. Then a is neither of type X_1 (Lemma 4.1) nor of type Y_1 (Lemma 3.8 and Corollary 3.14). If a is of type X_2 or Y_2 then, by Lemma 3.9, it has to be the second item of the last such bin since $\frac{1}{2}(\beta_i - 1) > \frac{1}{3}\beta_j \ge \frac{2}{3}r^2$. Hence we have $w(a) = s(a) - \Delta$, contradicting our earlier conclusion that none of a, b and c satisfies $w(\cdot) = s(\cdot) - \Delta$.

Case 1.2: $|P_i^*| = 2$. Let $P_i^* = (x, c)$. Since both P_i and P_i^* are normal, x is the normal, regular item of a 1-bin by Lemma 3.12. Suppose first that $P_i^* \cap X_1 \ne \emptyset$. Then $|P_i^* \cap X_1| = 1$ since $2(2r-1) > \alpha' > \alpha_i$. Hence $x \in P_j$ $(1 \le j \le l)$ and $w(c) \le s(c) - \varepsilon$. Let $P_j^* = (a, b)$ and a' be of maximum size among a, b and c. Then a' must be packed before the first normal regular 3-bin since otherwise the bin and P_j would dominate P_i^* and P_j^*. Let $a' \in P_k$. Then P_k is a 1-bin or 2-bin before P_i.

If $w(d_i) \ge 0$ or $w(d_i) + w(d_j) \ge 0$ then we are done since we can set $I \Leftarrow \{i, j\}$ and $\mathscr{I} \Leftarrow (\mathscr{I} - \{j\}) \cup I$. Therefore we suppose that it is otherwise. Then a' cannot be a fallback item. Otherwise either $w(a') = s(a') - \Delta$, which implies

$$w(d_i) + w(d_j) = d_i + \Delta - 3(\varepsilon + \delta) \ge d_i + \Delta - 3(\varepsilon + \delta) \ge 0,$$

since $d_i \ge \max\{3 - \alpha_i, (r-1)\alpha_i - 1\} \ge 3 - 4/r$, or P_k is an abnormal fallback 1-bin, which implies that $P_j \cup P_k$ dominates $P_j^* \cup P_i^*$. Therefore a' is regular.

Since $s(a') \le \max\{1 + 2\lambda, \mu_0\} = \mu_0 < \frac{2}{3}r^2$ owing to the fact that $s(x) > 2r - 1 + 2r\lambda$ by Lemma 4.1 and $s(x) + s(c) < \alpha' = 2r - 1 + 2r\lambda + \mu_0$, the same argument as at the end of Case 1.1 shows that $w(a') = s(a') - \Delta$, which contradicts our assumption as was shown above.

Suppose second that $P_i^* \cap X_1 = \emptyset$. Then, by Lemma 3.12, $x = P_j[1]$, where P_j is a normal fallback 1-bin. Hence $w(x)=s(x)-3\varepsilon$. As before, we let $P_j^*=(a,b)$ and let a' be of maximum size among a, b and c. Almost the same analysis as in the case where $P_i^*\cap X_1\neq\emptyset$ shows that a' satisfies $w(\cdot)=s(\cdot)-\Delta$, which implies that

$$w(d_i)+w(d_j)\geq d_i+3\varepsilon+\Delta-3(\varepsilon+\delta)\geq 3\varepsilon>\varepsilon$$

since $d_i\geq 3-4/r=3(\varepsilon+\delta)-\Delta$.

Therefore, in this case, we can set $I\Leftarrow\{i,j\}$ and $\mathcal{I}\Leftarrow(\mathcal{I}-\{j\})\cup I$. (Note: this is the only way that a normal fallback 1-bin is grouped with other bins.)

Case 2: $\alpha_i\geq\alpha'$, *P_i is normal.* (If P_i is the last such bin then we add 2δ of weight for compensation.) $w(P_i)=s(P_i)-3\varepsilon$. If $\alpha_i\geq\alpha''$ then $d_i\geq(r-1)\alpha''-1=4\varepsilon$ and hence $w(d_i)\geq d_i-3\varepsilon\geq 4\varepsilon-3\varepsilon=\varepsilon$. By setting $I\Leftarrow\{i\}$ and $\mathcal{I}\Leftarrow\mathcal{I}\cup I$ we are finished. Hence we suppose $\alpha_i<\alpha''$.

Case 2.1: $|P_i^*|\geq 3$. If $P_i^*\cap X_1\neq\emptyset$ then $\alpha_i\geq(2r-1)+2$, which implies $d_i\geq(r-1)\alpha_i-1\geq(2r+1)(r-1)-1>3\varepsilon$. Hence

$$w(d_i)\geq d_i-3\varepsilon\geq 0.$$

If $P_i^*\cap X_1=\emptyset$ then $w(P_i^*)\leq s(P_i^*)-3\varepsilon$ and hence $w(d_i)\geq d_i+3\varepsilon-3\varepsilon\geq 0$. In both cases we are allowed to set $I\Leftarrow\{i\}$ and $\mathcal{I}\Leftarrow\mathcal{I}\cup I$.

Case 2.2: $P_i^*=(x,c)$.

(i) $P_i^*\cap X_1=\emptyset$. If P_i^* is abnormal, then $\alpha_i>2r+1$. Thus $d_i\geq(r-1)\alpha_i-1\geq(r-1)(2r+1)-1>2\varepsilon$. Since $w(P_i^*)\leq s(P_i^*)-2\varepsilon$ we have

$$w(d_i)\geq d_i+2\varepsilon-3\varepsilon>\varepsilon.$$

If P_i^* is normal then, by Lemma 3.12, $w(x)=s(x)-3\varepsilon$. Considering $w(c)\leq s(c)-\varepsilon$, we then have

$$w(d_i)\geq d_i+4\varepsilon-3\varepsilon>\varepsilon.$$

Therefore, in either case, we can set $I\Leftarrow\{i\}$ and $\mathcal{I}\Leftarrow\mathcal{I}\cup I$.

(ii) $|P_i^*\cap X_1|=2$. Let $x\in P_{j_1}$ and $c\in P_{j_2}$ $(1\leq j_1,j_2\leq l)$. Since $\alpha_i>2(2r-1)$ and $d_i\geq(r-1)\alpha_i-1\geq 2(r-1)(2r-1)-1>2\varepsilon$, we have

$$w(d_i)\geq d_i-3\varepsilon>-\varepsilon,$$

or

$$w(\bar{d}_i)<\varepsilon.$$

Hence it is valid to set $I\Leftarrow\{j_1,j_2,i\}$ and $\mathcal{I}\Leftarrow(\mathcal{I}-\{j_1\}-\{j_2\})\cup I$.

(iii) $|P_i^*\cap X_1|=1$. If c is of type X_1 then x has to be abnormal, causing that $\alpha_i>2r+2r-1>\alpha''$, which contradicts our assumption that $\alpha_i<\alpha''$. Hence we let $x\in P_j$ $(1\leq j\leq l)$ and $P_j^*=(a,b)$. We are to show that $w(d_i)\geq 0$ or $w(d_i)+w(d_j)\geq 0$ so as to set $I\Leftarrow\{i,j\}$ and $\mathcal{I}\Leftarrow(\mathcal{I}-\{j\})\cup I$. Suppose to the contrary that $w(d_i)<0$ and $w(d_i)+w(d_j)<0$.

As usual we let a' be of maximum size among a, b and c. Since P_i is normal, $a'\in P_k$ was packed before the first normal regular 3-bin, where P_k is a 1-bin or 2-bin. a' cannot be of type F_1 or F_2. Otherwise either $w(a')\leq s(a')-2\varepsilon$, implying that

$$w(d_i) + w(d_j) \geq d_i + 2\varepsilon - 3\varepsilon = d_i - \varepsilon > 0$$

since

$$d_i \geq (r-1)\alpha_i - 1 \geq (r-1)\alpha' - 1 \geq (2r - 1 + \mu_0)(r-1) - 1 > \varepsilon,$$

or P_k is abnormal, causing that P_k dominates P_i^*. Therefore a' is regular. Since a' is normal and not of type X_1, the remaining possibilities for P_k are to be normal fallback 1-bin, normal regular 2-bin and normal fallback 2-bin.

If P_k is a normal fallback 1-bin or 2-bin (not the last), then, by Lemmas 3.8 and 3.9,

$$s(a') > \min\{\tfrac{2}{3}r^2, 2r-1\} = \tfrac{2}{3}r^2 > 1 + 2\lambda.$$

Noting that $w(a') \leq s(a') - (\varepsilon + \delta)$, we then have

$$\alpha_i > 2r - 1 + \tfrac{2}{3}r^2,$$

$$d_i > (r-1)\alpha_i - 1 \geq (2r - 1 + \tfrac{2}{3}r^2)(r-1) - 1 > 2\varepsilon - \delta.$$

Hence

$$w(d_i) + w(d_j) \geq d_i + \varepsilon + \delta - 3\varepsilon > 0,$$

a contradiction.

If P_k is a normal regular 2-bin (not the last) then, by Lemma 3.9,

$$s(a') > \tfrac{1}{2}(\beta_k - 1) \geq r^2 - \tfrac{1}{2}.$$

Considering that $1 + 2\lambda < r^2 - \frac{1}{2}$, we then have $a' = c$. Hence

$$\alpha_i \geq (2r-1) + (r^2 - \tfrac{1}{2}),$$

$$d_i \geq (r-1)(2r + r^2 - \tfrac{3}{2}) - 1 \geq 3\varepsilon - (r-1)^2,$$

and

$$w(a') \leq s(a') - (r-1)^2,$$

we then have

$$w(d_i) + w(d_j) \geq d_i + (r-1)^2 - 3\varepsilon \geq 0,$$

a contradiction.

As for the case where P_k is the last normal regular 2-bin or fallback 2-bin, either the same happens as above, or $w(a') = s(a') - \Delta$, which apparently implies

$$w(d_i) + w(d_j) \geq d_i + \Delta - 3\varepsilon \geq \Delta - 3\varepsilon > 0.$$

Therefore, whatever type P_k may be, we always have a contradiction.

4.6. P_i is a fallback 3-bin

Let $|P_i| = 3 + k$ $(k \geq 1)$. Then $w(P_i) \geq s(P_i) - (3+k)(\varepsilon + \delta)$. Since $s(P_i^*) \leq \alpha_i$ and, by Lemma 3.9, $s(P_i) \geq k + b(P_i) \geq k + \frac{3}{4}\beta_i$, we have

$$s(P_i) - s(P_i^*) \geq k + (\tfrac{3}{4}r - 1)\alpha_i,$$

and hence

$$w(d_i) \geq d_i - (3+k)(\varepsilon+\delta) \geq k + (\tfrac{3}{4}r - 1)2r - (3+k)(\varepsilon+\delta) \geq (k+1)\varepsilon.$$

The last inequality holds because it is equivalent to

$$k \geq \frac{4\varepsilon + 3\delta - (\frac{3}{4}r-1)2r}{1-2\varepsilon-\delta} = \frac{12 - 14r - \frac{3}{2}r^2 + 15/r - 16(r-1)\lambda}{6r - 5/r - 4 + 6(r-1)\lambda},$$

which is valid since the right-hand side is less than 1.

4.7. P_i is a regular 4-bin

If P_i is abnormal then $s(P_i) > 2r+3$. Hence $d_i \geq \max\{2r+3-\alpha_i, (r-1)\alpha_i - 1\} \geq 2r+1-4/r > 4(\varepsilon+\delta)$. Since $w(P_i) = s(P_i) - 4(\varepsilon+\delta)$, we are done by Lemma 4.3. In the following we suppose P_i is normal.

Case 1: $P_i^* \cap X_1 = \emptyset$, *$P_i$ normal.* Then $d_i \geq \max\{4-\alpha_i, (r-1)\alpha_i - 1\} \geq 4 - 5/r > 3\varepsilon$. $w(P_i) = s(P_i) - 4\varepsilon$. We show that $w(d_i) \geq 2\varepsilon$ so as to set $I \Leftarrow \{i\}$ and $\mathcal{I} \Leftarrow \mathcal{I} \cup I$.

If $|P_i^*| \geq 3$ then $w(P_i^*) \leq s(P_i^*) - 3\varepsilon$ and hence

$$w(d_i) \geq d_i + 3\varepsilon - 4\varepsilon \geq 2\varepsilon.$$

Suppose $|P_i^*| = 2$. If P_i^* is normal then, by Lemma 3.12, $w(P_i^*[1]) = s(P_i^*[1]) - 3\varepsilon$, and hence $w(P_i^*) \leq s(P_i^*) - 4\varepsilon$, making

$$w(d_i) \geq d_i + 4\varepsilon - 4\varepsilon = d_i \geq 3\varepsilon.$$

Therefore we consider the case where $P_i^* = (a, b)$ is abnormal.

Let $a \in P_j$. Then P_j is before P_i since a could fit in P_i. If P_j contains more than 3 items, then $\beta_i \geq \beta_j > 2r+3$. Hence $d_i \geq (r-1)\alpha_i - 1 \geq (r-1)(2+3/r) - 1 > 4\varepsilon$, which implies that

$$w(d_i) \geq d_i + 2\varepsilon - 4\varepsilon \geq 2\varepsilon,$$

we are done. Hence we assume $|P_j| \leq 3$. Since P_j is abnormal, from the table we know that a satisfies $w(\cdot) \leq s(\cdot) - (\varepsilon+\delta)$, and hence $w(P_i^*) \leq s(P_i^*) - (2\varepsilon+\delta)$. Considering that $\alpha_i > 2r+1$ and $d_i \geq (r-1)\alpha_i - 1 \geq (2r+1)(r-1) - 1 \geq 4\varepsilon - \delta$, we then have

$$w(d_i) \geq d_i + (2\varepsilon+\delta) - 4\varepsilon \geq 2\varepsilon.$$

Case 2: $|P_i^* \cap X_1| \geq 2$, *P_i is normal.*

Case 2.1: $|P_i^*| \geq 3$. Since $\alpha_i > 2(2r-1)+1$ we have $d_i \geq (r-1)\alpha_i - 1 \geq (r-1) \times (4r-1) - 1 > 5\varepsilon$. Hence

$$w(d_i) \geq d_i - 4\varepsilon > \varepsilon,$$

allowing us to set $I \Leftarrow \{i\}$ and $\mathcal{I} \Leftarrow \mathcal{I} \cup I$.

Case 2.2: $|P_i^*| = 2$. Suppose $P_i^* = (x_1, x_2)$, where $x_1 \in P_{i_1}$, $x_2 \in P_{i_2}$ $(1 \leq i_1, i_2 \leq l)$, and $P_{i_1}^* = (a_1, b_1)$, $P_{i_2}^* = (a_2, b_2)$. If $w(d_i) \geq 0$, or $w(d_i) + w(d_{i_1}) + w(d_{i_2}) \geq 0$, then we can set $I \Leftarrow \{i, i_1, i_2\}$ and let $\mathscr{I} \Leftarrow (\mathscr{I} - \{i_1\} - \{i_2\}) \cup I$. So suppose to the contrary that $w(d_i) < 0$, and $w(d_i) + w(d_{i_1}) + w(d_{i_2}) < 0$.

Then $\alpha_i < \alpha''$ since otherwise $d_i \geq (r-1)\alpha_i - 1 \geq 4\varepsilon$, causing $w(d_i) \geq d_i - 4\varepsilon \geq 0$.

Let $a' \in \{a_1, a_2\}$ be of larger size. Then $a' \in P_j$ where P_j is a 1-, 2- or 3-bin before the first normal regular 4-bin (otherwise the union of the 4-bin, P_{i_1}, P_{i_2} would dominate $P_{i_1}^* \cup P_{i_2}^* \cup P_i^*$). If $w(a') \leq s(a') - (\varepsilon + \delta)$, since $d_i \geq \max\{4 - \alpha_i, (r-1)\alpha_i - 1\} \geq 4 - 5/r$, we would have

$$w(d_i) + w(d_{i_1}) + w(d_{i_2}) \geq d_i + \delta - 4\varepsilon \geq \left(4 - \frac{5}{r}\right) + \delta - 4\varepsilon = 0,$$

a contradiction. Hence $w(a') > s(a') - (\varepsilon + \delta)$, which implies that P_j is not a fallback bin since any item in these bins satisfies $w(\cdot) \leq s(\cdot) - (\varepsilon + \delta)$. Since $s(a') \leq 1 + 2\lambda_0 < r^2 - \frac{1}{2} = \frac{1}{2}(2r^2 - 1)$, the only possibility for P_j is to be a normal regular 3-bin and $\alpha_j \geq \alpha'$. By Lemma 3.9 we then have

$$s(a') > \tfrac{1}{3}(\beta_j - 1).$$

(Note: if P_j is the last such bin then $a' = P_j[1]$ and hence the inequality also holds by Lemma 3.3.) Since $\alpha_j \geq \alpha'$ we then obtain

$$1 + 2\lambda \geq s(a') > \tfrac{1}{3}(\beta_j - 1) \geq \tfrac{1}{3}(r\alpha' - 1) = \tfrac{1}{3}(r(2r - 1 + \mu_0 + 2r\lambda) - 1),$$

which implies that

$$\lambda > \frac{2r^2 - r + r\mu_0 - 4}{6 - 2r^2} = \lambda_0,$$

contradicting our Lemma 4.2.

Case 3: $|P_i^* \cap X_1| = 1$, *P_i is normal.* If P_i^* is abnormal then $\alpha_i > 2r + (2r-1)$, hence $d_i \geq (r-1)(4r-1) - 1 > 6\varepsilon$. We have

$$w(d_i) \geq d_i - 4\varepsilon > 2\varepsilon.$$

If $|P_i^*| \geq 3$ then $w(P_i^*) \leq s(P_i^*) - 2\varepsilon$ and hence

$$w(d_i) \geq d_i + 2\varepsilon - 4\varepsilon \geq \left(4 - \frac{5}{r}\right) - 2\varepsilon > \varepsilon.$$

In either case we are allowed to set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Suppose then that P_i^* is normal and $|P_i^*| = 2$. Let $T_j \in P_i^*$ $(1 \leq j \leq l)$, and $a' \in \{P_j^*[1], P_j^*[2], P_i^*[2]\}$ be of maximum size. It can be readily seen, by Lemma 3.2, that a' was packed in a normal 1-bin or 2-bin before P_i.

We are to show that $w(d_i) + w(d_j) \geq \varepsilon$ so that we can set $I \Leftarrow \{i, j\}$ and $\mathscr{I} \Leftarrow (\mathscr{I} - \{j\}) \cup I$. Suppose to the contrary that $w(d_i) + w(d_j) < \varepsilon$.

Then $w(a') > s(a') - (\varepsilon + \delta)$ otherwise $w(d_i) \geq d_i + (\varepsilon + \delta) - 4\varepsilon \geq \varepsilon$ (noting that $d_i \geq \max\{4 - \alpha_i, (r-1)\alpha_i - 1\} \geq 4 - 5/r$). But we have

$$w(d_i) \geq d_i + \varepsilon - 4\varepsilon \geq \varepsilon - \delta.$$

Hence we obtain

$$\text{(i) } w(d_j) < \delta; \quad \text{and} \quad \text{(ii) } d_i < 4\varepsilon.$$

Noting that $\bar{d}_j \leq \alpha_j - s(T_j) \leq \alpha_l - s(T_j) = 2 + 2\lambda - s(T_j)$ and that $w(d_j) \geq 2\varepsilon - \bar{d}_j$, we, from (i), obtain

$$s(T_j) < 2(1+\lambda) + \delta - 2\varepsilon. \tag{4.1}$$

If $a' \in P_k$ then, because of the fact that $w(a') > s(a') - (\varepsilon + \delta)$ and the fact that P_k is a 1- or 2-bin, P_k is a normal regular 2-bin with $|P_k^* \cap X_1| = 1$ and that $a' = P_i^*[2]$ since each item in P_j^* is in size $\leq 1 + 2\lambda < r^2 - \frac{1}{2} \leq \frac{1}{2}(\beta_k - 1)$. But $s(a') > \frac{1}{2}(\beta_k - 1)$ by Lemma 3.9. Considering (4.1) and $d_i \geq 4 - s(P_i^*) \geq 4 - (s(T_j) + s(a'))$, we then, from (ii), get

$$s(a') > 4 - 4\varepsilon - s(T_j) \geq 4 - 4\varepsilon - (2 + 2\lambda + \delta - 2\varepsilon)$$

$$= 2 - 2\varepsilon - 2\lambda - \delta > (\delta + \tfrac{1}{2})\frac{r}{r-1} - \tfrac{1}{2}.$$

Hence $w(a') \leq s(a') - (\varepsilon + \frac{7}{10}\delta)$, implying that

$$w(d_i) \geq d_i + \varepsilon + \tfrac{7}{10}\delta - 4\varepsilon \geq \varepsilon - \tfrac{3}{10}\delta.$$

Hence we obtain

$$\text{(i}'\text{) } w(d_j) < \tfrac{3}{10}\delta; \quad \text{and} \quad \text{(ii}'\text{) } d_i < 4\varepsilon - \tfrac{7}{10}\delta.$$

Exactly the same argument as above allows us to conclude that

$$s(a') < 4 - (4\varepsilon - \tfrac{7}{10}\delta) - (2 + 2\lambda + \tfrac{3}{10}\delta - 2\varepsilon)$$
$$= 2 - 2\varepsilon - 2\lambda + \tfrac{2}{5}\delta > (\tfrac{13}{10}\delta + \tfrac{1}{2})\,\frac{r}{r-1} - \tfrac{1}{2},$$

which implies that $w(a') \leq s(a') - (\varepsilon + \delta)$, a contradiction.

4.8. P_i is a k-bin ($k > 3$) and $|P_i| > 4$

Case 1: $|P_i| = k_i \geq 7$. Since $w(P_i) = s(P_i) - k_i\varepsilon$, it is enough, by Lemma 4.3, to show that $d_i > (2k_i - 4)\varepsilon$. Suppose to the contrary that $d_i \leq (2k_i - 4)\varepsilon$. Since $d_i \geq \max\{k_i - \alpha_i, (r-1)\alpha_i - 1\} \geq (k_i + 1)(1 - 1/r) - 1$ we then have

$$(2k_i - 4)\varepsilon \geq (k_i + 1)\left(1 - \frac{1}{r}\right) - 1,$$

or

$$k_i \leq \frac{1 - 4r\varepsilon}{(r-1) - 2r\varepsilon} \leq \frac{4r^2 - 6r + 1}{2r^2 - 2r - 1} < 7,$$

a contradiction.

Case 2: $|P_i|=6$. We have $w(P_i)=s(P_i)-6\varepsilon$, $d_i \ge \max\{6-\alpha_i, (r-1)\alpha_i-1\} \ge 6-7/r > 8\varepsilon-\delta > 7\varepsilon$.

Case 2.1: $|P_i^*| \ge 3$. If $|P_i^* \cap X_1| \le 1$ then, since $w(P_i^*) \le s(P_i^*)-2\varepsilon$, we have

$$w(d_i) \ge d_i + 2\varepsilon - 6\varepsilon > 3\varepsilon.$$

Set $I \Leftarrow \{i\}$ and $\mathcal{I} \Leftarrow \mathcal{I} \cup I$. If $|P_i^* \cap X_1| \ge 2$ then let $T_{j_1}, T_{j_2}, \in P_i^*$ $(1 \le j_1, j_2 \le l)$. Since

$$w(d_i) \ge d_i - 6\varepsilon > \varepsilon,$$

we can set $I \Leftarrow \{i, j_1, j_2\}$ and $\mathcal{I} \Leftarrow (\mathcal{I} - \{j_1\} - \{j_2\}) \cup I$.

Case 2.2: $|P_i^*|=2$. If P_i^* is *normal* then $s(P_i^*) \le 2s(T_1) \le 2\alpha_1 \le 2\alpha_l = 2(2+2\lambda)$. Hence $d_i = s(P_i) - s(P_i^*) \ge 6-(4+4\lambda) > 10\varepsilon$. If P_i is *abnormal* then $s(P_i) > 2r+5$ and hence $d_i > \max\{2r+5-\alpha_i, (r-1)\alpha_i - 1\} \ge (r-1)(2+6/r)-1 \ge 10\varepsilon$. In either case we have

$$w(d_i) \ge d_i - 6\varepsilon \ge 4\varepsilon,$$

and hence are allowed to set $I \Leftarrow \{i\}$, $\mathcal{I} \Leftarrow \mathcal{I} \cup I$.

If $\alpha_i \ge (1/(r-1))(1+8\varepsilon)$ then $d_i > (r-1)\alpha_i - 1 \ge 8\varepsilon$ and hence we are done by Lemma 4.3.

In the following we then suppose that P_i is normal but P_i^* not, and $\alpha_i < (1/(r-1))(1+8\varepsilon)$.

Let $P_i^* = (a, b)$. Then b is normal else $\alpha_i > 2 \cdot 2r > (1/(r-1))(1+8\varepsilon)$. If at least one one of a and b satisfies $w(\cdot) \le s(\cdot) - (\varepsilon+\delta)$ then

$$w(d_i) \ge d_i + (2\varepsilon+\delta) - 6\varepsilon > (8\varepsilon-\delta) - (4\varepsilon-\delta) = 4\varepsilon,$$

we are done. Therefore we suppose to the contrary that both a and b satisfy $w(\cdot) > s(\cdot) - (\varepsilon+\delta)$. Let $a \in P_j$. Then P_j is before P_i and $P_j[1] = a$.

(i) $b \notin X_1$. Since P_j is abnormal, P_j is then a k-bin $(k \ge 4)$ and $|P_j| \ge 5$. Since $s(b) > s(P_j[2])$ (otherwise P_j would dominate P_i^*), b was packed before P_j. Let $b \in P_{j'}$. Then $\alpha_{j'} \ge 2r$ since $w(b) > s(b) - (\varepsilon+\delta)$ and $b \notin X_1$. Since that $s(a) > \beta_{j'}$ would imply that $\beta_i \ge \beta_j \ge s(a) + 4 \ge 2r^2 + 4$, or $\alpha_i \ge 2r + 4/r$, contradicting our assumption that $\alpha_i < (1/(r-1))(1+8\varepsilon)$, we then have $s(a) \le \beta_{j'}$, which implies that $P_{j'}$ is also abnormal and $s(P_{j'}[1]) \ge s(a)$. This fact violates Lemma 3.2 since $P_{j'}$ dominates P_i^*.

(ii) $b \in X_1$. Let $b = T_{i'}$ $(1 \le i' \le l)$ and $P_{i'}^* = (a', b')$. We assert that at least one of a, a' and b' satisfies $w(\cdot) \le s(\cdot) + (\varepsilon+\delta)$, so that

$$w(d_{i'}) + w(d_i) \ge d_i + (\varepsilon+\delta) - 6\varepsilon \ge 3\varepsilon,$$

and we then can set $I \Leftarrow \{i, i'\}$ and $\mathcal{I} \Leftarrow (\mathcal{I} - \{i'\}) \cup I$. Suppose our assertion were false.

Since P_j is a k-bin $(k \ge 4)$ we have $s(a') > s(P_j[3])$ otherwise $P_{i'} \cup P_j$ would dominate $P_i^* \cup P_{i'}^*$. Hence at least one of a' and b', say a', was packed before P_j. Let $a' \in P_{j'}$ $(j' < j)$. Then the same argument as in case (i) above suggests that $P_{j'}$ is an abnormal k'-bin ($k' \geq 4$) and $P_{j'}[1] < a$, which implies that all regular items of $P_{j'}$ are abnormal, causing that $P_{j'}$ dominates P_i^*, a contradiction.

Case 3: $|P_i| = 5$. We have $w(P_i) = s(P_i) - 5\varepsilon$, and $d_i \geq \max\{5 - \alpha_i, (r-1)\alpha_i - 1\} \geq 5 - 6/r > 6\varepsilon - \delta > 5\varepsilon$.

Case 3.1: $|P_i^*| \geq 3$. If $|P_i^* \cap X_1| \leq 1$ then, since $w(P_i^*) \leq s(P_i^*) - 2\varepsilon$, we have

$$w(d_i) \geq d_i + 2\varepsilon - 5\varepsilon \geq 2\varepsilon.$$

Set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

If $|P_i^* \cap X_1| \geq 2$ then let $T_{j_1}, T_{j_2} \in P_i^*$ $(1 \leq j_1, j_2 \leq l)$. Since

$$w(d_i) \geq d_i - 5\varepsilon \geq 0,$$

we can set $I \Leftarrow \{i, j_1, j_2\}$ and $\mathscr{I} \Leftarrow (\mathscr{I} - \{j_1\} - \{j_2\}) \cup I$.

Case 3.2: $|P_i^*| = 2$. Then the same argument as at the beginning of Case 2.2 allows us to suppose P_i is normal, P_i^* is abnormal and $\alpha_i < (1/(r-1))(1+6\varepsilon)$. Let $P_i^* = (a, b)$. Then a is abnormal, b is normal and $w(b) \leq s(b) - \varepsilon$ ($b \in X_1$ would imply $\alpha_i > 2r + (2r-1) > (1/(r-1))(1+6\varepsilon)$. The same analysis as in Case 2.2(i) shows that at least one of a and b satisfies $w(\cdot) \leq s(\cdot) - (\varepsilon + \delta)$. Hence

$$w(d_i) \geq d_i + (2\varepsilon + \delta) - 5\varepsilon \geq 3\varepsilon,$$

allowing us to set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Now we are ready to prove our main result.

Theorem 4.4. *There cannot be any r-counterexample.*

Proof. If the theorem fails then we have a minimal r-counterexample. Let $\mathscr{I}$ be our partition of $\{1, \ldots, m\}$ as was given in this section. We classify $\mathscr{I}$ into classes $\mathscr{I}_1$ and $\mathscr{I}_2$:

$$\mathscr{I}_1 = \left\{ I \in \mathscr{I} \colon \sum_{i \in I} w'(d_i) \geq k_I \varepsilon,\ k_I = \max\{0, \sum_{i \in I} (|P_i| - |P_i^*|)\} \right\},$$

$$\mathscr{I}_2 = \left\{ J \in \mathscr{I} \colon J \notin \mathscr{I}_1,\ \bar{k}_J = \sum_{j \in J} (|P_j^*| - |P_j|) > 0,\ \sum_{j \in J} w'(\bar{d}_j) < \bar{k}_J \varepsilon \right\},$$

where w' stands for the new weight after compensation.

As we have seen, after at most

$$p = (\Delta - (r-1)^2 + \tfrac{9}{100}) + (\Delta - (2\varepsilon + \delta + \tfrac{99}{1000})) + 2\delta$$

of weight compensation, we have $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$; namely,

$$\sum_{I \in \mathscr{I}_1} \sum_{i \in I} w'(d_i) + \sum_{J \in \mathscr{I}_2} \sum_{j \in J} w'(d_j) = \sum_{i=1}^{m} w'(d_i) \leq \sum_{i=1}^{m} w(d_i) + p. \tag{4.2}$$

Since

$$\sum_{I\in\mathcal{I}_1} k_I - \sum_{J\in\mathcal{I}_2} \bar{k}_J \ge \sum_{I\in\mathcal{I}_1}\sum_{i\in I}(|P_i|-|P_i^*|) - \sum_{J\in\mathcal{I}_2}\sum_{j\in J}(|P_j^*|-|P_j|)$$
$$= \sum_{i=1}^{m}|P_i| - \sum_{j=1}^{m}|P_j^*| = -1,$$

we have

$$\sum_{i=1}^{m} w'(d_i) = \sum_{I\in\mathcal{I}_1}\sum_{i\in I} w'(d_i) - \sum_{J\in\mathcal{I}_2}\sum_{j\in J} w'(\bar{d}_j)$$
$$\ge \sum_{I\in\mathcal{I}_1} k_I\varepsilon - \sum_{J\in\mathcal{I}_2}\bar{k}_J\varepsilon \ge -\varepsilon. \tag{4.3}$$

Since the last item has weight $1-\varepsilon$, we have

$$(1-\varepsilon) + \sum_{i=1}^{m} w(P_i) = \sum_{j=1}^{m} w(P_j^*),$$

or

$$\sum_{i=1}^{m} w(d_i) = -(1-\varepsilon). \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we then obtain

$$p-(1-\varepsilon)\ge -\varepsilon,$$

or

$$(\Delta-(r-1)^2+\tfrac{9}{100}) + (\Delta-(2\varepsilon+\delta+\tfrac{99}{1000})) + 2\delta \ge 1-2\varepsilon. \tag{4.5}$$

After simple calculation we know that (4.5) is impossible. Therefore we have a contradiction, which proves our theorem. □

5. Supplement: proofs of two lemmas

In this section we provide the proof sketch of two lemmas stated in the last section. Since the proofs run parallel to those in the last section, we are often referred back correspondingly. All notations remain the same unless otherwise specified.

Lemma 3.13. *Let $(\sqrt{3}+1)/2 = r_0 \le r \le 1.4$. Then in any minimal r-counterexample, $s(T_l) < 2$.*

Proof. We suppose there exists a minimal r-counterexample, in which $s(T_l)\ge 2$. Then, by the argument before Lemma 3.11, all items of type X_1 satisfy $s(\cdot)\ge 2$. We are to deduce a contradiction. Let $\varepsilon = 3r_0 - 4$. Without loss of generality we may assume $r=r_0$ since the changes in the proof are only those of some $=$'s to $\le$'s (e.g., $1/(r-1)=2r$ to $1/(r-1)\le 2r$). The weight function w is now given by Table 2.

We are to show that, for almost all $1\le i\le m$, we have $w(P_i)\ge w(P_i^*)$. Our desired contradiction can then be deduced.

Table 2

Item type	$\alpha_i < 2r$	$\alpha_i \geq 2r$
X_1	s	–
Y_1, F_1	$s-4\varepsilon$	$s-\varepsilon$, $s-2\varepsilon$
X_2[a]	–	$s-2\varepsilon$, if P_i normal, $\alpha_i<3$ $s-\varepsilon$, otherwise
Y_2, F_2	–	$s-2\varepsilon$
others	–	$s-\varepsilon$

[a] If a is the second item of the last normal regular 2-bin, then $w(a)=s-2\varepsilon$.

5.1. P_i is a regular 1-bin

Since $w(P_i^*) \leq s(P_i^*) - 2\varepsilon$ by Lemma 3.11 and

$$\bar{d}_i \leq \bar{d}_l \leq \min\{\alpha_l - s(T_l), 1-(r-1)\alpha_l\}$$

$$\leq \min\{\alpha_l - 2, 1-(r-1)\alpha_l\} \leq \frac{3}{r} - 2 = 2\varepsilon,$$

we have

$$w(d_i) \geq 2\varepsilon - \bar{d}_i \geq 0.$$

5.2. P_i is a fallback 1-bin

Case 1: $\alpha_i < 2r$. Then $w(P_i^*) \leq s(P_i^*) - 2\varepsilon$ by Lemma 3.11 and $w(P_i) = s(P_i) - 8\varepsilon$. Since, by Lemma 3.8, $d_i > 1-(3-2r)r = 2-r > 6\varepsilon$, we have

$$w(d_i) \geq d_i + 2\varepsilon - 8\varepsilon > 0.$$

Case 2: $\alpha_i \geq 2r$.

Case 2.1: $|P_i| = k \geq 3$. Then $w(P_i) = s(P_i) - (2k-1)\varepsilon$. Since $s(P_i) > (k-1) + \frac{1}{2}\beta_i$, we have

$$d_i \geq \max\{(k-1) - (1-\tfrac{1}{2}r)\alpha_i, (r-1)\alpha_i - 1\}$$

$$\geq 2k\frac{r-1}{r} - 1 > (2k-1)\varepsilon.$$

Case 2.2: $|P_i| = 2$. Then $w(P_i) = s(P_i) - 3\varepsilon$. Suppose to the contrary that $w(d_i) < 0$. Then we have $d_i < 3\varepsilon$. Further we have

(i) $|P_i^* \cap X_1| \leq 1$ otherwise $\alpha_i > 2 \cdot 2 = 4 > (1/(r-1))(1+3\varepsilon)$, which implies $d_i \geq (r-1)\alpha_i - 1 \geq 3\varepsilon$, contradicting $d_i < 3\varepsilon$.

(ii) $\alpha_i < (1/(r-1))(1+2\varepsilon)$ and $d_i < 2\varepsilon$. Otherwise $d_i \geq 2\varepsilon$. Since (i) implies that $w(P_i^*) \leq s(P_i^*) - \varepsilon$, we then have $w(d_i) \geq d_i + \varepsilon - 3\varepsilon \geq d_i - 2\varepsilon \geq 0$.

(iii) P_i is normal. Otherwise $d_i \geq \max\{2r+1-\alpha_i, (r-1)\alpha_i - 1\} \geq 2r - 1 - 2/r = 3 - 2r = 2r\varepsilon > 2\varepsilon$, contradicting (ii).

If $|P_i^*| \geq 3$ then by (ii), $P_i^* \cap X_1 = \emptyset$. Hence $w(P_i^*) \leq s(P_i^*) - 3\varepsilon$, causing $w(d_i) \geq d_i + 3\varepsilon - 3\varepsilon \geq 0$, a contradiction. Therefore, $|P_i^*| = 2$. Let $P_i^* = (a, b)$ and $P_i = (u, v)$.

Suppose first that $s(a) \leq s(u)$. Then $s(b) > s(v)$. b has to be after P_i since otherwise the bin it packed to would dominate P_i^*. Hence b could not fit in P_i:

$$s(u) + s(b) > \beta_i.$$

From (ii) we also have

$$s(u) + s(v) < \alpha_i + 2\varepsilon.$$

Combining the two inequalities we get

$$s(b) > s(v) - 2\varepsilon + (r-1)\alpha_i \geq 1 - 2\varepsilon + (r-1)2r = 2 - 2\varepsilon,$$

which implies $\alpha_i \geq 2s(b) > 2(2 - 2\varepsilon)$, contradicting (ii).

Suppose now $s(a) > s(u)$. Let $a \in P_j$. Since, by (iii), P_i is normal, P_j is a 1-bin and a is regular, that is, a is of type X_1 or Y_1.

If $a \in Y_1$ then $P_i^* \cap X_1 = \emptyset$, hence $w(P_i^*) \leq s(P_i^*) - 2\varepsilon$. We may assume $\alpha \geq 2r$. Then from the proof of Corollary 3.14 we can see that $s(u) > 2$, which is independent of Lemma 3.13.

Hence $\alpha_i \geq s(a) + s(b) > 3$, which implies that $d_i \geq (r-1)\alpha_i - 1 \geq 3(r-1) - \varepsilon$. We then have $w(d_i) \geq d_i + 2\varepsilon - 3\varepsilon \geq 0$, contradiction. Hence $a \in X_1$. Since $s(a) > s(u) > \frac{1}{2}\beta_i$, we have $\alpha_i > \frac{1}{2}r\alpha_i + 1$, implying that

$$\alpha_i > \frac{2}{2-r} \quad \text{and} \quad s(a) > \frac{r}{2-r}.$$

Then we have

$$\bar{d}_j \leq \min\{\alpha_j - s(a), 1 - (r-1)\alpha_j\}$$

$$\leq \min\left\{\alpha_j - \frac{r}{2-r}, 1 - (r-1)\alpha_j\right\} \leq \frac{2}{(2-r)r} - \frac{r}{2-r} = \frac{\varepsilon}{2-r},$$

and

$$d_i \geq (r-1)\alpha_i - 1 > \frac{2}{2-r}(r-1) - 1.$$

Considering

$$\frac{2}{2-r}(r-1) - 1 = \frac{1}{(2-r)r} - 1 = \frac{(r-1)^2}{(2-r)r} = \frac{\varepsilon}{2-r},$$

we obtain

$$w(d_i) + w(d_j) \geq d_i - \bar{d}_j + 3\varepsilon - 3\varepsilon = d_i - \bar{d}_j > 0.$$

We are then done.

5.3. P_i *is a regular 2-bin*

(At most ε of weight compensation is needed in this class of bins.)

If P_i is abnormal then, since $w(P_i) = s(P_i) - 2\varepsilon$ and

$$d_i \geq \max\{2r+1-\alpha_i, (r-1)\alpha_i\} \geq 2r-1-\frac{2}{r} = 3-2r > 2\varepsilon,$$

we have

$$w(d_i) \geq d_i - 2\varepsilon \geq 0.$$

Hence in the following we suppose P_i is normal.

Case 1: $\alpha_i < 3$. Then P_i^* is normal and $|P_i^*| = 2$. Considering that $P_i^* \cap X_1 = \emptyset$, we then have, by Lemma 3.12, $w(P_i^*[1]) = s(P_i^*[1]) - 4\varepsilon$. Hence $w(P_i^*) \leq s(P_i^*) - 5\varepsilon$ and thus

$$w(d_i) \geq d_i + 5\varepsilon - 4\varepsilon \geq 5\varepsilon - 4\varepsilon > 0.$$

Case 2: $\alpha_i \geq 3$. (If P_i is the last normal regular 2-bin then we add ε of weight to make $w(P_i) = s(P_i) - 2\varepsilon$.) If $\alpha_i \geq (1/(r-1))(1+2\varepsilon)$ then $d_i \geq 2\varepsilon$, which implies that $w(d_i) \geq d_i - 2\varepsilon \geq 0$. If $\alpha_i < (1/(r-1))(1+2\varepsilon)$ then $\alpha_i < 4$, implying that $|P_i^* \cap X_1| \leq 1$. Hence $w(P_i^*) \leq s(P_i^*) - \varepsilon$. But $d_i \geq 3(r-1) - 1 = \varepsilon$, we then have $w(d_i) \geq d_i + \varepsilon - 2\varepsilon \geq 0$.

5.4. P_i *is a fallback* 2-*bin*

Let $|P_i| = k+2$ $(k \geq 1)$. Then $w(P_i) = s(P_i) - 2(k+2)\varepsilon$. Since $s(P_i) \geq k + \frac{2}{3}\beta_i$ we have

$$d_i \geq \max\{k - (1 - \tfrac{2}{3}r)\alpha_i, (r-1)\alpha_i - 1\}$$

$$\geq 3(k+1)\frac{r-1}{r} - 1 > 2(k+2)\varepsilon.$$

Hence $w(d_i) \geq d_i - 2(k+2)\varepsilon > 0$.

5.5. P_i *is a regular* 3-*bin*

We may assume that $w(d_i) < 0$. Then $d_i < 3\varepsilon$ since $w(P_i) = s(P_i) - 3\varepsilon$. Then we have:

(i) Both P_i and P_i^* are normal. Otherwise $d_i \geq \min\{(2r+2)/r, 2r+1\} = 2 + 2/r = 4r - 2 > 3\varepsilon$.

(ii) $|P_i^* \cap X_1| \leq 1$. $|P_i^* \cap X_1| \geq 2$ implies that $\alpha_i > 4$ and thus $d_i > 4(r-1) - 1 \geq 3\varepsilon$.

(iii) $|P_i^*| = 2$. If $|P_i^*| \geq 3$ then, by (ii) $w(P_i^*) \leq s(P_i^*) - 2\varepsilon$. Noting that $d_i \geq 3(r-1) - 1 = \varepsilon$, we would have $w(d_i) \geq d_i + 2\varepsilon - 3\varepsilon \geq 0$.

Let $P_i^* = (x, c)$. x has to be of type X_1 since otherwise by Lemma 3.12, $w(x) = s(x) - 4\varepsilon$, implying $w(d_i) \geq d_i + 4\varepsilon - 3\varepsilon > 0$. Suppose $x \in P_j$ $(1 \leq j \leq l)$. Let $P_j^* = (a, b)$ and a' be of maximum size among a, b and c. If a' satisfies $w(\cdot) \geq s(\cdot) - 2\varepsilon$ then, considering $\alpha_i > 3$, which implies $d_i \geq \varepsilon$ we are done, since

$$w(d_i) + w(d_j) \geq d_i + 2\varepsilon - 3\varepsilon \geq 0.$$

Therefore we suppose $w(a') = s(a') - \varepsilon$.

Let $a' \in P_k$. Then P_k is before the first normal regular 3-bin, which implies that

P_k is a 1-bin or 2-bin. a' is a regular item by the fact that $w(a')=s(a')-\varepsilon$. Then P_k cannot be a 1-bin since otherwise a' is of type Y_1 and hence either $w(a')=s(a')-4\varepsilon$ if $\alpha_k<2r$, or $s(a')>2$ according to the proof of Corollary 3.14, which implies that $a'=c$ and $\alpha_i>4$ and thus $d_i>4(r-1)-1>3\varepsilon$. In either case we have a contradiction.

Hence P_k is a 2-bin. P_k cannot be abnormal else it would dominate P_i^*. P_k also cannot be fallback by the restriction $w(a')=s(a')-\varepsilon$. Therefore P_k can only be a normal regular 2-bin and $\alpha_k\geq 3$. Hence by Lemma 3.9,

$$s(a') > \tfrac{1}{2}(\beta_k - 1) \geq \tfrac{1}{2}(3r-1).$$

(Note: if P_k is the last such bin then $a'=P_i[1]$ and hence the above also holds.) Therefore we obtain

$$\alpha_i + \alpha_j \geq s(a)+s(b)+s(c)+s(x) > 2 + \frac{3r-1}{2} + 2 > 4r,$$

which implies that

$$d_i + d_j \geq (r-1)(\alpha_i+\alpha_j) - 2 > 4(r-1)r - 2 = 0,$$

or

$$w(d_i) + w(d_j) \geq d_i + d_j + 3\varepsilon - 3\varepsilon > 0.$$

5.6. $|P_i|>3$ and P_i is a k-bin ($k>2$)

Case 1: $|P_i|=k_i\geq 5$. Since $w(P_i)=s(P_i)-k_i\varepsilon$ and $d_i\geq\max\{k_i-\alpha_i,(r-1)\alpha_i-1\}\geq (k_i+1)(r-1)/r-1\geq k_i\varepsilon$, we have $w(d_i)\geq d_i-k_i\varepsilon>0$.

Case 2: $|P_i|=4$. Then $w(P_i)=s(P_i)-4\varepsilon$. If $w(P_i^*)\leq s(P_i^*)-\varepsilon$ then $w(d_i)\geq d_i+\varepsilon-4\varepsilon>0$ since we have $d_i\geq\max\{4-\alpha_i,(r-1)\alpha_i-1\}\geq 4-5/r>3\varepsilon$. If $w(P_i^*)=s(P_i^*)$ then $\alpha_i>4$, implying that $d_i\geq 4(r-1)-1>4\varepsilon$ and that $w(d_i)\geq d_i-4\varepsilon>0$.

In conclusion we have proved that

$$\varepsilon + \sum_{i=1}^{m} w(d_i) \geq 0.$$

Since the last item has weight $1-\varepsilon$ and the fact that

$$\sum_{i=1}^{m} w(P_i) + (1-\varepsilon) = \sum_{j=1}^{m} w(P_j^*),$$

we get

$$-(1-\varepsilon) = \sum_{i=1}^{m} w(d_i) \geq -\varepsilon,$$

or $1\leq 2\varepsilon$, which is our desired contradiction. Our lemma is then proved. □

Lemma 4.2. *Let r be the positive root of equation* $2r^3+4r^2-5r-6=0$ *and* $\lambda_0=(2r+1)/(4(r^2-1))-1$. *Then in any minimal r-counterexample,* $\lambda<\lambda_0$.

Table 3
$s_0=(\delta+\frac{1}{2})r/(r-1)-\frac{1}{2}$

Item type	Weight	
X_1	s	
Y_1, F_1, Y_2, F_2	$s-\Delta$	
X_2[a]	$s-(\varepsilon+\frac{1}{2}\delta),\ s\le s_0,$ $s-(\varepsilon+\delta),\ s>s_0,$	P_i normal
	$s-\Delta,$	otherwise
others	$s-\varepsilon$	

[a] If a is the second item of the last normal regular 2-bin, then $w(a)=s(a)-\Delta$.

Proof. We prove this lemma in essentially the same way as we did in the last section. So suppose we have a minimal r-counterexample, in which $\lambda\ge\lambda_0$. In the remaining of the paper, r is again exclusively used to represent the indicated value. Let

$$\varepsilon=\tfrac{3}{2}-r-(r-1)\lambda_0=\frac{1}{4(r+1)}=0.104975\ldots,$$

$$\delta=4\varepsilon-\left(4-\frac{5}{r}\right)=\frac{1}{r+1}-4+\frac{5}{r}=0.039153\ldots,$$

$$\Delta=3\varepsilon-\left(3-\frac{4}{r}\right)=\frac{3}{4(r+1)}-3+\frac{4}{r}=0.210327\ldots.$$

Our weight function w is now changed to be as in Table 3. Initially we set $\mathscr{I}\Leftarrow\emptyset$.

5.1′. P_i is a regular 1*-bin*

Since $w(P_i^*)\le s(P_i^*)-2\varepsilon$ by Lemma 3.11 and $\bar{d}_i\le 3-2r-2(r-1)\lambda\le 2\varepsilon$, we have $w(d_i)\ge 2\varepsilon-\bar{d}_i\ge 0$.

Set $I=\{i\}$ and $\mathscr{I}\Leftarrow\mathscr{I}\cup I$.

5.2′. P_i is a fallback 1*-bin*

Case 1: $\alpha_i<2r$. By Lemma 3.8 we have $w(P_i)=s(P_i)-2\Delta$ and $d_i>1-r(3-2r)>2\Delta$. Considering that $w(P_i^*)\le s(P_i^*)-2\varepsilon$, we have $w(d_i)\ge d_i+2\varepsilon-2\Delta>0$.

Set $I=\{i\}$ and $\mathscr{I}\Leftarrow\mathscr{I}\cup I$.

Case 2: $\alpha_i\ge 2r$.

Case 2.1: $|P_i|=k\ge 4$. Since $w(P_i)=s(P_i)-k\Delta$, we are done by Lemma 4.3 if $d_i>k\Delta+(k-4)\varepsilon$. So suppose $d_i\le k\Delta+(k-4)\varepsilon$. Then

$$(r-1)\alpha_i-1\le k(\Delta+\varepsilon)-4\varepsilon\quad\text{and}\quad 2(k-1)-\alpha_i\le k(\Delta+\varepsilon)-4\varepsilon.$$

From the above inequalities we get

$$k \le \frac{(r-1)(2-4\varepsilon)+(1-4\varepsilon)}{(r-1)(2-\Delta-\varepsilon)-(\Delta+\varepsilon)} < 4,$$

a contradiction.

Case 2.2: $|P_i| = 3$. Since $s(P_i) > 2r+2$ by Corollary 3.14, we have $d_i \ge \max\{2r+2-\alpha_i, (r-1)\alpha_i - 1\} \ge 2r - 3/r > 3\Delta - \varepsilon$. Considering that $w(P_i) = s(P_i) - 3\Delta$, we are done by Lemma 4.3.

Case 2.3: $|P_i| = 2$. Since $s(P_i) > 2r+1$ by Corollary 3.14, we have $d_i \ge \max\{2r+1-\alpha_i, (r-1)\alpha_i - 1\} \ge 2r - 1 - 2/r > 2\Delta - 2\varepsilon$. Considering that $w(P_i) = s(P_i) - 2\Delta$, we are done.

5.3′. P_i is a regular 2-bin

(At most Δ of weight is donated to this class for compensation.)

Let $P_i = (u_1, u_2)$ and

$$\theta(x) = \begin{cases} 1, & x>0, \\ 0, & x\le 0; \end{cases} \quad s_0 = (\delta + \tfrac{1}{2})\frac{r}{r-1} - \tfrac{1}{2};$$

$$\theta_t = \theta(s(u_t) - s_0) \quad (t = 1, 2).$$

Case 1: *P_i is not the last normal regular* 2-*bin*. Then

$$s(P_i) = s(P_i) - (2\varepsilon + \delta + \tfrac{1}{2}\delta(\theta_1 + \theta_2)),$$

and

$$d_i \ge \max\{s(u_1) + s(u_2) - \alpha_i, (r-1)\alpha_i - 1\} \ge (s(u_1) + s(u_2) + 1)\frac{r-1}{r} - 1$$

$$\ge \sum_{t=1}^{2} (s(u_t) - s_0)\frac{r-1}{r} + \left((2s_0 + 1)\frac{r-1}{r} - 1\right) = \sum_{t=1}^{2} (s(u_t) - s_0)\frac{r-1}{r} + 2\delta.$$

Since

$$s(u_t) > \tfrac{1}{2}(\beta_i - 1) \ge r^2 - \tfrac{1}{2} > (\tfrac{1}{2}\delta + \tfrac{1}{2})\frac{r}{r-1} - \tfrac{1}{2},$$

we have

$$-\tfrac{1}{2}\delta < (s(u_t) - s_0)\frac{r-1}{r} \le 0 \quad \text{iff} \quad \theta_t = 0 \quad (t = 1, 2).$$

Hence

$$d_i \ge \delta + \tfrac{1}{2}\delta(\theta_1 + \theta_2).$$

We are done by Lemma 4.3.

Case 2: *P_i is the last normal regular* 2-*bin*. Then by adding Δ weight to its second item we have $w(P_i) = s(P_i) - (\varepsilon + \tfrac{1}{2}\delta(1 + \theta_1))$. Since

$$d_i \ge \max\{s(u_1) + s(u_2) - \alpha_i, (r-1)\alpha_i - 1\} \ge (s(u_1) + s(u_2) + 1)\frac{r-1}{r} - 1$$

$$\geq (s(u_1)+2)\frac{r-1}{r}-1 = (s(u_1)-s_0)\frac{r-1}{r}+\left((s_0+\tfrac{1}{2})\frac{r-1}{r}-\tfrac{1}{2}\right)-\left(\frac{3}{2r}-1\right)$$

$$=\tfrac{1}{2}\delta(1+\theta_1)-\left(\frac{3}{2r}-1\right)>\tfrac{1}{2}\delta(1+\theta_1)-\varepsilon,$$

we are done by Lemma 4.3.

5.4′. P_i is a fallback 2-bin

Let $|P_i|=2+k$ $(k\geq 1)$. The fact that $s(P_i)\geq k+\frac{2}{3}\beta_k$ implies that

$$d_i \geq \max\{k-(1-\tfrac{2}{3}r)\alpha_i,(r-1)\alpha_i-1\}$$

$$\geq 3(k+1)\frac{r-1}{r}-1>(k+2)\Delta+(k-2)\varepsilon,$$

which, together with the fact that $w(P_i)=s(P_i)-(k+2)\Delta$ and Lemma 4.3, allows us to be done.

5.5′. P_i is a regular 3-bin

Case 1: $|P_i^*|\geq 3$. If $P_i^*\cap X_1\neq\emptyset$ then $\alpha_i>2r+1$, which implies $d_i\geq(2r+1)(r-1)-1>3\varepsilon$ and hence $w(d_i)\geq d_i-3\varepsilon>0$. If $P_i^*\cap X_1=\emptyset$ then $w(P_i^*)\leq s(P_i^*)-3\varepsilon$, which implies $w(d_i)\geq d_i+3\varepsilon-3\varepsilon>0$. In either case we are allowed to set $I\Leftarrow\{i\}$ and $\mathcal{I}\Leftarrow\mathcal{I}\cup I$.

Case 2: $|P_i^*|=2$. If $w(d_i)\geq\varepsilon$ then we can set $I\Leftarrow\{i\}$ and $\mathcal{I}\Leftarrow\mathcal{I}\cup I$. So we assume $w(d_i)<\varepsilon$. Then (i) P_i is normal else $d_i\geq\max\{2r+2-\alpha_i,(r-1)\alpha_i-1\}\geq 2r-3/r>\varepsilon$, which implies that $w(d_i)\geq d_i-3\varepsilon\geq\varepsilon$. And (ii) $P_i^*\cap X_1\neq\emptyset$ since otherwise, if P_i^* is abnormal, $\alpha_i>2r+1$, which implies that $d_i>(2r+1)(r-1)-1>2\varepsilon$ and thus $w(d_i)\geq d_i+2\varepsilon-3\varepsilon>\varepsilon$; or by Lemma 3.12, if P_i^* is normal, $w(P_i^*[1])=s(P_i^*[1])-\Delta$, which implies that $w(d_i)\geq d_i+(\Delta+\varepsilon)-3\varepsilon\geq d_i+\Delta-2\varepsilon\geq\max\{3-\alpha_i,(r-1)\alpha_i\}+(\Delta-2\varepsilon)\geq(3-4/r)+\Delta-2\varepsilon=\varepsilon$.

Case 2.1: $|P_i^*\cap X_1|=2$. Since $\alpha_i>2(2r-1)$ we have $d_i\geq 2(2r-1)(r-1)-1>2\varepsilon$. Hence we are done by Lemma 4.3.

Case 2.2: $|P_i^*\cap X_1|=1$. We may assume $P_i^*=(T_j,c)$ $(1\leq j\leq l)$. We show that $w(d_i)\geq 0$ or $w(d_i)+w(d_j)\geq 0$ so as to set $I\Leftarrow\{i,j\}$ and $\mathcal{I}\Leftarrow(\mathcal{I}-\{j\})\cup I$. Let $P_j^*=(a,b)$ and a' be of maximum size among a, b and c. Then a' was packed before the first normal regular 3-bin. Let $a'\in P_k$, where P_k is a 1- or 2-bin. If $w(a')=s(a')-\Delta$ then

$$w(d_i)+w(d_j)\geq d_i+\Delta-3\varepsilon\geq\left(3-\frac{4}{r}\right)+\Delta-3\varepsilon=0,$$

we are done. So we suppose $w(a')>s(a')-\Delta$. Then P_k is a normal regular 2-bin. By Lemma 3.9 we then have $s(a')>\frac{1}{2}(\beta_k-1)\geq r^2-\frac{1}{2}$. Hence

$$\alpha_i+\alpha_j \geq 2+(r^2-\tfrac{1}{2})+(2r-1+2r\lambda_0) = r^2+\frac{(2r+1)r}{2(r^2-1)}+\tfrac{1}{2},$$

which implies that

$$d_i+d_j \geq (r-1)\left(r^2+\frac{(2r+1)r}{2(r^2-1)}+\tfrac{1}{2}\right)-2 \geq 0,$$

and hence, considering that $\{a,b,c\} \cap X_1 = \emptyset$, we have

$$w(d_i)+w(d_j) \geq d_i+d_j+3\varepsilon-3\varepsilon \geq 0.$$

5.6′. P_i is a fallback 3-bin

Let $|P_i|=3+k$ $(k \geq 1)$. Since $s(P_i)>k+\frac{3}{4}\beta_i$ by Lemma 3.9, we have $d_i \geq k+(\frac{3}{4}r-1)\alpha_i \geq k+2r(\frac{3}{4}r-1)>(2k-1)\varepsilon$. Considering that $w(P_i)=s(P_i)-(k+3)\varepsilon$, we are done by Lemma 4.3.

5.7′. P_i is a regular 4-bin

If P_i is abnormal then $d_i>\max\{2r+3-\alpha_i,(r-1)\alpha_i-1\} \geq 2r+1-4/r>6\varepsilon$, implying $w(d_i) \geq d_i-4\varepsilon \geq 2\varepsilon$. If $|P_i^*| \geq 3$ then either, if $|P_i^* \cap X_1| \leq 1$, $w(P_i^*) \leq s(P_i^*)-2\varepsilon$, which implies $w(d_i) \geq d_i+2\varepsilon-4\varepsilon>(4-5/r)-2\varepsilon>\varepsilon$; or, if $|P_i^* \cap X_1| \geq 2$, $\alpha_i>2(2r-1)+1$, which implies that $d_i>(r-1)(4r-1)-1>5\varepsilon$ and hence $w(d_i) \geq d_i-4\varepsilon>\varepsilon$. In all above cases we can set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

So we suppose P_i is normal and $|P_i^*|=2$.

If $|P_i^* \cap X_1|=2$ then $\alpha_i>2(2r-1+2r\lambda_0)$, which implies that $d_i>2(r-1) \times (2r-1+2r\lambda_0)-1=4\varepsilon$, we are then done by Lemma 4.3. If $P_i^* \cap X_1=\emptyset$ then either, if P_i^* is abnormal, $\alpha_i>2r+1$, which implies that $d_i>(r-1)(2r+1)-1>4\varepsilon$ and thus $w(d_i) \geq d_i+2\varepsilon-4\varepsilon \geq 2\varepsilon$; or by Lemma 3.12, if P_i^* is normal, $w(P_i^*[1])= s(P_i^*[1])-\Delta$, which implies that $w(d_i) \geq d_i+(\Delta+\varepsilon)-4\varepsilon>(4-5/r)-\varepsilon>2\varepsilon$. In either case we are allowed to set $I \Leftarrow \{i\}$ and $\mathscr{I} \Leftarrow \mathscr{I} \cup I$.

Therefore we further suppose $|P_i^* \cap X_1|=1$. Then we may assume $P_i^*=(T_j,c)$ $(1 \leq j \leq l)$. Let $P_j^*=(a,b)$ and a' be of maximum size among a, b and c. Let $a' \in P_k$. Then P_k is a 1- or 2-bin before P_i. We are to show that $w(d_i)+w(d_j) \geq \varepsilon$ so that we can set $I \Leftarrow \{i,j\}$ and $\mathscr{I} \Leftarrow (\mathscr{I}-\{j\}) \cup I$.

Suppose to the contrary that $w(d_i)+w(d_j)<\varepsilon$. Then $w(a')>s(a')-(\varepsilon+\delta)$ since otherwise $w(d_i)+w(d_j) \geq d_i+(\varepsilon+\delta)-4\varepsilon \geq (4-5/r)+\delta-3\varepsilon=\varepsilon$. Hence P_k is a normal regular 2-bin. Considering that

$$w(d_i) \geq d_i+\varepsilon-4\varepsilon \geq (r-5/r)-3\varepsilon \geq \varepsilon-\delta,$$

we then obtain

(i) $w(d_j)<\delta$; and (ii) $d_i<4\varepsilon$.

Noting that $\lambda<3/(2r)-1$ by Corollary 3.15, we then, from the analysis of the same case in the last section, conclude that

$$s(a') > 4 - 4\varepsilon - s(T_j) \geq 4 - 4\varepsilon - (2 + 2\lambda + \delta - 2\varepsilon)$$
$$= 2 - 2\varepsilon - 2\lambda - \delta > 2 - 2\varepsilon - \frac{3-2r}{r} - \delta$$
$$= 8\left(1 - \frac{1}{r}\right) - \frac{3}{2(r+1)} > s_0 = (\delta + \tfrac{1}{2})\frac{r}{r-1} - \tfrac{1}{2},$$

which implies that $w(a') = s(a') - (\varepsilon + \delta)$, a contradiction.

5.8′. P_i is a k-bin ($k > 3$) and $|P_i| > 4$

Let $|P_i| = k \geq 5$. Since $d_i \geq \max\{k_i - a_i, (r-1)\alpha_i - 1\} \geq (k_i + 1)(r-1)/r - 1 > (2k_i - 4)\varepsilon$ and $w(P_i) = s(P_i) - k_i\varepsilon$, we are done by Lemma 4.3.

Now our desired contradiction to the assertion that Lemma 4.2 were false is easy to find. The same argument as in the proof of Theorem 4.4 allows us to conclude

$$\Delta - (1 - \varepsilon) \geq -\varepsilon,$$

which is obviously impossible. Hence our Lemma 4.2 is proved. □

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