Isotopies of Foliated 3-Manifolds without Holonomy

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Received May 20, 1996; accepted July 21, 1996

DEDICATED TO KY FAN

We offer a new proof of a deep result of Laudenbach and Blank. This proof is based on the Nielsen–Thurston classification of surface automorphisms.

1. INTRODUCTION

Let M be a compact, connected, oriented 3-manifold and 𝒇 a smooth ($C^\infty$), transversely oriented foliation of M without holonomy. If $\partial M \neq \emptyset$, we assume that the foliation is transverse to the boundary. For example, if a fibration $\pi: M \to S^1$ has fibers transverse to $\partial M$, these fibers are the leaves of a foliation 𝒇 of the required type. Somewhat more interesting examples include the linear foliations of $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, induced by families of parallel planes in $\mathbb{R}^3$. Such foliations can have all leaves 2-tori, in which case they fiber $T^3$ over $S^1$, or each leaf can be a cylinder, dense in $T^3$, or each leaf can be a plane, also dense in $T^3$. In all cases, these foliations are defined by closed, nowhere vanishing 1-forms $\omega = a \, dx + b \, dy + c \, dz$. More subtle examples are constructed on $T^3$ by using orientation preserving diffeomorphisms of $S^1$, constructed by V. I. Arnold [1, Sect. 1], which

* Research by the first author was partially supported by N.S.F. Contract DMS-9201213.
† Research by the second author was partially supported by N.S.F. Contract DMS-9201723.
have irrational rotation number but are not even $C^1$ conjugate to a rotation. Suspending such a diffeomorphism over $T^2$ produces a dense leaved foliation without holonomy which cannot be defined by a closed, nonsingular 1-form.

It is deep a result of R. Sacksteder [20] that, under our hypotheses, either every leaf is compact and $\mathcal{F}$ fibers $M$ over $S^1$, or every leaf is dense. As remarked above, in the dense leaved case the foliation may or may not be defined by a closed, nonsingular 1-form. When the foliation is defined by such a form, an elegant argument of D. Tischler [25] shows that the manifold fibers over $S^1$ and the foliation can be uniformly well approximated in the $C^\infty$ topology by such fibrations. In all cases, Sacksteder’s results imply that there is a continuous, nonsingular flow, transverse to $\mathcal{F}$ and carrying leaves to leaves. Equivalently, there is a transverse, holonomy invariant measure $\mu$ which is nonatomic, positive on open transverse arcs, and finite on compact transverse arcs. In the dense leaved case, the measure is unique up to a constant $\lambda$ of proportionality and it is smooth if and only if the foliation is defined by a closed, nonsingular 1-form. Arnold’s examples [1] show that $\mu$ may not even be absolutely continuous. At any rate, the measure defines a cohomology class $[\mu] \in H^1(M; \mathbb{R})$, just as a closed, nonsingular 1-form would, by integration around closed loops.

If $\mu$ is not smooth, one can change the differentiable structure of $M$ to make it so. The holonomy invariance of $\mu$ allows us to introduce new local coordinate charts $(U, x, y, z)$ on $M$, with $x$ and $y$ leaf coordinates and $z$ a transverse coordinate which is defined up to an additive constant by $\mu$. Thus, on overlapping charts $(U, x, y, z)$ and $(U', x', y', z')$, the coordinate changes take the form

$$x = x(x', y')$$
$$y = y(x', y')$$
$$z = z' + c,$$

where $c$ is constant. The changes in $x$ and $y$ are smooth by the smoothness of the foliation, so the new atlas defines a differentiable structure in which the foliation is still smooth, but is now defined by a closed, nonsingular 1-form $\omega = dz$. Essentially, the new differentiable structure converts $\mu$ to the closed form $\omega$ and $[\omega] = [\mu]$. This trick is due to Sacksteder [20].

We will call $[\mu]$ a foliated class in $H^1(M; \mathbb{R})$. In the special case that $\mathcal{F}$ is a fibration, $[\mu]$ will also be called a fibered class. The ray $\{\lambda [\mu]\}_{\lambda > 0}$ in $H^1(M; \mathbb{R})$ will be called a foliated (respectively, fibered) ray and denoted by $[\mathcal{F}]$. It can be thought of as an equivalence class of foliations.

The theorem of Laudenbach and Blank [13] asserts that nonsingular, cohomologous, closed 1-forms on $M$ are isotopic. Thus, if $[\mathcal{F}_0] = [\mathcal{F}_1]$ and
\( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are defined by closed 1-forms, then \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are smoothly isotopic. As we will see, if \( \mathcal{F} \) is a smooth, dense leaved foliation without holonomy, but the associated measure is not smooth, there is a \( C^0 \) isotopy of \( \mathcal{F} \) to a foliation that is defined by a closed, nonsingular 1-form. (This avoids changing the differentiable structure and is undoubtedly “well known to experts,” but we do not know of a proof in the literature.) Thus, [13] gives the following.

**Theorem 1.1 (Laudenbach–Blank).** Let \( M \) be a compact, oriented 3-manifold. If \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are smooth, transversely oriented foliations without holonomy, transverse to \( \partial M \) and such that \(< \mathcal{F}_0, \mathcal{F}_1 > = \mathcal{F}_1 \), then there is a \( C^0 \) isotopy of \( M \) carrying \( \mathcal{F}_0 \) to \( \mathcal{F}_1 \). If \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) are defined by closed, nowhere vanishing 1-forms \( \omega_0 \) and \( \omega_1 \), respectively, this isotopy is smooth. If \( \omega_0 \) and \( \omega_1 \) are scaled so that \([ \omega_0, \omega_1 ] = [ \omega, \omega ] \in H^1(M; \mathbb{R})\), the isotopy carries \( \omega_0 \) to \( \omega_1 \).

This fails in higher dimensions [12]. When the foliations are fibrations, the result is due to Waldhausen [26].

It should be noted that Laudenbach and Blank assume the foliation to be tangent to the boundary. If \( \partial M \neq \emptyset \), their work implies that \( \mathcal{F} \) is isotopic to a product and yields a new proof of a celebrated theorem of Cerf [5]. A proof of the Laudenbach–Blank theorem by Què and Roussarie [17] simplifies matters considerably by assuming Cerf’s theorem. In this paper, we offer a new proof of Theorem 1.1 which likewise assumes Cerf’s theorem, but differs considerably from [17]. One use of Cerf’s theorem will be to sketch a proof of Waldhausen’s theorem.

The main new ingredients in our discussion are the classification of surface homeomorphisms due to Nielsen and Thurston [8] and Sullivan’s theory of foliation cycles [22].

As in all proofs of this theorem, the hardest step is to find isotopies of \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) to foliations transverse to a common, oriented, 1-dimensional foliation \( \mathcal{L} \). Here, transversality includes the condition that the orientation of \( \mathcal{L} \) coincides with the transverse orientations of the two foliations. If these foliations are defined by closed 1-forms, a technique of Moser [15] is then used to deform \( \mathcal{F}_0 \) to \( \mathcal{F}_1 \) by a smooth isotopy along \( \mathcal{L} \). If one or both are not defined by closed forms, we will use the transverse measures to produce a \( C^0 \) isotopy along \( \mathcal{L} \).

The distribution of foliated classes in the vector space \( H^1(M; \mathbb{R}) \) is described by Thurston [24] via a (possibly degenerate) norm on this space. If the norm is nontrivial, it defines a (possibly noncompact) polyhedral unit ball with finitely many faces. Each foliated ray passes through the interior of some top dimensional face \( A \) and the interior of the cone over \( A \) consists entirely of foliated classes [24, Theorem 5]. The top dimensional faces with this property are called \textit{foliated faces}. 
If $M$ (assumed to fiber over $S^1$) admits a Seifert fibration $\mathcal{L}$, the Thurston norm vanishes on a subspace $P$ of codimension one and the Thurston ball has only two faces $A_{\pm \mathcal{L}}$, these being hyperplanes parallel to $P$. These faces are foliated and the foliations corresponding to rays through $A_{\pm \mathcal{L}}$ can be isotoped to be transverse to $\pm \mathcal{L}$ (Theorem 5.2).

Here, of course, $-\mathcal{L}$ is the oppositely oriented version of $\mathcal{L}$.

If the 3-manifold is hyperbolic, Fried shows [9] that each foliated face $A = A_{\mathcal{L}}$ corresponds to a pseudo-Anosov flow $\mathcal{L}$, unique up to isotopy. The flow is obtained by taking any fibration $\mathcal{F}$ in a fibered ray $[\mathcal{F}]$ through $\mathcal{A}$ and suspending the pseudo-Anosov monodromy of $\mathcal{F}$. Again, every foliation in a foliated ray through $A_{\mathcal{L}}$ is isotopic to a foliation transverse to $\mathcal{L}$ (Theorem 5.3).

All of the 3-manifolds we study fiber over $S^1$, each such fibration being completely determined by the isotopy class of its monodromy diffeomorphism $\varphi$. If the fiber $F$ has negative Euler characteristic, $\varphi: F \to F$ is isotopic either to a periodic, pseudo-Anosov, or reducible diffeomorphism [8]. Correspondingly, $M$ is either Seifert fibered, hyperbolic, or can be decomposed into such manifolds by removing an open normal neighborhood $N_0(T)$ of the union $T$ of a finite, disjoint family of incompressible tori and/or Klein bottles. Preliminary isotopies make $\mathcal{F}_0$ and $\mathcal{F}_1$ transverse to $\partial(M \setminus N_0(T))$ and the isotopies described above are carried out in each component of $M \setminus N_0(T)$. Care must be taken to extend the isotopies in each of these manifolds compatibly across $N_0(T)$ in such a way that $\mathcal{F}_0$ and $\mathcal{F}_1$ become everywhere transverse to the same 1-dimensional foliation $\mathcal{L}$. This is the other place in which we use Cerf’s theorem.

Finally, the fiber has nonnegative Euler characteristic if and only if the Thurston norm is identically zero. There are finitely many such cases and it will be easy to check that every nonzero class is foliated and that Theorem 1.1 holds.

2. INVARIANT MEASURES AND ISOTOPY

In this section, dim $M$ is arbitrary. We assume that the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ are smooth, without holonomy, transverse to a common 1-dimensional foliation $\mathcal{L}$ and that $[\mathcal{F}_0] = [\mathcal{F}_1]$. We will show that $\mathcal{F}_0$ and $\mathcal{F}_1$ are isotopic.

The easier case is that in which the foliations are defined by closed nonsingular 1-forms $\omega_0$ and $\omega_1$, respectively. For this case, we employ a construction of Moser [15].

We assume that $\mathcal{L}$ is integral to a $C^0$ line field and choose a nonsingular $C^0$ vector field $v$, tangent to $\mathcal{L}$ and oriented by the common transverse
orientation of $\mathcal{F}_0$ and $\mathcal{F}_1$. We can also scale the forms so that $[\omega_0] = [\omega_1] \in H^1(M; \mathbb{R})$. Let $f \in C^\infty(M)$ be such that $df = \omega_1 - \omega_0$ and set
\[
\Omega = t \omega_1 + (1-t) \omega_0 + f \, dt \in A^1(M \times [0, 1]).
\]
If we extend the vector field $v \in \mathfrak{X}(M)$ canonically to $v \in \mathfrak{X}(M \times [0, 1])$ by setting
\[
v_{(x, t)} = v_x, \quad \forall x \in M, \quad 0 \leq t \leq 1,
\]
our assumptions guarantee that $\Omega(v) > 0$ everywhere. In particular, $\Omega$ is nonsingular. Furthermore,
\[
d\Omega = dt \wedge (\omega_1 - \omega_0) + df \wedge dt = dt \wedge df + dt \wedge df = 0.
\]
The foliation $\mathcal{K}$ of $M \times [0, 1]$, defined by $\Omega$, is clearly transverse to the submanifolds $M \times \{t\}$, inducing foliations $\mathcal{K}_0, 0 \leq t \leq 1$, with $\mathcal{K}_0 = \mathcal{F}_0$ and $\mathcal{K}_1 = \mathcal{F}_1$. That is, $\mathcal{F}_0$ is integrably homotopic to $\mathcal{F}_1$ through the foliations $\mathcal{K}_t$. This integrable homotopy defines a smooth isotopy of $\mathcal{F}_0$ to $\mathcal{F}_1$. Indeed, one constructs a vector field $w$ on $M \times I$, tangent to $\mathcal{K}$ and having vertical component $\partial / \partial t$. Compactness of $M$ guarantees that the flow lines of $w$ do not “exit” $M \times I$ for time $0 \leq t \leq 1$, so this flow defines the isotopy.

**Proposition 2.1 (Moser).** If $\mathcal{F}_0$ and $\mathcal{F}_1$ are foliations defined by closed, nonsingular 1-forms $\omega_0$ and $\omega_1$, respectively, transverse to a common 1-dimensional foliation and determining the same foliated ray $[\mathcal{F}_0] = [\mathcal{F}_1]$, then $\mathcal{F}_0$ and $\mathcal{F}_1$ are smoothly isotopic. If the forms are scaled so that $[\omega_0] = [\omega_1]$, then $\omega_0$ and $\omega_1$ are isotopic.

In the more difficult case, we only know that $\mathcal{F}_0$ and $\mathcal{F}_1$ are dense leaved and admit transverse, holonomy invariant measures $\mu_0$ and $\mu_1$, respectively. The measures $\mu_0$ and $\mu_1$ are strictly positive and continuous. That is, they are nonatomic, finite on compact arcs of $\mathcal{L}$, and strictly positive on open arcs of $\mathcal{L}$. Because the foliations are dense-leaved, these measures are unique up to scaling.

Line integrals \( \int_{\sigma} d\mu_i, i = 0, 1 \), are defined along continuous, compact curves $\sigma$ in $M$ (generally not transverse to the foliations) and these numbers depend only on the homotopy class of $\sigma$ (mod the endpoints). Indeed, working in foliated charts that cover $\sigma$, one homotopes $\sigma$ to a path consisting of segments alternately transverse to $\mathcal{F}_t$ and tangent to it. The measure $\mu_i$ deposits zero mass on the tangent segments, positive mass on the transverse segments which are oriented compatibly with the transverse orientation of the foliation, and it deposits negative mass on the oppositely oriented, transverse segments. The holonomy invariance of the measure
implies that the resulting path integral is homotopy invariant, as advertised. (For a bit more detail, cf. [16, p. 345].)

In particular, these integrals define homomorphisms

\[
\mu_0 \text{ and } \mu_1 : H_1(M; \mathbb{Z}) \to \mathbb{R},
\]

which we view as elements of \( H^1(M; \mathbb{R}) \). The images of these homomorphisms are called the period groups \( P(\mu_0) \) and \( P(\mu_1) \) of measures. The assumption that \( [\mathcal{F}_0] = [\mathcal{F}_1] \) allows us to rescale these measures so that \( [\mu_0] = [\mu_1] \). We emphasize that the measures themselves need not be equal, although they do have equal line integrals round closed loops.

An alternative description of the period group is useful. The measures \( \mu_0 \) and \( \mu_1 \) define \( C^0 \) flows, \( \Phi^0 \) and \( \Phi^1 \), respectively, with the leaves of \( \mathcal{L} \) as flow lines. The flow \( \Phi^i \) carries leaves of \( \mathcal{F}_i \) onto leaves of \( \mathcal{F}_i \), \( i = 0, 1 \). The following lemma and its corollary are left as interesting exercises.

**Lemma 2.2.** The period group \( P(\mu_i) \) is the set of \( t \in \mathbb{R} \) such that \( \Phi^i_t \) carries each leaf of \( \mathcal{F}_i \) onto itself, \( i = 0, 1 \).

**Corollary 2.3.** A continuous, compact path \( \sigma \) in \( M \) has endpoints in the same leaf of \( \mathcal{F}_i \) if and only if \( \int_{\sigma} d\mu_i \in P(\mu_i) \), \( i = 0, 1 \).

For \( 0 \leq t \leq 1 \), \( \nu_t = t\mu_1 + (1 - t)\mu_0 \) is a continuous, strictly positive measure along \( \mathcal{L} \) and has line integrals

\[
\int_{\sigma} d\nu_t = t\int_{\sigma} d\mu_1 + (1 - t)\int_{\sigma} d\mu_0.
\]

This defines a cohomology class and \( [\mu_0] = [\nu_t] = [\mu_1] \), \( 0 \leq t \leq 1 \). It is natural to suspect that \( \nu_t \) is a transverse, invariant measure for a dense-leaved \( C^0 \) foliation \( \mathcal{F}_i \) without holonomy, \( 0 \leq t \leq 1 \), and that this homotopy of \( \mathcal{F}_0 = \mathcal{F}_0 \) with \( \mathcal{F}_1 = \mathcal{F}_1 \) is actually a continuous isotopy. The following discussion confirms this.

Fix a leaf \( L \) of \( \mathcal{F}_0 \) and a basepoint \( x \in L \). For fixed but arbitrary \( t \in [0, 1] \), we are going to define a continuous map

\[
\phi_t : L \to M.
\]

If \( z \in L \), choose a path \( \sigma \) in \( L \) from \( x \) to \( z \) and let \( a_t(z) = \int_{\sigma} d\nu_t \).

**Lemma 2.4.** The number \( a_t(z) \) depends only on \( z \) and \( t \), not on the choice of \( \sigma \), and the function \( a_t : L \to \mathbb{R} \) is continuous.
\textbf{Proof.} If \( \tau \) is another path in \( L \) from \( x \) to \( z \), \( \sigma - \tau \) is a loop, so
\[
[v_r][\sigma - \tau] = [\mu_0][\sigma - \tau] = 0.
\]
This proves that \( a_t \) is well defined. Continuity is evident. 

The measure \( v_r \) defines a flow \( ^t\Psi \) on \( M \) with flow lines the leaves of \( \mathcal{L} \).

Define
\[
\phi_t: L \to M
\]
\[
\phi_t(z) = ^t\Psi_{-a_t(z)}(z).
\]

\textbf{Lemma 2.5.} If \( \ell \) is a leaf of \( \mathcal{L} \) (with its 1-dimensional manifold topology), then \( \phi_t \) restricts to an order preserving bijection of the dense subset \( \ell \cap L \) onto a dense subset of \( \ell \).

\textbf{Proof.} The order on \( \ell \), of course, is that induced by the transverse orientation of \( \mathcal{F}_0 \). In the case that \( \ell \simeq S^1 \), this is a cyclic order. We carry out the argument for the case that \( \ell \simeq \mathbb{R} \), leaving to the reader the obvious adaptations for the case \( \ell \simeq S^1 \).

Let \( z, w \in \ell \cap L \) and suppose that \( z < w \). Let \( \tau \) be the oriented subarc of \( \ell \) connecting these points and let \( \sigma_w \) and \( \sigma_z \) be paths in \( L \) joining \( x \) to \( w \) and \( z \), respectively. Consider the loop \( \gamma = \sigma_z + \tau - \sigma_w \) in \( M \). Since
\[
[v_r] = [\mu_0],
\]
we obtain
\[
\int_{\gamma} \mu_0 = \int_{\gamma} dv_r,
\]
from which it follows that
\[
a_t(z) - a_t(w) + v_r(\tau) = \mu_0(\tau) > 0. \tag{*}
\]
We use the flow \( ^t\Psi \) to parametrize \( \ell \),
\[
x \leftrightarrow ^t\Psi(x),
\]
identifying \( \ell \) with \( \mathbb{R} \) so that \( z \) becomes the origin 0. Thus,
\[
w = v_r(\tau)
\]
\[
\phi_t(z) = -a_t(z)
\]
\[
\phi_t(w) = -a_t(w) + v_r(\tau).
\]
With this order preserving parameterization, \( (*) \) implies that
\[
\phi_t(w) - \phi_t(z) = \mu_0(\tau) > 0, \tag{**}
\]
proving that $\varphi_1$ is order preserving. This also shows that, if $z$ and $w$ are close along $\ell$, then $\mu_0(\tau)$ is small and $\varphi_1(z)$ is close to $\varphi_1(w)$ along $\ell$.

It follows that, for each leaf $\ell$ of $\mathcal{L}$, $\varphi_1$ extends canonically to a homeomorphism of $\ell$ onto itself. Furthermore, $\varphi_1 : L \to M$ will be an injective topological immersion extending canonically to a homeomorphism $\varphi_1 : M \to M$. The continuous dependence of this homeomorphism on $t \in [0, 1]$ is also elementary.

**Lemma 2.6.** The map

$$\varphi : M \times [0, 1] \to M,$$

defined by $\varphi(z, t) = \varphi_1(z)$, is an isotopy of $\varphi_1$ to $\varphi_0 = \text{id}$.

For each $t \in [0, 1]$, we obtain a topological foliation $\mathcal{F}_t = \varphi_t(\mathcal{F})$ which we view as an isotopy of $\mathcal{F}_0 = \mathcal{F}_0$ to $\mathcal{F}_1$. Remark that $\varphi_1$ pushes the invariant measure $\mu_0$ forward to the measure $\nu_1$. Indeed, $\mathcal{F}_t$ asserts that $\mu_0(\tau) = \nu_1(\varphi_1(\tau))$, where $\tau$ is an arc of $\mathcal{L}$ with endpoints in $L$. The general assertion follows by continuity of the measures and the fact that $L$ is dense in $M$. Thus, $\nu_t$ is a holonomy invariant measure for $\mathcal{F}_t$, $0 \leq t \leq 1$.

**Proposition 2.7.** The foliation $\mathcal{F}_1$ is identical with $\mathcal{F}_1$, hence $\mathcal{F}_0$ and $\mathcal{F}_1$ are topologically isotopic.

**Proof.** For $z \in L$, let $\tau_z$ be the directed subarc of the leaf $\ell$ of $\mathcal{L}$ through $z$ with initial point $z$ and terminal point $\varphi_1(z)$. Let $\sigma_z$ be a path in $L$ from $x$ to $z$ and set $\sigma = \sigma_z + \tau_z$. Then

$$\int_{\sigma} d\mu_1 = \int_{\sigma} d\nu_1 = a_1(z) - a_1(z) = 0$$

and, by Corollary 2.3, the endpoints $x$ and $\varphi_1(z)$ of $\sigma$ lie in the same leaf of $\mathcal{F}_1$. That is, $\varphi_1$ carries $L$ into the leaf $L'$ of $\mathcal{F}_1$ through $x$. By interchanging the roles of $\mathcal{F}_0$ and $\mathcal{F}_1$, we see that $\varphi_1^{-1}$ carries $L'$ into $L$. Since these leaves are dense in $M$, it follows that $\mathcal{F}_1 = \varphi_1(\mathcal{F}) = \mathcal{F}_1$.

This proposition and a theorem of Sullivan will have the following corollary.

**Proposition 2.8.** If $\mathcal{F}$ is a smooth foliation without holonomy, then $\mathcal{F}$ is topologically isotopic to a foliation defined by a closed, nonsingular 1-form.

The proof will use the theory of foliation cycles [22] for the transverse foliation $\mathcal{L}$. These cycles are closed de Rham 1-currents $T_1$, defined by
transverse, invariant measures $v$ for $L$. The current $T_v$ is evaluated on 1-forms $\eta$ as follows. In a flow box for $L$, integrate $\eta$ along the plaques of $L$ in the usual way, then integrate the resulting function along a transverse disk using the measure $v$. These local integrals are assembled by a partition of unity to a global 1-current. Holonomy invariance of the measure is used to prove independence of the choice of local flow boxes and of the partition of unity and Stokes’ theorem is used to check that $T_v(df) = 0$ (cf. [16, pp. 330–331]). These foliation cycles determine a closed, convex cone $C'_L = H_1(M; \mathbb{R})$ and the dual cone

$$C''_L = \{ z \in H^1(M; \mathbb{R}) | z(\cdot) \geq 0, \forall z \in C'_L \}.$$ 

is also closed and convex. By [22, Theorem 1.7] we have the following.

**Lemma 2.9.** A 1-dimensional foliation $L$ is transverse to a closed, nonsingular 1-form if and only if no nontrivial foliation cycle of $L$ bounds. In this case, $\text{int} C'_L$ consists exactly of those classes represented by closed 1-forms transverse to $L$.

In particular, every ray in $\text{int} C'_L$ is a foliated ray, represented by a foliation $F$ transverse to $L$. These foliations are defined by closed, nonsingular 1-forms. We also need to know that if $F$ is not defined by a closed form but is transverse to $L$, then no foliation cycle for $L$ bounds and $[F]$ is a ray in $\text{int} C'_L$. For this, we use a particularly simple spanning set of $C'_L$, the so-called “homology directions” [9, p. 260]. Assuming that $L$ has been parametrized as a nonsingular, $C^0$ flow $\Phi$, select a point $x \in M$ and let $t_k \rightarrow \infty$ be chosen so that $\{\Phi_t(x)\}_{t=1}^{t_k}$ converges in $M$. For large values of $N$, each segment $[\Phi_t(x)]_{t_N < t < t_{N+1}}$, $k \geq 1$, can be slightly perturbed near its ends so as to produce a closed loop $\Gamma_k$ which is tangent to $L$ except on a small subarc. Passing to a subsequence, if necessary, we can assume that, in the space of de Rham currents, the limit

$$\lim_{k \rightarrow \infty \frac{1}{t_{N+k} - t_N}} \Gamma_k = \Gamma,$$

exists and is a foliation cycle for $L$.

**Definition 2.10.** The foliation cycles $\Gamma$ (and their homology classes), obtained as above, are called homology directions of $L$.

An elementary application of ergodic theory proves the following (cf. [22, Proposition II.25]).
Lemma 2.11. An arbitrary foliation cycle \(0 \neq z\) of \(\mathcal{L}\) can be arbitrarily well approximated in the space of de Rham currents by cycles of the form

\[ z_n = \sum_{i=1}^{m_n} c_{n,i} \Gamma^\times_i, \quad n \geq 1, \]

where all \(\Gamma^\times_i\) are homology directions, the coefficients \(c_{n,i}\) are positive, and the numbers

\[ c_n = \sum_{i=1}^{m_n} c_{n,i} \]

are uniformly bounded away from 0, \(\forall n \geq 1\).

Lemma 2.12. If \(F\) is a smooth foliation without holonomy and transverse to \(L\), then no foliation cycle of \(L\) bounds and \([F]\) is a ray in \(\text{int} C^\times L\).

Proof. If \(F\) is defined by a closed, nonsingular 1-form, this is immediate. Otherwise, let \(\mu\) be a transverse, invariant measure for \(F\). Using \(\mu\) to parametrize \(L\) as a flow \(\Phi\), we see that \([\mu]\) takes the value 1 on every homology direction \(\Gamma\). By Lemma 2.11, \([\mu]\) is strictly positive on all nontrivial foliation cycles for \(L\), so these cannot bound and \([F]\) is a ray in \(\text{int} C^\times L\).

Proof of Proposition 2.8. If \(F\) is smooth and without holonomy, select any transverse 1-dimensional foliation \(L\). By Lemma 2.12, \([F]\) is a ray in \(\text{int} C^\times L\), and, by Lemma 2.9, there is a foliation \(F' \in [F]\), also transverse to \(L\) and defined by a closed, nonsingular 1-form. By Proposition 2.7, \(F\) is topologically isotopic to \(F'\).

Accordingly, in proving Theorem 1.1, we will restrict to the case of foliations defined by closed, nonsingular 1-forms.

3. FIBRATIONS

The following case of Theorem 1.1 plays a key role in proving the general case. We will give a brief presentation of the proof indicated by Thurston in [24]. On a first reading, one may prefer to skip this, accepting Theorem 3.1 as a given.

Theorem 3.1 (Waldhausen). Let \(M\) be a compact, oriented 3-manifold. Let \(F_0\) and \(F_1\) are foliations by the fibers of smooth fibrations of \(M\) over \(S^1\), transverse to \(\partial M\), and if \([F_0] = [F_1]\), then there is a smooth isotopy of \(M\) carrying \(F_0\) to \(F_1\).
The proof which we outline assumes that the fiber is neither the 2-disk nor the 2-sphere. In these two cases, the manifold is $D^2 \times S^1$ and $S^2 \times S^1$, respectively, and the hardest part of the proof is extending the following lemma to these cases. This involves standard but delicate general position arguments and we omit it.

**Lemma 3.2.** Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be as in Theorem 3.1 and let $F$ be a fiber of $\mathcal{F}_0$, assumed to be neither $D^2$ nor $S^2$. Then, there is a nonsingular flow on $M$, with underlying 1-dimensional foliation $\mathcal{L}$, tangent to $\partial M$ and transverse to $\mathcal{F}_1$, and a smooth isotopy of $F$ to a surface that is transverse to $\mathcal{L}$ and meets every leaf of $\mathcal{L}$ infinitely often both in forward and backward time.

**Proof.** The analogue of Waldhausen’s theorem holds on the 2-torus (Corollary 3.6). It allows us to assume, by a preliminary isotopy, that $\mathcal{F}_0 \mid \partial M = \mathcal{F}_1 \mid \partial M$. Thus, $F$ is a properly imbedded, incompressible surface in $M$ and each component of $\partial F$ lies in a leaf of the taut foliation $\mathcal{F}_1$. By the Roussarie–Thurston theorem [24, Theorem 4], $F$ is either isotopic to a leaf of $\mathcal{F}_1$ or to a surface having only saddle tangencies with $\mathcal{F}_1$ in int $M$. (This is where we need to assume that $F$ is neither $D^2$ nor $S^2$.) In the first case, any choice of 1-dimensional foliation $\mathcal{L}$, transverse to $\mathcal{F}_1$, will do. In the case of saddle tangencies, let $p_+$ denote the number of such tangencies at which the orientations of $F$ and the leaf of $\mathcal{F}_1$ through the tangency agree. Let $p_-$ be the number of saddles at which the two orientations are opposite. By the Poincaré–Hopf theorem, $\chi(F) = -p_+ - p_-$. If $F'$ is any fiber of $\mathcal{F}_1$, the fact that $[\mathcal{F}_0] = [\mathcal{F}_1]$ implies $[F, \partial F] = [F', \partial F'] \in H_2(M, \partial M; \mathbb{R})$.

If $e \in H^2(M, \partial M; \mathbb{R})$ is the Euler class of the tangent bundle of $\mathcal{F}_1$, we see that

$$-p_+ + p_- = e[F, \partial F] = e[F', \partial F'] = \chi(F').$$

But, by [24, Corollary 2, p. 119], $F$ and $F'$ both realize the Thurston norm of the class $[F, \partial F]$, so

$$-p_+ + p_- = \chi(F') = \chi(F) = -p_+ - p_-,$$

implying that $p_- = 0$. Since the orientations agree at the saddle tangencies, we can find a smooth, nonsingular vector field $v$ on $M$, tangent to $\partial M$ and transverse both to $F$ and to $\mathcal{F}_1$. Then the flow $\Phi_t$ of $v$ is transverse to $\mathcal{F}_1$ and to $F$ and $\mathcal{L}$ will be the underlying foliation by flow lines.

We must show that $F$ meets every leaf of $\mathcal{L}$ infinitely often as $t \searrow \infty$ and as $t \nearrow -\infty$. Say that this fails in forward (respectively, backward) time for some leaf $\ell$ of $\mathcal{L}$ and let $X$ denote the $\omega$-(respectively $\alpha$-) limit set of $\ell$. 

**ISOTOPIES OF FOLIATED 3-MANIFOLDS**
Since $F \cap \mathcal{L}$, the compact, $\mathcal{L}$-saturated set $X$ also misses $F$. There is a non-trivial foliation cycle $v$ for $\mathcal{L}$, supported in $X$ and having intersection product $v^*[F, \partial F] = 0$. But, $\mathcal{L} \cap \mathcal{F}_1$ implies that $\text{supp} \ v$ meets every fiber of $\mathcal{F}_1$. Thus, if $F'$ is such a fiber, $v^*[F, \partial F] = v^*[F', \partial F'] > 0$, a contradiction.

Let $\mathcal{D}^k = \text{Diff}(D^k, \partial D^k)$, the group of diffeomorphisms of the $k$-disk $D^k$ which leave the boundary pointwise fixed. It is standard and elementary that $\mathcal{D}^1$ is path connected. The same assertion for $\mathcal{D}^2$ is due to Smale [21] and, for $\mathcal{D}^3$, to Cerf [5].

**Lemma 3.3.** For $1 \leq k \leq 3$, $\pi_d(\mathcal{D}^k) = 0$.

This lemma is the key step in proving the following. Details will be found, for example, in [4, Sect. 5].

**Lemma 3.4.** Let $F$ be a compact, orientable manifold of dimension $\leq 2$ and let $\mathcal{F}_0$ be a foliation of $F \times I$ by leaves diffeomorphic to $F$, transverse to $\partial F \times I$, and tangent to $F \times \partial I$. Then there is an isotopy of $\mathcal{F}_0$, compactly supported in the interior of $F \times I$, to a foliation everywhere transverse to the factors $\{x\} \times I$, $\forall x \in F$.

The isotopy in Lemma 3.2 can be viewed as an ambient isotopy of the fibration $\mathcal{F}_0$. After performing this isotopy, one can cut $M$ along the fiber $F$, producing a manifold diffeomorphic to the product $F \times [-1, 1]$. Here, by the last assertion in Lemma 3.2, we can let the foliation by the factors $\{x\} \times [-1, 1]$ be induced by the foliation $\mathcal{L}$ of that lemma. By abuse of notation, we again denote this foliation by $\mathcal{L}$. The foliation of $F \times [-1, 1]$ induced by $\mathcal{F}_0$ will also be denoted by $\mathcal{F}_0$ and the isotopy of Lemma 3.4 can be viewed (after regluing) as an isotopy in $M$ moving $\mathcal{F}_0$ to a position transverse to $\mathcal{L}$.

At this point, $\mathcal{F}_0$ has been isotoped so that both $\mathcal{F}_0$ and $\mathcal{F}_1$ are transverse to a common 1-dimensional foliation $\mathcal{L}$. Proposition 2.1 completes the proof of Theorem 3.1.

**Remark.** We only used the case $\dim F = 2$ in Lemma 3.4. As an application of the 1-dimensional case, it is an elementary exercise to deduce the following.

**Lemma 3.5.** Let $\mathcal{F}$ be a smooth, 1-dimensional foliation without holonomy of the torus $T^2$, $\mathcal{L}$ a fibration of $T^2$ by circles. Then there is a smooth isotopy of $\mathcal{F}$ to a foliation which either coincides with $\pm \mathcal{L}$ or is transverse to $\pm \mathcal{L}$. 24 CANTWELL AND CONLON
By Propositions 2.1 and 2.8, we obtain

**Corollary 3.6.** The exact analogue of Theorem 1.1 holds for 1-dimensional foliations of $T^2$.

### 4. THE PRELIMINARY CASES

The manifold fibers over $S^1$. Our main concern will be the case in which the fiber carries a hyperbolic structure. The finitely many alternative cases are somewhat special, being exactly the cases, noted in the Introduction, in which the Thurston norm is totally trivial. Here we present a brief treatment of these cases.

If the fiber is the 2-sphere, then $M = S^2 \times S^1$ and the only foliations without holonomy are such fibrations. Indeed, $H^1(M; \mathbb{R}) = \mathbb{R}$. Consequently, there are only two foliated rays and these are fibered rays corresponding to the two transverse orientations of the fiber. In this case, Waldhausen’s theorem gives the result. The case in which the fiber is $D^2$ is similar.

If the fiber is an annulus $A$, there are two cases. Indeed, the monodromy map is an orientation preserving automorphism of $A = S^1 \times [-1, 1]$. Such a homeomorphism is either isotopic to the identity or to the map $\varphi(z, t) = (z^{-1}, -t)$. In the second case, $M$ is the total space of a nonorientable interval bundle over the Klein bottle, so $H^1(M; \mathbb{R}) = \mathbb{R}$ and we are reduced to Waldhausen’s theorem. In the first case, $M = A \times S^1$ is a “thickened torus.” If $\omega_0$ and $\omega_1$ are nonsingular, closed, cohomologous 1-forms, transverse to $\partial M$, Corollary 3.6 allows us to perform an isotopy of $\omega_0$, making $\omega_0|\partial M = \omega_1|\partial M$. We will see (Lemma 5.4) that, in this case, an isotopy makes $\omega_0$ transverse to a 1-dimensional foliation to which $\omega_1$ is also transverse. Propositions 2.1 and 2.7 complete the proof in this case. Notice that $H^1(M; \mathbb{R}) = \mathbb{R}^2$, the Thurston norm is trivial, and one easily constructs a nonsingular 1-form in each nonzero class.

If the fiber is a torus $T$, one sees that every fibration of $M$ has toral fiber $[10, p. 49]$. The dense leaved foliations without holonomy will have all leaves planes or all leaves cylinders. It is known (cf. [2, Introduction]) that every orientation preserving toral automorphism is isotopic to a linear automorphism $\varphi \in SL(2, \mathbb{Z})$, so we lose no generality in assuming that the monodromy $f = \varphi$ of the fibration is linear. According to whether $|\text{tr } \varphi|$ is $>2$, $=2$, or $\leq 1$, this linear automorphism is one of the following types:

1. an Anosov diffeomorphism;
2. a power of a Dehn twist or the composition of such with an automorphism of period 2;
3. a periodic automorphism (the possible periods are 1, 2, 3, 4, and 6).
A glance at the Serre spectral sequence shows that $H^1(M; \mathbb{R}) = \mathbb{R} \oplus \mathcal{F}$, where $\mathcal{F} \subseteq H^1(T; \mathbb{R})$ is the space of $+1$ eigenvectors for $\varphi^*$. In the Anosov case, this gives $H^1(M; \mathbb{R}) = \mathbb{R}$ and, as above, our theorem reduces to Waldhausen's.

In the Dehn twist case, the monodromy of $T$ has an invariant, essential circle on the fiber, which may be sent to itself with a reverse of orientation and transverse orientation. In suitable coordinates (mod $\mathbb{Z}^2$) on $T$, these cases are

$$\varphi(x, y) = (-x + ky, -y)$$

$$\varphi(x, y) = (x + ky, y),$$

where $k$ is an integer. In the first case, $H^1(M; \mathbb{R}) = \mathbb{R}$ and we are again reduced to Waldhausen's theorem. Otherwise, $H^1(M; \mathbb{R}) = \mathbb{R}^2$ and the group of periods of a foliated class $\alpha$ must either be infinite cyclic (hence $\alpha$ is a fibered class) or of rank 2. In the latter case, the leaves cannot be planes, as this would force $M$ to be homeomorphic to $T^3$ [10, p. 49]. Thus, these leaves are cylinders and the period group $\mathbb{Z}^2$ is generated by the (lift of) the fundamental class of the base space and another class. The period group, therefore, restricts to an infinite cyclic group on the toral fiber $T$.

Since any fixed choice of fiber $T$ is incompressible, it can be isotoped to be transverse to any dense-leaved foliation $\mathcal{F}$ [19]. This can be viewed as an ambient isotopy and the inverse isotopy can be used to deform the whole foliation to be transverse to $T$. Given a dense leaved class $[\mathcal{F}_0] = [\mathcal{F}_1]$, deform $\mathcal{F}_0$ to a foliation $\mathcal{F}'$ transverse to $T$, then use an ambient isotopy to deform $T$ to a torus transverse to $\mathcal{F}_0$, carrying along both the fibration by tori and the foliation $\mathcal{F}'$. That is, without loss of generality, we assume that both $\mathcal{F}_0$ and $\mathcal{F}_1$ are transverse to the toral fiber $T$. They induce foliations without holonomy, $\mathcal{F}_0|T$ and $\mathcal{F}_1|T$, on $T$ and $[\mathcal{F}_0|T] = [\mathcal{F}_1|T]$ in $H^1(T; \mathbb{R})$. Since the period group of $\mathcal{F}_0$ restricts to an infinite cyclic group on $T$, $\mathcal{F}_0|T$ (and $\mathcal{F}_1|T$) must be a foliation of $T$ be circles. By Corollary 3.6, we can deform $\mathcal{F}_1|T$ to coincide with $\mathcal{F}_0|T$. This extends to an ambient isotopy of $M$, compactly supported in a neighborhood of $T$, so we lose no generality in assuming that $\mathcal{F}_0|T = \mathcal{F}_1|T$. The next move is to cut $M$ open along $T$, producing $A \times S^1$, where $A$ is the annulus, and two foliations, still called $\mathcal{F}_0$ and $\mathcal{F}_1$, meeting the boundary tori transversely and agreeing there, with $[\mathcal{F}_0] = [\mathcal{F}_1]$. These foliations are now fibrations by annuli, hence they are isotopic by Waldhausen's theorem. Since the isotopy is free at the boundary, there is a problem about regluing. Actually, one should remove a whole open, normal neighborhood $N_0(T)$, and extend the isotopies compatibly over $N_0(T)$ so that $\mathcal{F}_0$ and $\mathcal{F}_1$, while not coinciding there, are transverse to a common 1-dimensional foliation. The elementary
technique for this is described in Subsection 5.3 and Proposition 2.1 completes the isotopy. Remark that, in this case, even if \( F_0 \) and \( F_1 \) are dense leaved they are smoothly isotopic. These foliations are always defined by closed forms.

In the periodic case, unless the monodromy is the identity map, one again finds that \( H^1(M; \mathbb{R}) = \mathbb{R} \) and our theorem reduces to Waldhausen’s.

If the monodromy is isotopic to the identity, then \( M = T^3 \), the Thurston norm is trivial, and all nonzero classes are foliated. Let \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) be dense leaved foliations of \( T^3 \) such that \([\mathcal{F}_0] = [\mathcal{F}_1]\). If \( T \subset T^3 \) is an incompressible torus, isotopies as described above allow us to assume that \( T \) is transverse both to \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \). Removing a normal neighborhood of \( N_d(T) \) produces foliations of \( T^3 \setminus N_d(T) = A \times S^1 \). Carrying out the isotopy there, as indicated earlier, and extending over \( N_d(T) \), one completes the isotopy as in the previous paragraph. For the case that the leaves are planes, also cf. [18].

The above discussion establishes the following.

**Proposition 4.1.** If \( M \) admits a fibration over \( S^1 \), transverse to \( \partial M \) and such that the fiber has nonnegative Euler characteristic, then all fibrations have such fibers, the Thurston norm vanishes identically, all nonzero classes in \( H^1(M; \mathbb{R}) \) are foliated, and the assertions of Theorem 1.1 hold.

5. HYPERBOLIC FIBERS

By Section 4, we restrict to the case in which for some, hence every, fibration of \( M \) over \( S^1 \), the fiber has a hyperbolic structure. Fix any such fibration with connected fiber \( F \) and let \( \varphi : F \to F \) be the monodromy. By the classification, up to isotopy, of the orientation preserving homeomorphism of \( F \) [8], we can assume that \( \varphi \) is either periodic, pseudo-Anosov, or reducible.

5.1. The Periodic Case. If \( \varphi \) is periodic, \( M \) admits a Seifert fibration \( \mathcal{L} \) obtained by suspending \( \varphi \). As usual, we denote the oppositely oriented version of \( \mathcal{L} \) by \(-\mathcal{L}\).

**Lemma 5.1.** In the periodic case, the Thurston ball is bounded by a pair of parallel hyperplanes \( A_{\pm \varphi} \). The foliated ray \([\mathcal{F}]\) meets \( A_{\pm \varphi} \), if and only if there is \( \mathcal{F} \in [\mathcal{F}] \) transverse to \( \pm \mathcal{L} \).

**Proof.** If \( \mathcal{L} \) is a Seifert fibration, it is obvious that the homology directions for \( \mathcal{L} \) define classes in \( H_1(M; \mathbb{R}) \) which are positive multiples of one another. That is, \( C_\varphi \) is a single ray, so \( C_{-\varphi} \) is a halfspace in \( H^1(M; \mathbb{R}) \) and a ray is in the interior of this halfspace if and only if it is represented by
a closed form transverse to $L$ [22, Theorem I.7]. Since every element 
$\xi \in \text{int} C_{\gamma}$ is a foliated class and $-\xi \in C_{\gamma}$, Thurston's classification of the 
foliated classes [24, Theorem 5] implies the assertion.

**Theorem 5.2.** In the periodic case, if the foliated ray $[F]$ meets $A_{\gamma}$ and $F \in [F]$ is defined by a closed, nonsingular 1-form, then $F$ is smoothly isotopic to a foliation transverse to $\pm L$.

This result will be proven in Section 6, completing the proof of 
Theorem 1.1 in the periodic case.

**Remark.** In fact, it is known [14, 7] that if $L$ is a Seifert fibration of 
$M$ transverse to a surface of genus $\geq 2$ and $\partial M = \emptyset$, every foliation of $M$ 
without a compact leaf is isotopic to a foliation transverse to $\pm L$. There 
seems to be no obstruction to extending this result to manifolds with 
boundary, but our proof will not need this.

5.2. The Pseudo-Anosov Case. This is exactly the case in which $M$ is 
hyperbolic [23], hence every fibration of $M$ over $S^1$ will have pseudo-
Anosov monodromy. As shown in [9], the foliated faces $A_{\gamma}$ of the 
Thurston ball are each defined by a pseudo-Anosov flow $L$. Indeed, if 
$[F]$ is a fibered ray through (the interior of) $A_{\gamma}$, choose a fibration 
$F \in [F]$ and let $\varphi: F \to F$ be the pseudo-Anosov monodromy. The 
suspension $L$ of $\varphi$ is a pseudo-Anosov flow and, by [9, Theorem 7], the 
face $A_{\gamma}$ exactly subtends the cone $C_{\gamma}$, so every foliated ray meeting $A_{\gamma}$ 
can be represented by foliations that are defined by closed, nonsingular 
1-forms transverse to $L$. The following assertion is much stronger and will 
complete the proof of Theorem 1.1 in the pseudo-Anosov case.

**Theorem 5.3.** Let $M$ be hyperbolic, $[F]$ a foliated ray meeting the inte-
rior of $A_{\gamma}$, and let $F \in [F]$ be defined by a closed, nonsingular 1-form. 
Then $F$ is smoothly isotopic to a foliation transverse to the pseudo-Anosov 
flow $L$.

Section 6 will be devoted to the proof of Theorem 5.3. Indeed, except 
for some smoothness problems, dealt with in the Appendix, the proof is 
identical with that of Theorem 5.2.

5.3. The Reducible Case. Finally, we suppose that $M$ admits a fibration 
over $S^1$ with reducible monodromy. The following lemma is all that 
remains to be proven in order to handle this case.

**Lemma 5.4.** Let $\omega_0$ and $\omega_1$ be closed, nonsingular 1-forms on the thick-
ened torus $N = T^2 \times I$ which are cohomologous, transverse to $\partial N$, and equal 
there. Then there is a smooth 1-dimensional foliation $L$ of $N$, tangent to $\partial N$
and transverse to $\omega_1$, and a smooth isotopy, compactly supported in $\text{int } N$, that carries $\omega_0$ to a form transverse to $\mathcal{L}$.

We fix cohomologous forms $\omega_0$ and $\omega_1$ on $M$, defining the foliations $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively.

**Proof of Theorem 1.1 using Lemma 5.4.** Let $F$ be a fiber with reducible monodromy $\varphi: F \to F$ relative to a suitable choice of transverse, 1-dimensional foliation $\mathcal{L}$. There is a family of disjoint, essential, nonperipheral circles on $F$, no two of which are isotopic in $F$, which are permuted by $\varphi$ and decompose $F$ into pieces $\{F_1, ..., F_n\}$ that are permuted by $\varphi$. Furthermore, for a smallest integer $k_i > 0$, $\varphi^k(F_i) = F_i$ and $\varphi^k|_{F_i}$ is isotopic to a periodic or pseudo-Anosov map. Each subset of these decomposing circles that is permuted cyclically by $\varphi$ has $\mathcal{L}$-saturation an incompressible torus or Klein bottle. In the second case, a normal neighborhood of the Klein bottle is the (orientable) total space of a nonorientable interval bundle, the boundary of which is an incompressible torus. (We will call this normal neighborhood a “thickened Klein bottle.”)

The disjoint union $T$ of all the incompressible tori obtained as above is isotopic to a submanifold transverse to $\mathcal{F}_0$ [19]. Actually, [19] envisions only one incompressible torus, so we proceed inductively on the number $n$ of components of $T$. An ambient isotopy, applied to $T$, makes one of the components, say $T_1$, of $T$ transverse to $\mathcal{F}_0$. After cutting $M$ apart along $T_1$, we are reduced to the case in which $T$ has $n-1$ components and apply the inductive hypothesis. Reversing the ambient isotopy that makes $T$ transverse to $\mathcal{F}_0$, we obtain a smooth isotopy of $\mathcal{F}_0$ to a foliation transverse to $T$. Similarly, $\mathcal{F}_1$ is isotoped to be transverse to $T$.

Let $N(T)$ denote a closed normal neighborhood of $T$ in $M$, $N_0(T)$ its interior. Each component $M'$ of $M \setminus N_0(T)$ is either Seifert fibered, has the property that all of its fibrations over $S^1$ have pseudo-Anosov monodromy, or is a thickened Klein bottle. In this first two cases, Theorems 5.2 and 5.3, together with Proposition 2.1, allow us to isotope $\mathcal{F}_0|_{M'}$ to $\mathcal{F}_1|_{M'}$. In the third case, we again have an isotopy of $\mathcal{F}_0|_{M'}$ to $\mathcal{F}_1|_{M'}$ by Proposition 4.1. These isotopies, carried out in each component of $M \setminus N_0(T)$, are free at the boundary. In standard fashion, they are extended to an isotopy of $\mathcal{F}_0|_{N(T)}$, compactly supported in a neighborhood of the boundary. At this point, $\mathcal{F}_0$ and $\mathcal{F}_1$ agree on $M \setminus N_0(T)$ and satisfy the hypotheses of Lemma 5.4 in each component of $N(T)$. A vector field $v$ can be constructed on $M$, everywhere transverse to $\mathcal{F}_1$, tangent to $\partial N(T)$ and, in each component of $N(T)$, tangent to the foliation $\mathcal{L}$ of Lemma 5.4. Since $\mathcal{F}_0$ coincides with $\mathcal{F}_1$ outside of $N_0(T)$, $v$ is transverse to $\mathcal{F}_0$ there and Lemma 5.4 provides an isotopy of $\mathcal{F}_0$, compactly supported in $N_0(T)$, making...
\[ F \] everywhere transverse to \( v \). Finally, an application of Proposition 2.1 completes the proof.

**Proof of Lemma 5.4.** Let \( F_{i} \) be the foliation of \( N \) defined by \( \omega_{i}, \ i = 0, 1 \). The forms induce a linear foliation of the same constant slope on each component of \( \partial N \). Thus, choose an essential loop on \( T^{2} \times \{ 0 \} \) which is transverse to the restriction of \( \omega_{1} \) to that torus. Then there is an annulus \( A \subset N \) having this loop as a boundary component, the other component of \( \partial A \) being a loop on \( T^{2} \times \{ 1 \} \), also transverse to the restriction of \( \omega_{1} \). This annulus is incompressible, hence the constructions in [19] provide an isotopy that fixes \( \partial A \) pointwise and makes \( A \) transverse to \( \omega_{1} \). Thus, \( F_{i} \) is a foliation of the annulus, transverse to \( \partial A \) and without holonomy. Such foliations are well understood, being interval bundles over \( S^{1} \). We can choose a foliation \( L \) of \( A \) by circles transverse to \( F_{1} \), \( A \) being two of the leaves. By cutting \( N \) apart along \( A \), we obtain a manifold (with corners) homeomorphic to \( D^{2} \times S^{1} \) with a foliation transverse to the boundary and without holonomy. By Theorem 3.1, this is isotopic to the foliation by disks \( D^{2} \times \{ z \}, z \in S^{1} \). The 1-dimensional foliation by the \( S^{1} \) factors can be assumed to coincide with the foliation \( L' \) on the two copies of \( A \), hence regluing these copies of \( A \), we obtain a foliation \( L \) of \( N \) by circles, everywhere transverse to \( \omega_{1} \).

Another isotopy of \( A \), compactly supported in \( \text{int} \ N \), makes the annulus transverse to \( F_{0} \). Viewing this as an ambient isotopy and reversing it, we isotope \( F_{0} \) to be transverse to \( A \). One easily isotopes a leaf \( J \) of \( F_{0} \) to be transverse to \( L' = L \mid A \) and the isotopy can be required to be compactly supported in \( \text{int} \ A \). This extends to an ambient isotopy of \( F_{0} \), compactly supported in \( \text{int} \ N \). Cut \( A \) along \( J \) and apply Lemma 3.4 to isotope \( F_{0} \mid A \) to a foliation transverse to \( L' \). Again, the isotopy extends to an ambient isotopy of \( F_{0} \), compactly supported in \( \text{int} \ N \). Cutting open along \( A \) gives \( D^{2} \times S^{1} \) with a foliation \( F' \) by disks induced by \( F_{0} \). These disks are transverse to the circle fibration (induced by) \( L' \) near their boundaries.

Consider a fixed leaf \( D' \) of \( F' \). One readily sees that \( \partial D' \) is also the boundary of an imbedded disk \( D \), transverse to the circle fibration \( L' \). Indeed, \( \partial D' \) is isotopic in \( \partial D^{2} \times S^{1} \), through circles transverse to \( L' \), to the boundary of a leaf of (the foliation induced by) \( F_{0} \). This disk is isotopic through properly imbedded disks, transverse to \( L' \), to a disk \( D \) having the same boundary as \( D' \). Since \( D' \) is also transverse to \( L' \) near the boundary, we can assume that \( D \) and \( D' \) coincide on a compact annular neighborhood of their common boundary and are in general position to each other outside of that annulus. In standard fashion, the irreducibility of \( D^{2} \times S^{1} \) allows us to perform an isotopy of \( D' \), compactly supported in the interior of \( D^{2} \times S^{1} \), to a position that coincides with \( D \). One can now
cut apart along this leaf $D$ and apply Lemma 3.4, so as to make the whole foliation $\mathcal{F}'$ transverse to $L$. Since this isotopy is supported away from $\partial D^2 \times S^1$, it can be interpreted as an isotopy in $N$, compactly supported away from $A \cup \partial N$, which makes $\mathcal{F}_0$ transverse to $\mathcal{L}$.

6. THE IRREDUCIBLE CASES

We complete the proof of Theorem 1.1 by proving Theorems 5.2 and 5.3. Thus, $\mathcal{F} \in [\mathcal{F}]$ is defined by a closed, nonsingular 1-form and the foliated ray meets the interior of $A_\mathcal{F}$, where $\mathcal{L}$ is, respectively, a Seifert fibration or a pseudo-Anosov foliation. We first show that $\mathcal{F}$ is transverse to a foliation $\mathcal{L}'$, respectively, a Seifert fibration or a pseudo-Anosov foliation, then that $\mathcal{L}'$ is smoothly isotopic to $\mathcal{L}$. This smooth, ambient isotopy moves $\mathcal{F}$ to a foliation transverse to $\mathcal{L}$.

Naively, one might try to construct $\mathcal{L}'$ by choosing a 1-dimensional foliation $\mathcal{L}_0$ transverse to $\mathcal{F}$ and a fibration $\mathcal{F}'$ close to $\mathcal{F}$ and transverse to $\mathcal{L}_0$. Since $\mathcal{L}_0$ is the suspension of the monodromy $\varphi_0$ of a fiber $F$ of $\mathcal{F}'$, an isotopy of $\varphi_0$ to a periodic (respectively, pseudo-Anosov) automorphism $\varphi_1$ defines a homotopy of $\mathcal{L}_0$ to $\mathcal{L}'$ through foliations $\mathcal{L}_s$ transverse to $\mathcal{F}'$. The problem is that this is a large homotopy and there seems to be no natural way to keep it transverse to $\mathcal{F}$. Accordingly, we will construct the homotopy by a “suspension” construction of the dense-leaved foliation $\mathcal{F}$ itself, simultaneously isotoping $\mathcal{F}'$ to stay transverse to $\mathcal{L}_s$, $0 \leq s \leq 1$.

6.1. Tischler Fibrations and Coverings. Let $\mathcal{F}$ be a dense leaved foliation of $M$ defined by a closed, nonsingular 1-form. Such a foliation $\mathcal{F}$ can be arbitrarily well approximated in the $C^\infty$ topology by fibrations $\mathcal{F}'$ [25]. Furthermore, these approximating fibrations can be chosen so that each leaf $L$ of $\mathcal{F}$ is a regular covering of each fiber $F$ of $\mathcal{F}'$ by a projection along the leaves of an arbitrary 1-dimensional foliation $\mathcal{L}$ that is transverse to $\mathcal{F}$ and $\mathcal{F}'$ [10, p. 47].

Fix a foliation $\mathcal{L}$, transverse to $\mathcal{F}$ and integral to a $C^0$ line field. A closed, nonsingular 1-form $\omega$ defining $\mathcal{F}$ determines a flow,

$$\Phi: \mathbb{R} \times M \to M,$$

preserving $\mathcal{F}$ and having as flow lines the leaves of $\mathcal{L}$. This flow has continuous first derivative relative to the time parameter. We fix the choice of $\omega$, $\mathcal{L}$, and $\Phi$.

**Definition 6.1.** A fibration $\mathcal{F}'$ transverse to $\mathcal{L}$ is covered by $\mathcal{F}$ if, given a leaf $L$ of $\mathcal{F}$ and a fiber $F$ of $\mathcal{F}'$, there is a continuous function $\tau: L \to \mathbb{R}$ such that $\Phi_{\tau(x)}(x) \in F$, $\forall x \in L$. 

The map $p: L \rightarrow F$, defined by

$$p(x) = \Phi_{t(x)}(x), \quad \forall x \in L,$$

is a regular covering and the covering group $G$ is canonically a direct summand of $P_t \cong \mathbb{Z}^{\rho(e)}$ of corank one [3].

It is rather well known and elementary that this property is equivalent to the requirement that, for some (hence every) $x \in M$,

$$i_*(\pi_1(L, x)) \subset j_*(\pi_1(F, x)),$$

where

$$i: L \hookrightarrow M$$

$$j: F \hookrightarrow M$$

are the inclusions, respectively, of the leaf $L$ of $\mathcal{F}$ through $x$ and the fiber $F$ of $\mathcal{F}'$ through $x$ (For example, cf. [3], where a generalization is proven for open, connected, saturated sets without holonomy.) Another equivalent condition is that $\ker[\omega] \subseteq \ker[\omega']$, where we view the cohomology classes $[\omega]$ and $[\omega']$ as homomorphisms of $\pi_1(M)$ into $\mathbb{R}$.

These remarks prove that the covering property is purely homological. More precisely, we have the following.

**Lemma 6.2.** Let $[\mathcal{F}]$ be a foliated ray, $[\mathcal{F}']$ a fibered ray, and assume that $\mathcal{F}$ is dense-leaved. Then $[\mathcal{F}']$ contains a fibration $\mathcal{F}$ covered by a foliation $\mathcal{F} \in [\mathcal{F}]$ (defined by a closed, nonsingular 1-form) if and only if the following conditions are both satisfied:

1. There is a 1-dimensional foliation $\mathcal{L}$, integral to a $C^0$ line field, such that $[\mathcal{F}] \cap [\mathcal{F}'] \subseteq \text{int} C'_{\mathcal{F}}$.

2. If $[\omega] \in [\mathcal{F}]$ and $[\omega'] \in [\mathcal{F}']$, then $\ker[\omega] \subseteq \ker[\omega']$.

In this case, we say that $[\mathcal{F}']$ is covered by $[\mathcal{F}]$.

**Proposition 6.3.** There is a sequence $\{[\mathcal{F}_n]\}_{n=1}^\infty$ of fibered rays in $H^1(M; \mathbb{R})$, clustering at $[\mathcal{F}]$ and each covered by $[\mathcal{F}]$.

Fix a choice of smooth fibration $\mathcal{F}'$ which is transverse to $\mathcal{L}$ and covered by $\mathcal{F}$. Let $\omega'$ be a closed, nonsingular 1-form defining $\mathcal{F}'$. Choose $x_0 \in M$ and let $L$ be the leaf of $\mathcal{F}$ through $x_0$ and $F$ the fiber of $\mathcal{F}'$ through $x_0$. The monomorphisms of fundamental groups induced by the inclusions
allow us to identify \(\pi_1(L, x_0)\) and \(\pi_1(F, x_0)\) as subgroups of \(\pi_1(M, x_0)\). By the covering property,

\[
\pi_1(L, x_0) \subset \pi_1(F, x_0)
\]

and the group of covering transformations is identified canonically as

\[
G = \pi_1(F, x_0)/\pi_1(L, x_0).
\]

The short exact sequences.

\[
0 \to \pi_1(L, x_0) \subset \pi_1(M, x_0) \xrightarrow{\alpha} P(\alpha) \to 0
\]

\[
0 \to \pi_1(F, x_0) \subset \pi_1(M, x_0) \xrightarrow{\beta} P(\beta) \to 0
\]

give identifications

\[
P(\alpha) = \pi_1(M, x_0)/\pi_1(L, x_0)
\]

\[
P(\beta) = \pi_1(M, x_0)/\pi_1(F, x_0),
\]

so we obtain a short, exact sequence

\[
0 \to G \to P(\alpha) \to P(\beta) \to 0. \tag{2}
\]

Since \(P(\alpha) \cong \mathbb{Z}^{p(\alpha)}\), where \(p(\alpha) \geq 2\), and \(P(\beta) \cong \mathbb{Z}\), the sequence splits and \(P(\alpha) \cong G \oplus \mathbb{Z}\).

The algebra just sketched has a nice geometric realization. The period group \(P(\alpha)\) has a representation as a group of homeomorphisms of \(L\). Indeed, \(t \in P(\alpha)\) if and only if \(\Phi_t(L) = L\). We will view \(P(\alpha)\) as a subgroup of \(\text{Homeo}_+(L)\) or as an additive subgroup of \(\mathbb{R}\), according to context. If \(t \in P(\alpha)\) and \(\gamma = \Phi_t|L\), we will denote \(t\) by \(t_\gamma\). The covering map \(p: L \to F\) is of the form

\[
p(x) = \Phi_{t(x)}(x),
\]

where \(t: L \to \mathbb{R}\) is continuous, normalized so that \(t(x_0) = 0\). The covering group \(G \subset P(\alpha)\) can be identified as

\[
G = \{ \tau(x) - \tau(y) \mid x, y \in L \text{ and } p(x) = p(y) \}
\]

[3, Sect. 3]. The monodromy of the fibration \(\mathcal{F}^*\) relative to \(\mathcal{F}\) is the first return map on \(F\). Precisely, the monodromy \(\varphi: F \to F\) has the form

\[
\varphi(z) = \Phi_{p(\alpha)}(z),
\]
where $\rho : F \to (0, \infty)$ is a continuous map such that
\[
\Phi_{\rho(z)}(z) \in F, \quad \forall z \in F
\]
\[
\Phi_t(z) \notin F, \quad 0 < t < \rho(z).
\]
A reparametrization of $\mathcal{L}$ produces a flow $\Psi$ on $M$, preserving $\mathcal{F}'$ and such that $\varphi = \Psi_t$. This amounts to rescaling $\omega'$ so that $P(\omega') \subset \mathbb{R}$ is the subgroup of integers and gives a representation of this group in $\text{Homeo}_+(F)$. Again, we use these identifications without further comment.

**Lemma 6.4.** The monodromy $\varphi$ admits lifts $\hat{\varphi} : L \to L$ and any such lift is an element of $P(\omega)$. The choice of lift corresponds to a choice of splitting of the exact sequence (2).

**Proof.** The subgroup $\pi_1(L, x_0) \subset \pi_1(F, x_0)$ is exactly the subset represented by loops $\sigma \subset F$, based at $x_0$, for which $\int_\sigma \omega = 0$. But $\varphi(\sigma)$ is freely homotopic to $\sigma$ in $M$, so this subgroup is carried by $\varphi_*$ onto the corresponding subgroup of $\pi_1(F, \varphi(x_0))$. This implies that the lifts $\hat{\varphi}$ exist and are uniquely determined by the value of $\hat{\varphi}(x)$ at a specified point $x$. Since $\hat{\varphi}(x)$ and $x$ always lie on the same leaf of $\mathcal{L}$, it follows that
\[
\hat{\varphi}(x) = \Phi_{\rho(x)}(x), \quad \forall x \in L,
\]
for a continuous function $\hat{\rho} : L \to \mathbb{R}$. This implies that $\hat{\rho}(x) \in P(\omega) \subset \mathbb{R}$, $\forall x \in L$. The group $P(\omega)$ is discrete and $L$ is connected, implying that $\hat{\rho}$ is constant. Finally, the group $P(\omega')$ is generated by $\varphi$ and the third map in the exact sequence (2) carries an element $\psi$ to $\varphi$ if and only if $\hat{\psi}$ is a lift of $\varphi$. ■

6.2. Constructing the Homotopy. We now assume that $M$ is either Seifert fibered or hyperbolic and let $\mathcal{F}$ and $\mathcal{F}'$ be as above. The monodromy $\varphi_0$ of $\mathcal{F}'$ is either isotopic to a periodic diffeomorphism $\varphi_1$ or to a pseudo-Anosov automorphism $\varphi_1$ of the fiber $F$. In the latter case, $\varphi_1$ fails to be a diffeomorphism exactly at the finite set $\Sigma_0 \subset \text{int} F$ of interior singularities. (Contrary to the remark on p. 217 of [8], $\varphi_1$ is smooth at the boundary—see part A of the Appendix.) In this case, the isotopy $\varphi$ is continuous everywhere and smooth on $F \times I \setminus \Sigma_0 \times \{1\}$. We will call $\varphi_1$ a regular pseudo-Anosov automorphism and $\varphi$ a regular isotopy. For details, see the Appendix. In either case, we write $\varphi | F \times \{s\} = \varphi_s$, $0 \leq s \leq 1$, and, by a slight abuse, let $\varphi$ denote also the map
\[
\tilde{\varphi} : F \times I \to F \times I
\]
\[
\tilde{\varphi}(x, s) = (\varphi_s(x), s).
\]
Fix a choice of lift $\psi_0: L \rightarrow L$ of $\phi_0$ and let

$$\tilde{\psi}: L \times I \rightarrow L \times I$$

be the lift of $\psi$ agreeing with $\psi_0$ on $L \times \{0\}$. Let $\psi_s = \tilde{\psi} | L \times \{s\}$, $0 \leq s \leq 1$. Since $\psi_0 \in P(\omega)$, it commutes with all $\gamma \in G$.

**Lemma 6.5.** For $0 \leq s \leq 1$ and all $\gamma \in G$, $\psi_s \cdot \gamma = \gamma \cdot \psi_s$.

**Proof.** Fix $x \in L$ and $\gamma \in G$. Since $p \cdot \gamma = p$ and $p \cdot \psi_s = \phi_s \cdot p$, $0 \leq s \leq 1$, we have

$$p(\gamma(\psi_s(x))) = \phi_s(\gamma(x)) = p(\psi_s(\gamma(x))), \quad 0 \leq s \leq 1.$$ 

Consequently, there is $\gamma_0 \in G$ such that

$$\gamma(\gamma_s(x)) = \psi_s(\gamma(x)), \quad 0 \leq s \leq 1.$$ 

But $G$ is discrete and $\gamma_0$ is continuous in $s$, so $\gamma_s = \gamma_0$ is constant. Therefore, $\gamma_0 = \gamma_0 = \text{id}$ and $x \in L$ is arbitrary, as is $\gamma \in G$.  

Since $P(\omega) = G \oplus \mathbb{Z}$, Lemma 6.5 allows us to define a representation

$$P(\omega) \times (\mathbb{R} \times L \times I) \rightarrow \mathbb{R} \times L \times I$$

of $P(\omega)$ as a group of homeomorphisms of the 4-manifold $\mathbb{R} \times L \times I$ by setting

$$\gamma \cdot (t, x, s) = (t - t_\gamma, \gamma(x), s), \quad \forall \gamma \in G,$$

$$k \cdot (t, x, s) = (t - kt_\phi, \psi_s(x), s), \quad \forall k \in \mathbb{Z}.$$ 

This is a properly discontinuous action having compact fundamental domain, hence the quotient space, denoted by $W_\phi$, is a compact 4-manifold. The submanifolds $\mathbb{R} \times L \times \{s\}$ pass to submanifolds of $W_\phi$, denoted by $M_s$, $0 \leq s \leq 1$, which are leaves of a foliation without holonomy. The manifolds $\{t\} \times L$ pass to one–one immersed copies of themselves in $W_\phi$, also leaves of a foliation $\mathcal{F}$ without holonomy, and the manifolds $\mathbb{R} \times \{(x, s)\}$ pass to the leaves of a 1-dimensional foliation $\mathcal{L}$ transverse to $\mathcal{F}$.

**Lemma 6.6.** The manifold $W_\phi$ has a smooth structure relative to which it is diffeomorphic to the product $M \times I$ in such a way that the factors $M \times \{s\}$ are identified with $M_s$, $0 \leq s \leq 1$, the factors $\{x\} \times I$ are tangent to the leaves of $\mathcal{F}$, and the foliation $\mathcal{L}$ is tangent to the factors $M \times \{s\}$. In particular, $\mathcal{F}$ identified with the foliation $\mathcal{F} \times I$ and $\mathcal{L}$ is a homotopy of $\mathcal{L}_0$ to $\mathcal{L}_1$ through foliations transverse to $\mathcal{F}$.
Proof. In the periodic case, everything in sight is smooth. In order to see that $W_{\Phi}$ is smooth in the pseudo-Anosov case, it will be convenient to pass to the quotient $W_{\Phi}$ in two stages, first dividing out the action of $G$, then the action of the factor $Z$. The quotient by $G$ is the total space of a principal $\mathbb{R}$-bundle

$$\pi: \tilde{W} \to F \times I,$$

and the submanifolds $\{t\} \times L \times I$ pass to leaves of a foliation $\tilde{\mathcal{F}}$ of $\tilde{W}$ which is transverse to the fibers and has leaves diffeomorphic to $L \times I$. So far, everything in sight is smooth. The action of $Z$ is generated by the homeomorphism

$$\tilde{\psi}: \tilde{W} \to \tilde{W}$$

$$\tilde{\psi}[t, x, s] = [t - t_{\Phi}, \psi_t(x), s],$$

where the notation $\tilde{\psi}$ is a slight abuse. Evidently, the diagram

$$\begin{array}{ccc}
\tilde{W} & \xrightarrow{\tilde{\psi}} & \tilde{W} \\
\downarrow \pi & & \downarrow \pi \\
F \times I & \xrightarrow{\psi} & F \times I
\end{array}$$

(#)

commutes, where $\pi$ is the bundle projection. Thus $\tilde{\psi}$ permutes the finitely many fibers $\pi^{-1}(x, 1), x \in \Sigma_\alpha$, and the union of these fibers quotients to a finite union $\Sigma$ of closed leaves of $\mathcal{F}$. These leaves have tubular neighborhoods that can be smoothly coordinatized by the local model described in part B of the Appendix, making $W_{\Phi}$ a smooth manifold and $\mathcal{F}$ a smooth foliation.

In all cases, the map

$$\mathbb{R} \times L \times \{0\} \to M$$

$$(t, x, 0) \mapsto \Phi_t(x)$$

passes to a canonical diffeomorphism of $M_{\Phi_0}$ to $M$. Since this is a compact leaf of a foliation without holonomy, it follows that $W_{\Phi} \cong M \times I$ as asserted and that $\mathcal{F}$ is tangent to the factors $M \times \{s\}$. In the product structure on $W_{\Phi}$, the $I$-factors are not yet specified, but the projection

$$p: W_{\Phi} \to I$$

is canonically determined by the requirement that $p(M_{\Phi}) = s, 0 \leq s \leq 1$.

Finally, the leaves of $\tilde{\mathcal{F}}$ are transverse to the submanifolds $M_{\Phi_0}$, so one
chooses a vector field \( v \), tangent to \( F \) and such that \( p_\ast(v) = d/ds \). The flow lines of \( v \), \( 0 \leq s \leq 1 \), are the \( I \)-factors.

Thus, we have homotoped the original 1-dimensional foliation \( \mathcal{L} = \mathcal{L}_0 \) through foliations transverse to \( \mathcal{F} \) to a foliation \( \mathcal{L}_1 \) which we expect to be a Seifert fibration or pseudo-Anosov, depending on which case we are studying. Generally, this is a large homotopy which may not stay transverse to the fibration \( \mathcal{F} \), but we will prove the following.

**Proposition 6.7.** There is a smooth isotopy of the fibration \( \mathcal{F}' \) to a fibration \( \mathcal{F}'_1 \), transverse to \( \mathcal{L}_1 \), and the first return map induced by \( \mathcal{L}_1 \) on the fibers is smoothly conjugate to \( \varphi_1 \).

In the proof, it will be useful to lift \( \mathcal{F}' \) to the \( k \)-fold cover \( p: \hat{M} \to M \) induced by the standard \( k \)-fold covering map \( q: S^1 \to S^1, k \geq 1 \). The covering map

\[ p \times \text{id}: \hat{M} \times I \to M \times I \]

lifts \( \mathcal{D} \) to a 1-dimensional foliation \( \mathcal{D}' \).

**Lemma 6.8.** Let \( [\eta] \in H^1(M \times I; \mathbb{R}) \) be an arbitrary class. If \( (p \times \text{id})^\ast [\eta] \in \text{int} \, C_{\mathcal{D}} \), then \( [\eta] \in \text{int} \, C_{\mathcal{D}} \).

**Proof.** Foliation cycles are transverse invariant measures [22]. Let \( v \) be such a measure for \( \mathcal{D} \) and \( v' \) its lifts to such a measure for \( \mathcal{D}' \). Since \( p \times \text{id} \) is a \( k \)-fold covering map,

\[ \int (p \times \text{id})^\ast (\eta) \, dv' = k \int \eta \, dv. \]

By assumption, the left hand side is strictly positive. But \( v \) is arbitrary, implying that \( [\eta] \in \text{int} \, C_{\mathcal{D}} \).  

**Lemma 6.9.** The fibered ray \( [\mathcal{F}' \times I] \) is contained in the cone \( C_{\mathcal{D}} \).

**Proof.** By Lemma 6.8, it will be enough to show that, for some integer \( k \geq 1 \),

\[ (p \times \text{id})^\ast [\mathcal{F}' \times I] \subset C_{\mathcal{D}}. \]

Let \( \tilde{W} \) be as in the proof of Lemma 6.6. The quotient map

\[ \tilde{W}|F \times \{0\} \to M_{\eta_0} = M \]

lifts the fibration \( \mathcal{F}' \) to a fibration by sections of the principal bundle \( \tilde{W}|F \times \{0\} \). Let \( \eta_0 : F \times \{0\} \hookrightarrow \pi^{-1}(F \times \{0\}) \) be one of these sections. Since the structure group \( \mathcal{R} \) of the principal bundle \( \tilde{W} \) is contractible, we can find
a smooth section \( i: F \times I \subset \tilde{W} \) which extends \( i_0 \). For a large enough integer \( k \), \( \tilde{\psi}^k(\tilde{\alpha}(F \times I)) \cap \tilde{\alpha}(F \times I) = \emptyset \) and these two copies of \( F \times I \) are sections cobounding a compact manifold diffeomorphic to \([0, -k t_0] \times F \times I\).

Iterating the action of \( \tilde{\psi}^k \) and its inverse propagates this product structure, defining an explicit diffeomorphism of \( \tilde{W} \) to \( \mathbb{R} \times F \times I \). In the pseudo-Anosov case, we can arrange that each point of \( \Sigma_0 \times \{1\} \subset F \times \{1\} \) has a small neighborhood in \( F \times I \) over which the factors \( \{t\} \times F \times I \) are tangent to the leaves of \( \tilde{\mathcal{F}} \). The quotient by the action of \( \{\tilde{\psi}^k\}_{k \in \mathbb{Z}} \) is the \( k \)-fold cover \( \tilde{M} \times I \) of \( M \times I \) with an induced fibration \( \tilde{\mathcal{F}}' \) of \( \tilde{M} \times I \) which is transverse to \( \tilde{\mathcal{F}} \). The tangency condition over neighborhoods of the singular points guarantees the smoothness of \( \tilde{\mathcal{F}}' \) in \( \tilde{M} \times I \). We have also guaranteed that \( \tilde{\mathcal{F}} | \tilde{M} \times \{0\} \) has a fiber in common with \( p^{-1}(\mathcal{F}') \) and it follows that \((p \times \text{id})^* \tilde{\mathcal{F}}' = \tilde{\mathcal{F}}' \).

**Proof of Proposition 6.7.** By Lemma 6.9 there is a fibration \( \mathcal{F}' \) of \( M \times I \), transverse to \( \mathcal{F} \), with \( [\mathcal{F}'] = [\mathcal{F} \times I] \). Since \( \mathcal{F} \) is tangent to the factors \( M \times \{s\} \), \( \mathcal{F}' \) is also transverse to these factors, hence defines an integrable homotopy of \( \mathcal{F}_0 = \mathcal{F}' | M \times \{0\} \) through fibrations \( \mathcal{F}'_s \) transverse to \( \mathcal{F}'_s = \mathcal{F} | M \times \{s\}, 0 \leq s \leq 1 \). Furthermore, \( [\mathcal{F}'] = [\mathcal{F}'_0] \), and both of these fibrations are transverse to \( \mathcal{F}_0 \), so Proposition 2.1 implies that \( \mathcal{F}' \) is smoothly isotopic to \( \mathcal{F}_0' \).

The quotient map \( \tilde{W} \rightarrow M \times I \) injects \( \pi_1(\tilde{W}) \) as the subgroup \( \pi_1(F \times I) \). It follows that the fibered \( \mathcal{F}' \) pulls back to a foliation \( \mathcal{F}' \) by sections of the projection \( \pi: \tilde{W} \rightarrow F \times I \). Commutativity of the diagram (22) then implies that the first return map on a fiber of \( \mathcal{F}' \) is smoothly conjugate to \( \phi \).

6.3. The Proof of Theorems 5.2 and 5.3. Given \( \mathcal{F} \) as above, we have shown that there is a fibration \( \mathcal{F}' \) covered by \( \mathcal{F} \) and a 1-dimensional foliation \( \mathcal{L}' \), transverse both to \( \mathcal{F} \) and \( \mathcal{F}' \) and inducing the periodic or regular pseudo-Anosov monodromy \( \varphi_1 \) on a fiber of \( \mathcal{F}' \). Furthermore, the fibered ray \( [\mathcal{F}'] \) can be chosen as close to \( [\mathcal{F}] \) as desired. In particular, if \( \mathcal{A} \) is the face of the Thurston ball pierced by \( [\mathcal{F}] \), we can assume that \( [\mathcal{F}'] \) also meets the interior of this face. By Waldhausen’s theorem, there is a smooth isotopy of \( \mathcal{F}' \) to a fibration transverse to \( \mathcal{L}' \). Moving \( \mathcal{F} \) and \( \mathcal{L}' \) by this ambient isotopy, we lose no generality in assuming that \( \mathcal{F}' \) is simultaneously transverse both to \( \mathcal{L}' \) and \( \mathcal{L} \). In the respective cases, both of these 1-dimensional foliations induce isotopic periodic or regular pseudo-Anosov monodromy on the fibers of \( \mathcal{F}' \).

**Lemma 6.10.** If \( \varphi_i: F \rightarrow F, \ i = 1, 2, \) are periodic diffeomorphisms or regular pseudo-Anosov automorphisms in the same isotopy class, then there is a diffeomorphism \( g: F \rightarrow F \), isotopic to the identity, such that \( g \varphi_2 g^{-1} = \varphi_1 \).
Proof. In the pseudo-Anosov case, if \( \partial F = \emptyset \) the existence of \( g \) is proven in [8, pp. 238–241]. The proof adapts fairly easily to the case \( \partial F \neq \emptyset \) (cf. Lemma 4 in the Appendix). Assume, then, that \( \varphi_1 \) and \( \varphi_2 \) are periodic. The isotopy class \([\varphi_1] = [\varphi_2]\) induces a continuous map of the Thurston compactification of Teichmüller space to itself. In the Thurston classification, the periodic case is exactly the one in which this isotopy class fixes a unique point \( x \) of Teichmüller space itself [8, Exposé 9 and 11].

Viewing Teichmüller space as the set of isotopy classes of hyperbolic metrics, we see that \( \varphi_i \) is an isometry of a unique metric \( h_i \), \( i = 1, 2 \). Let \( g \in \text{Diff}_0(F) \) be such that \( g^*(h_2) = h_1 \), remarking that \( g \varphi_2 g^{-1} \) is an isometry of \( h_1 \). This isometry is isotopic to \( \varphi_1 \), hence is identical with it by [8, Exposé 3, Theorem 18].

For the case of pseudo-Anosov monodromy, the following is proved by Fried [9, p. 262] in the topological category. The proof works in the periodic case as well and a result of Earle and Schatz [6] makes the isotopy smooth. For notational convenience, we set \( L = L_2 \) in the following.

**Lemma 6.11.** There is a smooth isotopy of \( M \), carrying \( L_1 \) to \( L_2 \) and leaving \( \mathcal{F}' \) leafwise invariant.

**Proof.** Fix a fiber \( F \) of \( \mathcal{F}' \) and let \( \Phi_i^t \) be a flow, parametrizing \( \mathcal{L}_i \), \( i = 1, 2 \), so that \( \Phi_1^t(F) = \Phi_2^t(F) = F_i \) is a fiber, \( \forall t \in \mathbb{R} \), and \( F_1 = F \). Then the automorphisms \( \varphi_i = \Phi_i^1 | F_i \), \( i = 1, 2 \), are periodic or regular pseudo-Anosov automorphisms of \( F \) in the same isotopy class. By Lemma 6.10, there is a preliminary isotopy of \( M \), leaving \( \mathcal{F}' \) leafwise invariant, after which \( \varphi_1 = \varphi_2 \).

For \( 0 < t < 1 \), define \( g_t : F_i \to F_i \) by

\[
g_t(\Phi_i^t(x)) = \Phi_i^1(x), \quad \forall x \in F.
\]

Then \( g_0 = g_1 = \text{id} \) and we claim that each \( g_t \) is a diffeomorphism, isotopic to the identity. In the periodic case, this is clear since the flows are everywhere smooth. In the pseudo-Anosov case, remark first that, on \( F_i \),

\[
g_t \Phi_i^1 = g_t \Phi_i^1 = g_t \Phi_i^2 = g_t \Phi_i^2 \Phi_i^1 = g_t \Phi_i^1 \Phi_i^1 = \Phi_i^1.
\]

Remark also that \( \Phi_i^1 | F_i \) and \( \Phi_i^2 | F_i \) are regular pseudo-Anosov automorphisms. Write the general point of \( F_i \) as \( y = \Phi_i^1(x), x \in F \), and compute

\[
g_t \Phi_i^2 g_t^{-1}(y) = g_t \Phi_i^2 \Phi_i^1(x) = g_t \Phi_i^2 \Phi_i^1(x) = \Phi_i^1(y).
\]

Thus, \( g_t \) conjugates one regular pseudo-Anosov automorphism to another and the assertion follows by Lemma 5 in the Appendix.
The diffeomorphisms \( g_t \) assemble to define a diffeomorphism \( G: M \to M \) which preserves each fiber \( F_t \). This is clear in the periodic case. In the pseudo-Anosov case, \( G \) is a diffeomorphism in the complement \( M \setminus \Sigma \) of the interior singular orbits. If \( \sigma \) is such an orbit of \( \Phi^t \) and \( G(\sigma) \) the corresponding orbit of \( \Phi^t \), the standard model (part A of the Appendix) gives smooth coordinates \( (r, \theta, z) \) in the neighborhoods of these orbits, relative to which \( G \) is a rotation of the \( \theta \) coordinate which is independent of \( (r, z) \). Indeed, the proof of smoothness of each \( g_t \) at interior singularities [8, p. 241] shows that there are just a finite number of possible angles of rotation. Since \( g_t \) is continuous in \( t \), the angle of rotation is a constant and \( G \) is a diffeomorphism.

Finally, since the identity component \( \text{Diff}_0(\mathcal{F}) \) of the group of diffeomorphisms of \( \mathcal{F} \) is simply connected [6], \( G \) is isotopic to the identity through fiber-preserving diffeomorphisms. The isotopy of this lemma moves \( \mathcal{F} \) to a foliation transverse to \( \mathcal{L} \) and the proofs of Theorems 5.2 and 5.3 are complete.

**APPENDIX: PSEUDO-ANOSOV TECHNICALITIES**

We establish some folklore about the smoothness problems associated to pseudo-Anosov maps and flows. Although this material is well known to experts, the smoothness assertions in our proof of Theorem 5.3 require that these problems be handled carefully and it seems best to give details.

A. Pseudo-Anosov Maps. Pseudo-Anosov maps cannot be honest diffeomorphisms, failing to be \( C^1 \) at the interior singularities. This creates technical difficulties and we must show how to sidestep them. It will also be necessary to choose the pseudo-Anosov map to be smooth at the boundary. The fact that this can be done is obscured in the literature (cf. [8, pp. 216–217, erratum, p. 286]). Finally, the uniqueness of pseudo-Anosov maps in their isotopy class (up to conjugation by diffeomorphisms isotopic to the identity), proven in [8, pp. 238–241] for closed surfaces, must be extended to compact surfaces with boundary. The proof in [8] goes through without serious changes.

Let \( \varphi: \mathcal{F} \to \mathcal{F} \) be a pseudo-Anosov homeomorphism of a compact surface \( \mathcal{F} \) with associated stable and unstable foliations \( \mathcal{T}^s \) and \( \mathcal{T}^u \), respectively. These are mutually transverse, unoriented, measured foliations by curves, having isolated “\( p \)-prong” singularities, \( p \geq 3 \). In the interior of \( \mathcal{F} \), the singularities of \( \mathcal{T}^s \) and \( \mathcal{T}^u \) coincide, transversality being interpreted as indicated in Fig. 1, where \( p = 3 \). The solid curves represent leaves of \( \mathcal{T}^u \) and the dotted one leaves of \( \mathcal{T}^s \). If \( S \) is a component of \( \partial \mathcal{F} \), each foliation
FIG. 1. A 3-prong singularity for the stable and unstable foliations.

has the same number \( p \geq 1 \) of singular points on \( S \) and the singularities of \( \mathcal{H}^s \) on \( S \) are disjoint from those of \( \mathcal{H}^u \). Each singularity on \( S \) has three separatrices, two lying in \( S \) and one in \( \text{int} \, F \). Transversality of the two foliations fails on \( S \) where, outside of the singular points, they coincide. This situation is indicated in Fig. 2.

Initially, we assume no smoothness for the above structures. The pseudo-Anosov map \( \varphi \) preserves the stable and unstable foliations. If \( \mu^u \) is the

FIG. 2. The stable and unstable foliations near the boundary.
transverse, invariant measure for $\mathcal{H}^u$, then there is a constant $\lambda > 1$ such that the pull-back of $\mu^u$ by $\varphi$ is
$$\varphi^*\mu^u = \lambda \mu^u.$$ That is, an application of $\varphi$ expands the $\mu^u$ distance between the leaves of $\mathcal{H}^u$ by a factor $\lambda$. Similarly, if $\mu^s$ is the transverse, invariant measure for $\mathcal{H}^s$, $\varphi$ shrinks the $\mu^s$ distance between the stable leaves by the factor $1/\lambda$.

In much of the literature (e.g., [2]) smoothness questions are ignored, $\varphi$ being a homeomorphism, $\mathcal{H}^u$ and $\mathcal{H}^s$ being $C^0$ foliations, $\mu^u$ and $\mu^s$ continuous measures.

**Definitions**

1. The quintuple $(\varphi, \mathcal{H}^u, \mathcal{H}^s, \mu^u, \mu^s)$ is called a topological pseudo-Anosov system.

   For our constructions, purely topological pseudo-Anosov systems are inadequate. Accordingly, we follow [8] in requiring that the transverse, holonomy invariant measures $\mu^u$ and $\mu^s$ be smooth. That is, in the neighborhood of regular points, the foliations are defined by closed 1-forms $\omega^u$ and $\omega^s$ such that $\omega^u \wedge \omega^s$ is nowhere zero. These forms are only defined up to sign since the foliations generally are not transversely orientable. The forms can also be defined in a neighborhood of an interior singularity and in a neighborhood of the boundary. They vanish exactly at isolated singular points. Both forms also vanish on vectors tangent to $\mathcal{F}$, but they only vanish on vectors transverse to the boundary at their respective singular points. Being defined by smooth forms, the foliations are smooth.

   Finally, the pseudo-Anosov map $\varphi$ is to be a diffeomorphism on the complement of the interior singularities.

2. If $(\varphi, \mathcal{H}^u, \mathcal{H}^s, \mu^u, \mu^s)$ satisfies the above smoothness properties, it is said to be regular. In this case, $\varphi$ is said to be a regular pseudo-Anosov automorphism of $\mathcal{F}$.

   We are going to show that, even if the pseudo-Anosov system is purely topological, satisfying none of the smoothness requirements, there is a choice of differentiable structure on $\mathcal{F}$ relative to which $(\varphi, \mathcal{H}^u, \mathcal{H}^s; \mu^u, \mu^s)$ is regular. It will follow easily that, without a change of differentiable structure, one can find a $C^0$ isotopy of $\mathcal{F}$ that deforms the pseudo-Anosov system to a regular one. For reasons already mentioned, we will be particularly careful about behavior at $\partial \mathcal{F}$.

   Let $z \in \text{int } \mathcal{F}$ be a regular point and construct a coordinate chart about $z$ as follows. Give $z$ arbitrary coordinates $(u_0, v_0)$, select orientations of $\mathcal{H}^u$ and $\mathcal{H}^s$ in a product neighborhood of $z$, define the $u$-coordinate along the local leaf of $\mathcal{H}^u$ through $z$ to be constantly equal to $u_0$, and lay out the $u$-coordinate along the local leaves of $\mathcal{H}^s$ via the continuous measure $\mu^s$. 

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**References**

2. Fathi, Aoude; Herman, Marcel; Lasota, Adam. *V. Arnold.*
Similarly, define the \( v \)-coordinate along the local leaves of \( \mathcal{H}^r \) via \( \mu' \) so as to be the constant \( v_0 \) along the local leaf of \( \mathcal{H}^s \) through \( z \). By the holonomy invariance of these measures, we obtain a continuous coordinate chart. On any connected component of the intersection of two such charts, say \((U, u, v)\) and \((U', u', v)\), the change of coordinates will be

\[
\begin{align*}
    u'(u, v) &= eu + c \\
    v'(u, v) &= e'v + d,
\end{align*}
\]

where \(|e| = |e'| = 1\) and \(e\) and \(d\) are constants. Choosing such a chart around each regular point defines a \( C^\infty \) atlas on the open surface \( \text{int} F \setminus \Sigma_0 \), where \( \Sigma_0 \) is the set of interior singularities. In this atlas, the local formula of the pseudo-Anosov homeomorphism is

\[
\varphi(u, v) = \left( \begin{array}{c}
    u + a \\
    v + b
\end{array} \right),
\]

for suitable locally defined constants \(a\) and \(b\).

Consider a region \( U \subset \text{int} F \), adjacent to a component \( S \) of \( \partial F \) and strictly between two \( \mathcal{H}^s \)-separatrices which issue into \( \text{int} F \) from consecutive \( \mathcal{H}^r \)-singularities on \( S \) (Fig. 3). One can construct a coordinate chart \((U, u, v)\), as above, so that \( u \neq 0 \) on \( U \) and approaches 0 at \( S \) and the bounding separatrices. We also arrange that \( v \equiv 0 \) on along the \( \mathcal{H}^s \)-separatrix in \( U \), hence assumes both positive and negative values, and approaches 0 at \( S \). We define “polar coordinates” in \( U \) by the formulas

\[
\begin{align*}
    u &= r \cos \theta \\
    v &= r \sin \theta,
\end{align*}
\]

where \( c > r > 0 \) and \((2n-1)\pi/2 < \theta < (2n+1)\pi/2\), for some positive constant \( c \) and some \( n \in \mathbb{Z} \). Figure 3 can be drawn so that \((r, \theta)\) defines an

\[
\text{FIG. 3. A chart adjacent to the boundary.}
\]
orthogonal grid with \( S \) the line \( r = 0 \) and the vertical bounding separatrices the lines \( \theta = (2n \pm 1) \pi/2 \). That is, the \((r, \theta)\)-coordinates are extended to coordinates of an open neighborhood

\[
U_n = \{(r, \theta) \mid (2n - 1) \pi/2 < \theta < (2n + 1) \pi/2 \text{ and } 0 \leq r < \epsilon \}
\]

in \( F \) of an open segment of \( S \). In these coordinates, the leaves of \( \mathcal{H}^u \mid U_n \) are level sets of \( r \cos \theta \) and the leaves of \( \mathcal{H}^u \mid U_n \) are level sets of \( r \sin \theta \). Since \((U_n \setminus S, u, v)\) is smoothly compatible with the other interior charts already constructed, so is \((U_n, r, \theta)\). A similar construction works between two consecutive \( \mathcal{H}^u \)-separatrices issuing out of \( S \), providing a neighborhood

\[
V_n = \{(r, \theta) \mid 2n\pi/2 < \theta < (2n + 2) \pi/2 \text{ and } 0 \leq r < \epsilon \}.
\]

If each foliation has \( p \) singular points on \( S \), we let \( n = 0, 1, \ldots, p - 1 \) and obtain smoothly compatible charts covering a collar neighborhood of \( S \). Indeed, if we take \( \theta \mod p \pi \) and \( n \mod p \), the change of coordinates on overlaps \( U_n \cap V_n \) and \( V_n \cap U_{n+1} \) is the identity.

We must check the smoothness of \( \varphi \) at the boundary in the above coordinates. Choosing such coordinates in both a domain and range, and appealing to the fact that \( u \) and \( v \) vanish at \( \partial F \), we obtain the \((u, v)\)-formula

\[
\varphi(u, v) = (\lambda u, v/\lambda)
\]

in \( U_n \setminus S \). Passing to the associated polar coordinates and replacing the coordinate \( \theta \) with \( w = \tan \theta = v/u \), \( -\infty < w < \infty \), we readily compute

\[
\varphi(r, w) = \left( r \sqrt{\frac{w^2 + w}{1 + w^2}}, \frac{w}{\lambda} \right)
\]

in \( U_n \). In \( V_n \), we use \( w = \cot \theta = u/v \) and produce a similar formula. We emphasize that this formula holds along \( S \). In fact, when \( w = \tan \theta \),

\[
\varphi(0, w) = (0, w/\lambda^2)
\]

and, in the alternative case,

\[
\varphi(0, w) = (0, \lambda^2 w),
\]

so \( \varphi \mid \partial F \) expands at the singular points of \( \mathcal{H}^u \) in \( \partial F \) and contracts at those of \( \mathcal{H}^u \). Of course, \( \varphi \) generally permutes the components of \( \partial F \), but some positive power of \( \varphi \mid \partial F \) has the boundary singularities as its sole fixed points and is alternatively a contraction or expansion as asserted.
At an interior singularity \( z \), one can use essentially the same coordinates as at the boundary, collapsing the line \( r = 0 \) to a point as with the usual polar coordinates. Here \( p \geq 3 \) and we obtain a “polar coordinate” neighborhood of \( z \) in which the polar angle is taken mod \( p\pi \) rather than the standard \( 2\pi \). The resulting charts are \( C^\infty \) compatible with the ones already constructed, but smoothness of \( \varphi \) at the singularity is destroyed. Indeed, since the number of separatrices at \( z \) for each foliation is \( \geq 3 \) the Jacobian of some power of \( \varphi \) at \( z \) would have two linearly independent eigenvectors with eigenvalue \( \lambda > 1 \) and two more with eigenvalue \( 1/\lambda \).

**Lemma 3.** Let \((\varphi, \mathcal{H}^u, \mathcal{H}^s, \mu^u, \mu^s)\) be a topological pseudo-Anosov system on the compact, smooth surface \( F \). Then there is a \( C^0 \) isotopy of \( F \) that deforms this system to a regular one.

**Proof.** Let the given differentiable structure on \( F \) be denoted by \( \mathcal{A} \) and, as above, choose a differentiable structure \( \mathcal{B} \) on \( F \) relative to which \((\varphi, \mathcal{H}^u, \mathcal{H}^s, \mu^u, \mu^s)\) is regular. Since the classification, up to diffeomorphism, of compact, connected, orientable surfaces \( F \) depends only on the topology of \( F \) [11, Chap. 9], there is a diffeomorphism \( h : (F, \mathcal{A}) \rightarrow (F, \mathcal{B}) \).

Evidently, \( h \) pulls \((\varphi, \mathcal{H}^u, \mathcal{H}^s, \mu^u, \mu^s)\) back to a regular pseudo-Anosov system. If \( h \), as a homeomorphism of the underlying topological manifold \( F \), is isotopic to the identity, we are done. Otherwise, choose a diffeomorphism \( g \) of \((F, \mathcal{B})\) which is in the isotopy class of \( h \) and replace \( h \) with \( h \cdot g^{-1} \).

In particular, if the monodromy of a fibration is isotopic to a pseudo-Anosov homeomorphism, it is isotopic to a regular one.

**Lemma 4.** If \( \varphi_i, i = 1, 2, \) are regular pseudo-Anosov automorphisms of \( F \) in the same isotopy class, then there is a diffeomorphism \( g : F \rightarrow F \) isotopic to the identity, such that \( g\varphi_1g^{-1} = \varphi_2 \).

This is proven in [8, pp. 238–241], but the proof only considers explicitly the case of closed surfaces. A preliminary isotopy reduces to the case that both diffeomorphisms have the same unstable measured foliation:

\[(\mathcal{H}^u, \mu^u) = (\mathcal{H}'^u, \mu'^u)\]  

The proof of this works equally well in the presence of boundary. It is then shown that the sequence \( \{\varphi_2 \cdot \varphi_1^n\}_{n \geq 0} \) converges uniformly on \( F \) to a homeomorphism \( g \) such that \( g\varphi_1g^{-1} = \varphi_2 \) and \( g \) is isotopic to the identity. In this argument, the convergence is proven relative to the topological
metric induced by $ds^2 = (d\mu_1^2 + (d\mu_2)^2$. The metric degenerates at the boundary, so this argument only gives uniform convergence on compact subsets of $\text{int } F$. In this way, we obtain a homeomorphism on $\text{int } F$ with the desired property there. The final step in the above reference is to prove the following.

**Lemma 5.** Any topological conjugation $g$ between regular pseudo-Anosov automorphisms is automatically a diffeomorphism.

Here too, the proof goes through in $\text{int } F$. Near the boundary, one computes $g$ using the $(u_1, v_1)$-coordinates in the domain and the $(u_2, v_2)$-coordinates in the range. (Here, the indices indicate which pseudo-Anosov system is used to compute the coordinates and the measures have been scaled appropriately.) Relative to the associated polar coordinates, it is easily checked that $g$ is a translation in the $\theta$ coordinate and fixes $r$. Thus, $g$ extends to the boundary to a diffeomorphism of $F$ with all the required properties. (This is essentially the same argument indicated in [8, p. 241] to prove smoothness at the interior singularities.) In particular, Lemma 4 and Lemma 5 continue to hold for surfaces with boundary.

**B. Pseudo-Anosov Flows.** If $\varphi : F \to F$ is a regular pseudo-Anosov automorphism, the suspension of $\varphi$ defines a 3-manifold $M_\varphi = \mathbb{R} \times F / \{(t, x) \sim (t - 1, \varphi(x))\}$ and a fibration $M_\varphi \to S^1$, together with a 1-dimensional foliation $\mathcal{F}$ transverse to the fibers. Indeed, the leaves of $\mathcal{F}$ are parametrized to define a fiber-preserving flow $\Phi_t$ on $M_\varphi$ such that the restriction of $\Phi_t$ to a fiber $F$ is $\varphi$. This construction is smooth on the complement of the union $\Sigma$ of the finitely many closed leaves of $\mathcal{F}$ corresponding to the singular set $\Sigma_0 \subset \text{int } F$. In particular, $M_\varphi \setminus \Sigma$ inherits a canonical smooth structure.

There is a standard model in $\mathbb{R}^2 \times S^1$ of a pseudo-Anosov flow corresponding to a $p$-prong singularity at the origin of $\mathbb{R}^2$. The “polar coordinates” $(r, \theta)$, with $\theta$ taken mod $p\pi$, give rise to overlapping sectors $U_n$ and $V_n$, as above. These sectors contain the boundary point $r = 0$, but otherwise are open. In the $(u, v)$-coordinates, the flow on $U_n \times S^1$ (respectively, on $V_n \times S^1$) is given by

$$\Psi_t(u, v, z) = (e^{\alpha t}u, e^{-\alpha t}v, e^{2\pi t}z).$$

This formula is invariant under the change of $(u, v)$-coordinates on overlaps and defines a flow transverse to the factors $\mathbb{R}^2 \times \{z\}$. If $\alpha = \log \lambda$, this models the case in which each separatrix is $\varphi$-invariant. The flow commutes with arbitrary rotations in the $S^1$-coordinate and with rotations of $\mathbb{R}^2$ through $2\pi k/p$ radians, so we can choose $\alpha$ appropriately and quotient out
a smooth action of a subgroup $H \leq \mathbb{Z}_p$ to model the general situation. The flow is integral to a $C^0$ velocity field which is smooth except at the closed orbit $\{(0,0)\} \times S^1$. Also, the projection $\mathbb{R}^2 \times S^1 \to S^1$ is clearly a smooth fibration.

Given the suspension of a regular pseudo-Anosov automorphism $\varphi$, the above model is used to coordinatize a tubular neighborhood of each closed leaf $\sigma \leq \mathcal{L}$. Indeed, the germinal holonomy of $\mathcal{L}$ around $\sigma$ is isomorphic to that of our model around $\{(0,0)\} \times S^1$. Since the germinal holonomy of a compact leaf determines the germ of the foliation at that leaf, there is an isomorphism of foliated (but not $\mathcal{L}$-saturated) tubular neighborhoods. This can be chosen to respect the given fibrations of these neighborhoods over $S^1$. Since the holonomy is smooth except at the origin, the isomorphism is a diffeomorphism except at $\sigma$, yielding a coordinatization which is $C^\infty$-compatible with the smooth structure on $M_\varphi \setminus \mathcal{L}$. The total space $M_\varphi$ of the suspension is now a smooth manifold, smoothly fibered over $S^1$, with a pseudo-Anosov flow $\Phi_t$ having a nonsingular $C^0$ velocity field $X$ which is smooth on $M_\varphi \setminus \mathcal{L}$. The underlying 1-dimensional foliation $\mathcal{L}$ is called a pseudo-Anosov foliation.

Given suitable $0 << \varepsilon < 1$, a small perturbation of $X$, supported in a neighborhood of $\Sigma$, defines a homotopy $X_s$ of vector fields, $1 - \varepsilon \leq s \leq 1$, which is smooth on $M_\varphi \times [1 - \varepsilon, 1] \setminus \{1\}$ and such that $X=X_1$ and each $X_s$ agrees with $X$ outside of a tubular neighborhood of $\mathcal{L}$ of radius $1 - \varepsilon$. The first return map $\varphi_s : F \to F$ of the flow of $X_s$, $1 - \varepsilon \leq s \leq 1$, defines an isotopy of $\varphi = \varphi_1$ which is smooth on $F \times [1 - \varepsilon, 1] \setminus \{1\}$. Since any diffeomorphism $\varphi_0$ in the isotopy class of $\varphi_1$ is smoothly isotopic to $\varphi_{1-\varepsilon}$, we obtain the following.

**Lemma 6.** If $\varphi_0 : F \to F$ is a diffeomorphism isotopic to a pseudo-Anosov homeomorphism, then there is an isotopy $\tilde{\varphi}$ of $\varphi_0$ through automorphisms $\varphi_s$ of $F$, $0 \leq s \leq 1$, to a regular pseudo-Anosov automorphism $\varphi_1$. This isotopy is smooth on $F \times [0, 1] \setminus \{1\}$.

**Definition 7.** The isotopy in Lemma 6 will be called a regular isotopy.

Now suppose that the foliation $\mathcal{F}'$ of a compact, atoroidal 3-manifold $M$ fibers $M$ over $S^1$ and let $\mathcal{L}_0$ be a smooth, 1-dimensional foliation transverse to $\mathcal{F}'$. Properly parametrized, $\mathcal{L}_0$ is the underlying foliation of a smooth, nonsingular flow $\Phi_t$ which preserves $\mathcal{F}'$ and defines first return map $\varphi_0 = \Phi_1 | F$ on a fiber $F \subset M$. If we fix the choice of fiber, the map

$$
\mathbb{R} \times F \to M
\quad (t, x) \mapsto \Phi_t(x)
$$
passes to a diffeomorphism of the suspension $M_0$ to $M$. Another choice of fiber conjugates this diffeomorphism by an element of the flow, so we can safely identify $M$ and $M_0$.

Fix a regular isotopy $\varphi$ of $\varphi_0$ to a regular pseudo-Anosov automorphism $\varphi_1$. By a slight abuse of notation, we also denote by $\varphi$ the map

$$F \times I \to F \times I$$

$$(x, s) \mapsto (\varphi(x, s), s) = (\varphi_s(x), s).$$

The suspension of $\varphi$ is a 4-manifold

$$W_\varphi = \mathbb{R} \times F \times I \setminus \{(t, x, s) \sim (t - 1, \varphi_s(x), s)\},$$

canonically fibered over $S^1$ by fibers $F_s$, together with a canonical 1-dimensional foliation $\mathcal{D}$ transverse to the fibration. Again, everything is smooth on $W_\varphi \setminus \Sigma$, where $\Sigma$ is the union of the finitely many closed orbits covered by $\mathbb{R} \times S_0 \times \{1\}$. The homotopies $X_s$, used above, can be constructed on the standard model of the neighborhood of a singular orbit, allowing us to complete the differentiable structure of $W_\varphi$ so that the fibration $W_\varphi \to S^1$ is smooth and $\mathcal{D}$ is integral to a $C^0$ vector field which is smooth on $W_\varphi \setminus \Sigma$.

**Lemma 8.** The manifold $W_\varphi$ is diffeomorphic to the product $M \times I$ in such a way that the factors $M \times \{s\}$ are identified with $M_\varphi$, $0 \leq s \leq 1$, the factors $\{x\} \times I$ are tangent to the fibers $F \times I$, and the foliation $\mathcal{D}$ is tangent to the factors $M \times \{s\}$.

**Proof.** As above, we identify $M = M_{\varphi_0}$ canonically. The manifolds $\mathbb{R} \times F \times \{s\}$ pass canonically to $M_\varphi \subset W_\varphi$ and smoothly foliate $W_\varphi$. These leaves are compact and the foliation is without holonomy, so $W_\varphi \cong M \times I$ with $M \times \{s\} = M_\varphi$, $0 \leq s \leq 1$. In the product structure on $W_\varphi$, the $I$-factors are not yet specified, but the projection $p: W_\varphi \to I$ is canonically determined by the requirement that $p(M_\varphi) = s$, $0 \leq s \leq 1$. The smooth fibration of $W_\varphi$ over $S^1$ is also canonically fixed by the suspension construction and the fibers are transverse to the submanifolds $M_\varphi$, so the $I$-factors can be defined as the integral curves of a vector field $v$, tangent to the fibers, such that $p_s(v) = dv$. 

Thus, the foliation $\mathcal{D}$ can be viewed as a homotopy of the foliation $\mathcal{D}_0$ to the pseudo-Anosov foliation $\mathcal{D}_1$ through foliations $\mathcal{D}_s = \mathcal{D} \mid M \times \{s\}$. 

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