ELSEVIER
Journal of Pure and Applied Algebra 98 (1995) 25–44
JOURNAL OF PURE AND APPLIED ALGEBRA

Butler groups of infinite rank

L. Fuchs

Department of Mathematics, Tulane University, New Orleans, LA 70118, USA

Communicated by K.W. Gruenberg; received 3 March 1993

Abstract

Butler groups are torsion-free abelian groups which – in the infinite rank case – can be defined in two different ways. One definition requires that all the balanced extensions of torsion groups by them are splitting, while the other stipulates that they admit continuous transfinite chains (with finite rank factors) of so-called decent subgroups.

This paper is devoted to the three major questions for Butler groups of infinite rank: Are the two definitions equivalent? Are balanced subgroups of completely decomposable torsion-free groups always Butler groups? Which pure subgroups of Butler groups are again Butler groups? In attacking these problems, a new approach is used by utilizing $\aleph_0$-prebalanced chains and relative balanced-projective resolutions introduced by Bican and Fuchs [5].

A noteworthy feature is that no additional set-theoretical hypotheses are needed.

0. Introduction

All groups in this paper are abelian groups, written additively. For unexplained terminology and notation, we refer to Fuchs [12].

A torsion-free group B of finite rank is said to be a *Butler group* if it is a pure subgroup of a completely decomposable group (of finite rank); or, equivalently, it is an epimorphic image of a completely decomposable group of finite rank [7]. The equivalence of these properties to the condition that $\mathrm{Bext}^1(B, T) = 0$ for all torsion groups T (where $\mathrm{Bext}^1(B, T)$ denotes the group of all balanced extensions of T by B) led Bican and Salce [6] to initiate the theory of Butler groups of infinite rank. They call a torsion-free group B of infinite rank a

(1) B_1-*group* (or *Butler group*) if $\mathrm{Bext}^1(B, T) = 0$ holds for all torsion groups T;

(2) B_2-*group* if, for some ordinal τ, there is a continuous well-ordered ascending chain of pure subgroups,

$$0 = B_0 < B_1 < \cdots < B_\alpha < \cdots < B_\tau = B = \bigcup B_\alpha \tag{1}$$

with finite rank factors such that, for each $\alpha < \tau$, $B_{\alpha+1} = B_\alpha + G_\alpha$ holds for some finite rank Butler group G_α; i.e. B_α is *decent* in $B_{\alpha+1}$ in the sense of Albrecht and Hill [1]. It is easy to see that the chain (1) may be assumed to have rank 1 factors.

SSDI 0022-4049(94)00016-C

It is readily checked that all B_2-groups are B_1-groups [6].

Butler groups, whether of finite or of infinite rank, have been considered by several writers within recent years and are currently in the center of interest in abelian group theory. The infinite rank case is especially challenging in view of the fact that for groups of cardinality $> \aleph_1$ several results can be proved (or are true) only in certain models of set theory.

In the infinite rank case, three major questions are of paramount interest:

1. Are B_1-groups necessarily B_2-groups?
2. Does $\mathrm{Bext}^2(G, T) = 0$ hold for all torsion-free groups G and torsion groups T? I.e., are balanced subgroups of completely decomposable groups always B_1-groups?
3. Which pure subgroups of B_2-groups are again B_2-groups?

In the countable case, Bican and Salce [6] gave complete answers to these questions. However, for uncountable groups only fragmentary results are known.

The answer to Questions 1 and 2 is "yes" in the following cases: for groups of cardinality $\aleph_1$ [1, 9]; for groups of cardinality $\leq \aleph_\omega$, provided CH is assumed [8]; and for all groups without any cardinality restriction in the constructible universe L [13]. In contrast, Dugas and Thomé [11] proved that the negation of CH leads to a negative answer to Question 2 already for groups of cardinality $\aleph_2$. It is still an open problem whether or not Question 1 can have a negative answer in any model of ZFC.

As far as Question 3 is concerned; for groups of cardinality $\aleph_1$ sufficient condition for a pure subgroup of a B_2-group to be again a B_2-group is its "separativeness' (or, in the terminology of Albrecht and Hill [1], "separability in Hill's sense"). For groups of cardinality $\leq \aleph_\omega$, CH implies that the balancedness of the subgroup suffices (see [8]), while in case $V = L$ a sufficient condition is the balancedness, or even the prebalancedness, of the subgroup (see [13] and [5], respectively). If CH is assumed, $\aleph_0$-prebalancedness turns out to be a necessary condition (cf. [5]).

In this paper, we use the newly developed machinery of Bican and Fuchs [5] in the search for answers to the posed questions. Though we have been unable to answer Questions 1 and 2 in general by simply saying "yes" or "no", nevertheless we shall obtain a better insight into these problems by slightly modifying the questions posed above: when is a B_1-group a B_2-group? and when does a torsion-free group G satisfy $\mathrm{Bext}^2(G, T) = 0$ for all torsion groups T? We are in a position to offer satisfactory answers to these questions and to Question 3 even in ZFC.

Two important concepts introduced by Bican and Fuchs [5] will play a leading role in our study.

One is "κ-prebalancedness", in particular, $\aleph_0$-prebalancedness (the latter replaces P. Hill's notion of "separativeness" as a main tool). It opens the door to a different and more natural approach to the theory of Butler groups of infinite rank.

The other basic concept is the balanced-projective resolution of a group relative to a pure subgroup. It was implicit in [4] and in [13], and became an indispensable tool in [5]; as we shall see, it is a most fitting device in the study of Butler groups.

Special emphasis is placed on results which can be dealt with without assuming CH; this is a drastic change to earlier publications about Butler groups of cardinality $> \aleph_1$ in which CH was ubiquitous (and indispensable). Some of our results will confirm the undisputable significance of set theory in dealing with Butler groups; actually, its role is more vital than previously expected.

We stress the need of being mindful of the fine distinction between B_1-groups and B_2-groups. Most of our results involve B_2-groups, but their validity for B_1-groups has not been confirmed. For a reader familiar with infinite rank Butler groups it should not be surprising that B_2-groups are more tractable than B_1-groups.

The main results in this paper can be summarized as follows (for the relevant definitions we refer to Sections 1 and 2).

The central theme is the problem as to when a torsion-free group G admits an $\aleph_0$-prebalanced chain. We prove in Corollary 2.4 that it does if and only if in a balanced-projective resolution $0 \to B \to C \to G \to 0$ of G (i.e. a balanced-exact sequence with completely decomposable C) the subgroup B is a B_2-group. Using this criterion, we are able to show that a B_1-group G is a B_2-group if and only if it admits an $\aleph_0$-prebalanced chain (see Theorem 4.1). If CH holds, then these conditions are equivalent to the vanishing of $\mathrm{Bext}^2(G, T)$ for all torsion groups T.

In Theorem 4.5 we shall prove that, in any model of ZFC, $\mathrm{Bext}^2(G, T) = 0$ holds for all torsion-free groups G and all torsion groups T if and only if every torsion-free group admits an $\aleph_0$-prebalanced chain.

Some of our results (see Theorems 5.2 and 7.3) deal with the situation when instead of CH we only suppose that $2^{\aleph_0} = \aleph_n$ for some integer $n \geq 1$. In this case, $\mathrm{Bext}^{n+2}(G, T) = 0$ holds for all torsion-free groups G and torsion groups T. Furthermore, a torsion-free group B turns out to be a B_2-group if and only if it satisfies $\mathrm{Bext}^1(B, T) = \mathrm{Bext}^2(B, T) = \cdots = \mathrm{Bext}^{n+1}(B, T) = 0$ for all torsion groups T. (This extends a significant result recently proved by Rangaswamy [21].)

In Theorem 8.1 we give a satisfactory answer to Question 3 posed above: a pure subgroup A of a B_2-group G is again a B_2-group if and only if there is an $\aleph_0$-prebalanced chain from A to G if and only if there is a continuous well-ordered ascending chain of B_2-subgroups from A to G with rank 1 (or, equivalently, with countable) factors. (This result is analogous to Hill's criterion for an isotype subgroup of a totally projective p-group to be again totally projective [17].)

1. Preliminaries

Let B be any torsion-free group and A a pure subgroup of corank 1 in B. Consider the types $\mathbf{t}(J)$ of those pure rank 1 subgroups J of B which are not contained in A. In the lattice of all types, we form the ideal $\mathscr{I}_{B|A}$ generated by all these types $\mathbf{t}(J)$. Let κ be a cardinal such that $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. A is said to be *κ-prebalanced* in B, if $\mathscr{I}_{B|A}$ is at most κ-generated (i.e., it has a generating set of cardinality $\leq \kappa$). More explicitly, this means that we can write

$$B = A + \sum_{\alpha \in I} J_\alpha$$

where I is an index set of cardinality $\leq \kappa$ and the groups J_α are rank 1 pure subgroups of B not in A such that every rank 1 pure subgroup J of B not in A satisfies $\mathbf{t}(J) \leq \mathbf{t}(J_{\alpha_1}) \cup \cdots \cup \mathbf{t}(J_{\alpha_m})$ for some integer m, $\alpha_j \in I$. (The case $\kappa = \aleph_{-1}$ can be interpreted as *prebalanced*ness, that is to say, $B = A + J_1 + \cdots + J_m$ for some integer m; cf. [14].)

Let us point out rightaway that, moreover, we may assume that the J_α are selected such that for every rank 1 pure subgroup J of B not in A, there are a finite number of J_α satisfying $J \leq J_{\alpha_1} + \cdots + J_{\alpha_j}$. If we write $C = \oplus_{\alpha \in I} C_\alpha$ where $C_\alpha \cong J_\alpha$, then the map $C \to B$ induced by the chosen isomorphisms between C_α and J_α, along with the inclusion map of A in B, gives rise to a balanced-exact sequence

$$0 \to H \to A \oplus C \xrightarrow{\varphi} B \to 0 \tag{2}$$

where the kernel H is isomorphic to a corank 1 pure subgroup of C; see [5]. It is readily seen that such a balanced-exact sequence (which may be called a *balanced-projective resolution of B relative to A*) can be constructed even if A has arbitrary corank in B, though then nothing similar can be said of H.

One of the motivations stems from the following result which was proved by Fuchs et al. [15] and which will be needed later on.

Lemma 1.1. *A corank* 1 *pure subgroup G of $A = \bigoplus_{\alpha < \tau} A_\alpha$ that does not contain any A_α is a B_2-group if and only if the lattice ideal $\mathcal{I}$ generated by the types $\mathbf{t}(A_\alpha)$ is countably generated. (In the presence of CH, the same criterion applies to G being a B_1-group.)*

In dealing with the most important special case of κ-prebalancedness, namely, when $\kappa = \aleph_0$, it is useful to have the following lemma.

Lemma 1.2 (Bican and Fuchs [5]). *Let A be a pure subgroup of corank* 1 *in a torsion-free group B. A is $\aleph_0$-prebalanced in B exactly if there is a prebalanced-exact sequence* (2) *where both C and H are countable B_2-groups, and $\phi | A$ is the inclusion map.*

Our next lemma is about pure-exact sequences. Recall that if λ is an infinite cardinal, then by a *$G(\lambda)$-family* of subgroups in the group A is meant a collection $\mathcal{G}$ of subgroups of A such that (i) $0, A \in \mathcal{G}$ (ii) $\mathcal{G}$ is closed under unions of chains; (iii) if $B \in \mathcal{G}$ and X is any subset of A of cardinality $\leq \lambda$, then there is a $B' \in \mathcal{G}$ that contains both B and X, and satisfies $|B'/B| \leq \lambda$. (See [18].)

Lemma 1.3 (Dugas et al. [8]). *Let*

$$0 \to A \to C \xrightarrow{\gamma} G \to 0 \tag{3}$$

be a pure-exact sequence where C *is a completely decomposable group of cardinality* $\lambda > 2^{\aleph_0}$. *Then there are* $G(2^{\aleph_0})$*-families* $\mathscr{A} = \{A_\alpha\}, \mathscr{C} = \{C_\alpha\}, \mathscr{G} = \{G_\alpha\}$ *of subgroups in* A, C *and* G, *respectively, such that*

(i) *the subgroups* $C_\alpha \in \mathscr{C}$ *are summands of* C;
(ii) *the subgroups in* $\mathscr{A} = \{A_\alpha = A \cap C_\alpha | C_\alpha \in \mathscr{C}\}$ *are balanced in* A;
(iii) *the subgroups in* $\mathscr{G} = \{G_\alpha = \gamma C_\alpha | C_\alpha \in \mathscr{C}\}$ *are pure in* G;
(iv) $0 \to A \cap C_\alpha \to C_\alpha \to \gamma C_\alpha \to 0$ *is exact for each* $C_\alpha \in \mathscr{C}$.

We shall require a weakened version of Lemma 1.3 which works also for uncountable cardinals below the power of the continuum – if such cardinals exist in our model.

Lemma 1.4 (Dugas et al. [8]). *Let* (3) *be balanced-exact where* C *is a completely decomposable group of cardinality* κ, κ *an uncountable regular cardinal. Then there are continuous well-ordered ascending chains of subgroups.*

$$0 = A_0 < A_1 < \cdots < A_\alpha < \cdots; \quad 0 = C_0 < C_1 < \cdots < C_\alpha < \cdots;$$
$$0 = G_0 < G_1 < \cdots < G_\alpha < \cdots \quad (\alpha < \kappa)$$

with unions A, C, *and* G, *respectively, such that, for each* $\alpha < \kappa$,

(i) *the subgroups* C_α *are summands of* C *and have cardinality* $< \kappa$;
(ii) *the subgroups* $A_\alpha = A \cap C_\alpha$ *are balanced in* A (*and hence in* C);
(iii) *the subgroups* $G_\alpha = \gamma C_\alpha$ *are pure in* G;
(iv) $0 \to A_\alpha \to C_\alpha \to G_\alpha \to 0$ *is exact.*

2. κ-prebalanced chains

It turns out that there is an intimate relation between B_2-groups and $\aleph_0$-prebalanced chains. This relationship provides us with a powerful tool in the study of Butler groups.

We start with a general definition. Let A be a pure subgroup of the torsion-free group G. By a *κ-prebalanced chain* from A to G we mean a continuous well-ordered ascending chain

$$A = G_0 < G_1 < \cdots < G_\alpha < \cdots < G_\tau = G \quad \text{(for some ordinal } \tau)$$

of κ-prebalanced subgroups of G, where all the factors $G_{\alpha+1}/G_\alpha$ are of rank 1. We say that G admits a *κ-prebalanced chain* if there is a κ-prebalanced chain from 0 to G. As countable extensions of κ-prebalanced subgroups are again κ-prebalanced, it suffices to require that the factors $G_{\alpha+1}/G_\alpha$ be countable (or of cardinality $\leq \aleph_1$).

Our first goal is to find criteria for torsion-free groups to admit κ-prebalanced chains. The starting point is an elementary lemma.

Lemma 2.1. *Let G be a completely decomposable group, whose typeset has cardinality κ^+ for an infinite cardinal κ. Every pure subgroup of G admits a κ-prebalanced chain.*

Proof. A completely decomposable group G with κ^+ types can be written as a direct sum $\oplus H_\nu$ with $\nu < \kappa^+$ where the summands H_ν are homogeneous completely decomposable groups. Hence G can be thought of as the union of a continuous well-ordered ascending chain $0 = G_0 < G_1 < \cdots < G_\nu < \cdots < G$ of pure subgroups G_ν ($\nu < \kappa^+$) each of which has at most κ types; e.g. we can choose $G_{\nu+1} = G_\nu \oplus H_\nu$. This chain can be refined to a chain of summands of G with rank 1 factors. The typeset of every member of this chain is of cardinality at most κ.

If A is a pure subgroup of G, then intersecting A with the above constructed chain of G and dropping repetitions we obtain a κ-prebalanced chain for A. $\square$

It is worth while noting a simple consequence of Lemma 2.1: CH implies that pure subgroups of completely decomposable groups admit $\aleph_0$-prebalanced chains.

We proceed to verify the following more substantial result.

Theorem 2.2. *If there is a κ^+-prebalanced chain from a pure subgroup A of a torsion-free group G to the group G itself, then in a relative balanced-projective resolution $0 \to B \to A \oplus C \to G \to 0$ (C is completely decomposable) the group B admits a κ-prebalanced chain.*

Proof. Suppose G admits a κ^+-prebalanced chain from A to G, say, $A = G_0 < \cdots < G_\alpha < G_{\alpha+1} < \cdots$ is such a chain. We build a relative balanced-projective resolution of G with the aid of this chain as follows: from a relative balanced-projective resolution $0 \to B_\alpha \to A \oplus C_\alpha \to G_\alpha \to 0$ of G_α we form a relative balanced-projective resolution $0 \to B_{\alpha+1} \to A \oplus C_{\alpha+1} \to G_{\alpha+1} \to 0$ of $G_{\alpha+1}$ by choosing $C_{\alpha+1} = C_\alpha \oplus C'$ for a suitable completely decomposable group C' and mapping C' into $G_{\alpha+1}$ such that in a fixed direct decomposition of C', no rank 1 summand maps into G_α; and – of course – we take unions at limit ordinals (note that the direct limit is again balanced-exact).

In this way, we obtain the following commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_\alpha & \longrightarrow & A \oplus C_\alpha & \longrightarrow & G_\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{\alpha+1} & \longrightarrow & A \oplus C_\alpha \oplus C' & \longrightarrow & G_{\alpha+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_{\alpha+1}/B_\alpha & \longrightarrow & C' & \longrightarrow & G_{\alpha+1}/G_\alpha & \longrightarrow & 0
\end{array}$$

Evidently, if G_α is a κ^+-prebalanced subgroup in $G_{\alpha+1}$, then the ideal $\mathscr{I}$ generated by the types of summands in C' has a set of κ^+ generators. In the bottom row of the diagram, the subgroup $B_{\alpha+1}/B_\alpha$ is a corank 1 pure subgroup of the completely decomposable group C' where the rank 1 summands in the chosen decomposition of C' map non-trivially upon $G_{\alpha+1}/G_\alpha$. Since $\mathscr{I}$ is κ^+-generated, Lemma 2.1 implies that $B_{\alpha+1}/B_\alpha$ admits a κ-prebalanced chain. Consequently, B is the union of a continuous well-ordered ascending chain $0 = B_0 < \cdots < B_\alpha < B_{\alpha+1} < \cdots$ of balanced subgroups where each of the factors $B_{\alpha+1}/B_\alpha$ admits a κ-prebalanced chain. It follows at once that these chains lift to B creating a κ-prebalanced chain for B, as desired. □

For reasons that will become apparent in the sequel, our primary concern is the case $\kappa = \aleph_0$. It is significant that in this case a much stronger version of Theorem 2.2 holds true.

Recall that by an *Axiom-3 family* of subgroups in G is meant a collection $\mathscr{C}$ of subgroups of G such that (i) $0, G \in \mathscr{C}$; (ii) $\mathscr{C}$ is closed under taking arbitrary unions of its members; (iii) if $A \in \mathscr{C}$ and X is any countable subset of G, then there is an $A' \in \mathscr{C}$ that contains both A and X such that A'/A is countable (see [18]). A TEP-*subgroup* of G (torsion extension property, see [9]) is a (necessarily) pure subgroup A of G such that every homomorphism of A into a torsion group T extends to a homomorphism from G into T.

Theorem 2.3. *There exists an $\aleph_0$-prebalanced chain from a pure subgroup A of a torsion-free group G to G itself if and only if in a relative balanced-projective resolution $0 \to B \to A \oplus C \to G \to 0$ (where C is completely decomposable), B is a B_2-group.*

Proof. The proof of Theorem 2.2 shows that if there is an $\aleph_0$-prebalanced chain from A to G, then the group B is the union of a well-ordered continuous ascending chain $0 = B_0 < \cdots < B_\alpha < B_{\alpha+1} < \cdots$ of its balanced subgroups where the factors $B_{\alpha+1}/B_\alpha$ are pure subgroups of countable completely decomposable groups. These factors are therefore B_2-groups, and then B is likewise a B_2-group.

To verify sufficiency, let $0 \to B \to A \oplus C \to G \to 0$ be a balanced-projective resolution of G relative to A where C is completely decomposable and B is a B_2-group. As a B_2-group, B admits an Axiom-3 family $\mathscr{B}$ of decent, TEP subgroups (see [1, 13]). C is a completely decomposable group, so it has an Axiom-3 family $\mathscr{C}$ of summands; furthermore, the torsion-free group G/A has an Axiom-3 family $\mathscr{G}$ of pure subgroups [18]. Dropping to suitable subfamilies by the usual back-and-forth argument, filtrations of B, C and G/A can be found with members in the respective families such that the exact sequences $0 \to B_\alpha \to A \oplus C_\alpha \to G_\alpha \to 0$ are all prebalanced-exact and the factor groups $C_{\alpha+1}/C_\alpha$ are all countable. We can write $C_{\alpha+1} = C_\alpha \oplus C'$, and form a prebalanced exact sequence $0 \to H \to G_\alpha \oplus C' \to G_{\alpha+1} \to 0$ with the obvious maps

(H is defined as the kernel). This leads to a commutative diagram with prebalanced-exact rows

$$\begin{array}{ccccccccc}
 & & B_\alpha & = & B_\alpha & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B_{\alpha+1} & \longrightarrow & A \oplus C_\alpha \oplus C' & \longrightarrow & G_{\alpha+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & G_\alpha \oplus C' & \longrightarrow & G_{\alpha+1} & \longrightarrow & 0
\end{array}$$

where the first column is exact in view of the 3×3-lemma. B_α being a TEP-subgroup of the B_2-group $B_{\alpha+1}$, it follows that $H \cong B_{\alpha+1}/B_\alpha$ is a B_2-group [8].

Let G' be a pure subgroup of $G_{\alpha+1}$ that contains G_α as a corank one subgroup. The bottom row induces a prebalanced-exact sequence $0 \to H \to G_\alpha \oplus D \to G' \to 0$ where D is a pure subgroup of the countable completely decomposable group C'. Therefore, D is a countable B_2-group. Lemma 1.2 implies that G_α is $\aleph_0$-prebalanced in G'. We conclude that G_α is $\aleph_0$-prebalanced in $G_{\alpha+1}$, and the proof is complete. □

Note that in the last theorem the condition that B be a B_2-group is independent of the choice of the relative balanced-projective resolution. Indeed, suppose that $0 \to B \to A \oplus C \xrightarrow{\varphi} G \to 0$ and $0 \to B' \to A \oplus C' \xrightarrow{\psi} G \to 0$ are two balanced-projective resolutions of G relative to A where C and C' are completely decomposable, and $\phi | A$, $\psi | A$ are the inclusion maps. The top row in the following pullback diagram splits, since by the complete decomposability of C there is a map $\rho: A \oplus C \to A \oplus C'$ such that $\psi\rho = \phi$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B' & \longrightarrow & M & \longrightarrow & A \oplus C & \longrightarrow & 0 \\
 & & \| & & \downarrow & \swarrow^{\rho} & \downarrow \varphi & & \\
0 & \longrightarrow & B' & \longrightarrow & A \oplus C' & \xrightarrow{\psi} & G & \longrightarrow & 0.
\end{array}$$

Therefore, $M \cong B' \oplus A \oplus C$, and analogously, $M \cong B \oplus A \oplus C'$. In the arising isomorphism $B' \oplus A \oplus C \cong B \oplus A \oplus C'$ we may cancel the A's on both sides, since the map between them is the identity. Thus $B' \oplus C \cong B \oplus C'$. Since summands of B_2-groups are B_2-groups [5], the last isomorphism shows that if one of B and B' is a B_2-group, then so is the other.

It would be tempting to conjecture that Theorem 2.3 generalizes straightforwardly to higher degrees of prebalancedness. However, it is easy to construct a counter-example. In fact, if $2^{\aleph_0} \geq \aleph_3$, then Theorem 6.1 yields a corank one pure subgroup G of a completely decomposable group that has no $\aleph_1$-prebalanced chain, but the kernel of its balanced-projective resolution admits an $\aleph_0$-prebalanced chain.

The case $A = 0$ in the preceding theorem yields at once an important criterion we have been looking for:

Corollary 2.4. *A torsion-free group G admits an $\aleph_0$-prebalanced chain if and only if in a balanced-projective resolution* $0 \to B \to C \to G \to 0$ *of G (where C is completely decomposable) the subgroup B is a* B_2*-group.*

Recall that a pure subgroup A of G is called *separative* – of separable in the sense of Hill [1] – if for each $g \in G$ there is a countable subset $\{a_n | n < \omega\} \subset A$ such that $\{\chi(g + a_n) | n < \omega\}$ is cofinal in the partially ordered set $\{\chi(g + a) | a \in A\}$ of characteristics. From the definitions it is evident that separative subgroups are $\aleph_0$-prebalanced.

Fuchs and Magidor [13] have shown that $V = L$ guarantees that every torsion-free group admits a separative, and hence an $\aleph_0$-prebalanced chain. It is not known if $V = L$ is a necessary condition for the existence of either chain, but CH is certainly necessary; cf. Theorem 6.4. Let us point out that necessary conditions for the existence of these two kinds of chains might be different; in fact Example 6.5 will show that ZFC $+ \neg$CH implies the existence of torsion-free groups with $\aleph_0$-prebalanced chains that fail to admit separative chains.

3. Axiom-3 families of $\aleph_0$-prebalanced subgroups

Fuchs and Magidor [13] have shown that if a torsion-free group G admits a single separative chain, then it also admits a whole Axiom-3 family of separative subgroups. An analogous claim can be established for $\aleph_0$-prebalanced chains, as is shown by our next result. The proof is similar, based on an idea of Paul Hill's [17], but the situation here is sufficiently different to warrant details. Actually, we prove a more general result.

We say that a family $\mathscr{F}$ of subgroups of G is an Axiom-3 family over a subgroup A and G if every member F of $\mathscr{F}$ contains A and $\{F/A | F \in \mathscr{F}\}$ is an Axiom-3 family in G/A; cf. [1].

Theorem 3.1. *A torsion-free group G that has an $\aleph_0$-prebalanced chain from a pure subgroup A to G admits an Axiom-3 family over A consisting of $\aleph_0$-prebalanced subgroups of G.*

Proof. Let G be a torsion-free group of rank λ and assume that G is the union of a continuous well-ordered ascending chain of $\aleph_0$-prebalanced subgroups G_ν $(\nu < \lambda)$ of ranks $< \lambda$ with rank 1 quotients $G_{\nu+1}/G_\nu$ and $G_0 = A$. For each $\nu < \lambda$, and for each coset $g + G_\nu$ in $G_{\nu+1}$, consider a countable set $\{g + a_n | n < \omega\}$ with $a_n \in G_\nu$ such that the types $\{\mathbf{t}(g + a_n | n < \omega\}$ generate the ideal generated by the set $\{\mathbf{t}(g + a) | a \in G_\nu\}$. Let B_ν denote the subgroup of $G_{\nu+1}$ which is generated by the pure subgroups $\langle g + a_n \rangle_*$ $(n < \omega)$ for each coset $g + G_\nu$ in $G_{\nu+1}$. Evidently, B_ν is a countable subgroup of $G_{\nu+1}$ such that $G_{\nu+1} = G_\nu + B_\nu$.

A subset S of λ is said to be *closed*, if, for each μ in S,

$$G_\mu \cap B_\mu \leq \langle B_\nu | \nu \in S, \nu < \mu \rangle.$$

Lemmas 5.5–5.6 in [1] show, respectively, that

(1) the union of any number of closed subsets of λ is closed,

(2) for a closed subset S of λ, the subgroup $G(S) = \langle A, B_\nu | \nu \in S \rangle$ is pure in G,

(3) every countable subset F of λ is contained in a countable closed subset S of λ.

Define the family $\mathscr{C}$ of subgroups of G to consist of all subgroups of the form $G(S)$ with S closed in λ. This $\mathscr{C}$ will be a desired family provided that $G(S)$ is a $\aleph_0$-prebalanced subgroup in G for S closed in λ.

Evidently, every $g \neq 0$ in G defines an ordinal $\nu(g) < \lambda$ such that $G_{\nu(g)+1}$ is the first member of the chain G_ν $(\nu < \lambda)$ that contains g. We use induction on $\nu(g)$ to find, for each $g \in G \backslash G(S)$, a countable subset in the set $\{\mathbf{t}(g + x) | x \in G(S)\}$ which generates the same ideal as this set.

Let $\nu(g) = \mu$ where μ is the smallest index such that $\mu \notin S$. There is a countable subset $\{\mathbf{t}(g + a_n) | n < \omega\}$ in $\{\mathbf{t}(g + x) | x \in G_\mu\}$ which generates the same ideal. Obviously $G_\mu \leq G(S)$, so $a_n \in G(S)$. We claim that $\{\mathbf{t}(g + a_n) | n < \omega\}$ generates the same ideal as $\{\mathbf{t}(g + x) | x \in G(S)\}$. Given $x \in G(S)$, in order to find an integer $n < \omega$ satisfying $\mathbf{t}(g + x) \leq \bigcup_{i<n} \mathbf{t}(g + a_i)$, we induct on $\nu(x) = \sigma$.

If $\sigma < \mu$, we are done by the choice of a_n.

The case $\sigma = \mu$ is ruled out, since this would mean $x = a + x_0 + x_1$ with $a \in A$, $x_0 \in \Sigma B_\nu$ $(\nu < \mu)$ and $0 \neq x_1 \in \Sigma B_\nu$ $(\nu > \mu)$, thus $x_1 = x - a - x_0 \in G_\mu$. Write $x_1 = y_1 + \cdots + y_k$ with $\sigma_i = \nu(y_i) \in S$ and $\sigma_1 < \cdots < \sigma_k$ and assume that the y_j are chosen such that σ_k is minimal. Then $y_k = G_{\sigma_k} \cap B_{\sigma_k} \leq \langle B_\nu | \nu \in S,\ \nu < \sigma_k \rangle$ leads to a contradiction.

Finally, let $\sigma > \mu$; then necessarily $\sigma \in S$. Hence there are $b_j \in G_\sigma$ such that $x + b_j \in B_\sigma$ and $\mathbf{t}(g + x) \leq \bigcup_{j<m} \mathbf{t}(g + b_j)$. We obtain $\mathbf{t}(g + x) \leq \bigcup_{j<m} \mathbf{t}(g + b_j)$. Here $b_j = (x + b_j) - x \in G(S)$, $\nu(b_j) < \sigma$, thus, by induction hypothesis, for each $j < m$, $\mathbf{t}(g - b_j) \leq \bigcup_{i<n} \mathbf{t}(g + a_i)$ holds for some n. We have reached the desired conclusion.

Continuing our induction, we can assume that g is chosen such that $\nu(g)$ is minimal for the elements in the coset $g + G(S)$. Then $\nu(g) \notin S$, since for the coset $g + G_{\nu(g)}$ we have selected a countable set $\{g + a_n | n < \omega\}$ and $\nu(g) \in S$ would imply that a_n was a representative of the same coset with a smaller $\nu(a_n)$. By induction hypothesis, for each $n < \omega$, there is a countable subset $\{\mathbf{t}(a_n + a_{nk}) | k < \omega\}$ of $\{\mathbf{t}(a_n + x) | x \in G(S)\}$ which generates the same ideal. We claim that the set $\{\mathbf{t}(g - a_{nk}) | n, k < \omega\}$ generates the same ideal as the set $\{\mathbf{t}(g + x) | x \in G(S)\}$. Again, we use induction on $\nu(x) = \sigma$. If $\sigma < \mu(g)$, then we have $\mathbf{t}(g + x) \leq \bigcup_{i<n} \mathbf{t}(g + a_i)$ for some $n < \omega$; hence $\mathbf{t}(g + x) \leq \bigcup_{i<n} \mathbf{t}(a_i - x)$. Since for $i < n$ we have $\mathbf{t}(a_i - x) \leq \bigcup_{j<k} \mathbf{t}(a_i + a_{ij})$, and hence $\mathbf{t}(a_i + x) \leq \bigcup_{j<k} \mathbf{t}(x + a_{ij})$ holds for some k, we infer that $\mathbf{t}(g + x) \leq \bigcup_{i<n} \bigcup_{j<k} \mathbf{t}(x + a_{ij})$, and consequently, $\mathbf{t}(g + x) \leq \bigcup_{i<n} \bigcup_{j<k} \mathbf{t}(g - a_{ij})$. The proof in the last but one paragraph applies to show that $\sigma = \mu(g)$ cannot occur. The case $\sigma > \mu(g)$ can be settled as above to convince ourselves that the set $\{\mathbf{t}(g + a_{nk}) | n, k < \omega\}$ generates the same lattice ideal as the set $\{\mathbf{t}(g + x) | x \in G(S)\}$. $\square$

4. B_1 -groups with $\aleph_0$-prebalanced chains

In general, it is a hard problem to decide whether or not a given B_1-group is a B_2-group – unless additional set-theoretical hypotheses guarantee that all B_1-groups of that cardinality are B_2-groups or the typeset of the group is restricted [16]. From the very definition of B_2-groups it is evident that B_2-groups admit (prebalanced, and hence) $\aleph_0$-prebalanced chains. Our next purpose is to show that – in every model of ZFC – this property alone characterizes B_2-groups within the class of B_1-groups.

Theorem 4.1. *A B_1-group is a B_2-group if and only if it admits an $\aleph_0$-prebalanced chain.*

Proof. It remains to prove sufficiency. Let B be a B_1-group of cardinality λ, and $0 = B_0 < \cdots < B_\alpha < B_{\alpha+1} < \cdots (\alpha < \lambda)$ an $\aleph_0$-prebalanced chain up to B with rank one factors. From [5] we know that all the subgroups B_α in the chain are B_1-groups.

The claim is verified by induction on λ. For $\lambda \leq \aleph_1$, the assertion is true; in fact, in view of [8], all B_1-groups of cardinality $\leq \aleph_1$ are B_2-groups.

First, assume λ is a regular cardinal. Because of Theorem 7.1 in [8], there is a cub E in λ such that B_α is a TEP-subgroup of B, for all $\alpha \in E$. It follows from [14] that, for all $\alpha \in E$, the subgroups B_α are prebalanced in B. Omitting all the B_α with $\alpha \notin E$, the remaining chain contains only subgroups which are prebalanced in B and all the factors are B_1-groups of cardinality $< \lambda$. The omitted B_α ($\alpha \notin E$) map upon $\aleph_0$-prebalanced subgroups of the factors of the new chain [5], so by induction hypothesis, these factors are B_2-groups. We conclude that B itself is a B_2-group.

Next, suppose that λ is a singular cardinal. By Theorem 3.1, B has an Axiom-3 family $\mathscr{B}$ of $\aleph_0$-prebalanced subgroups. The proof of Theorem 3.1 shows that those subgroups in $\mathscr{B}$ that are contained in B_α yield an Axiom-3 family of $\aleph_0$-prebalanced subgroups of B_α. Every member of $\mathscr{B}$ belongs to an $\aleph_0$-prebalanced chain of B, and therefore all the subgroups in $\mathscr{B}$ are B_1-groups. Let $\mathscr{B}^*$ consist of all the members of $\mathscr{B}$ whose cardinality is $< \lambda$. By induction hypothesis, the subgroups in $\mathscr{B}^*$ are B_2-groups. It is straightforward to check that $\mathscr{B}^*$ is a *λ-family* in the sense of [8], i.e. it satisfies the following four conditions:

(i) $0 \in \mathscr{B}^*$;

(ii) $|A| < \lambda$ for all $A \in \mathscr{B}^*$;

(iii) if $\mu < \lambda$ is a regular cardinal, and if A_α ($\alpha < \mu$) is an ascending chain of subgroups in $\mathscr{B}^*$ such that $|A_\alpha| < \kappa$ for some $\kappa < \lambda$, then $\bigcup_{\alpha<\mu} A_\alpha \in \mathscr{B}^*$;

(iv) if $A \in \mathscr{B}^*$ and X is a subset of B of cardinality $< \lambda$, then some $A' \in \mathscr{B}^*$ contains both A and X and satisfies $|A'| \leq |A|\,|X|$.

It only remains to appeal to Corollary 6.3 in [8] to conclude that B itself is a B_2-group. (There is a gap in the proof of the quoted Corollary which can easily be corrected; see a forthcoming paper by Fuchs and Rangaswamy.) □

Let λ be an infinite cardinal. A continuous well-ordered ascending chain of subgroups of G, $0 = G_0 < \cdots < G_\alpha < G_{\alpha+1} < \cdots$ with union G, will be called a λ-*filtration* of G if the cardinalities of the quotients $G_{\alpha+1}/G_\alpha$ do not exceed λ.

Corollary 4.2. *A B_1-group that admits* an *$\aleph_1$-filtration with (pre)balanced subgroups is a B_2*-group.

Proof. An $\aleph_1$-filtration with (pre)balanced subgroups can be refined by putting, between consecutive terms, chains of type $\leq \omega_1$ consisting of pure subgroups with rank one factors. The arising chain will be an $\aleph_0$-prebalanced chain, since countable extensions of $\aleph_0$-prebalanced subgroups are likewise $\aleph_0$-prebalanced [5]. A simple reference to Theorem 4.1 completes the proof. □

From Corollary 4.2 and the remark made after Lemma 2.1 we derive a result of Fuchs and Rangaswamy [16].

Lemma 4.3. (CH) *A pure subgroup of a completely decomposable group is a B_2-group if and only if it is a B_1-group.*

Consequently, from Theorem 2.3 we obtain at once the following corollary.

Corollary 4.4. (CH) *A torsion-free group G admits an $\aleph_0$-prebalanced chain exactly if* $\mathrm{Bext}^2(G, T) = 0$ *for all torsion groups T.*

We are in a position to characterize (even without assuming CH) those models of ZFC in which the groups $\mathrm{Bext}^2(G, T)$ vanish for all torsion-free groups G and torsion groups T.

Theorem 4.5. *In a model of* ZFC,

$$\mathrm{Bext}^2(G, T) = 0$$

holds for all torsion-free groups G and torsion groups T if and only if every torsion-free group admits an $\aleph_0$-prebalanced chain.

Proof. Sufficiency is an immediate consequence of Theorem 2.3. In order to verify necessity, assume that $\mathrm{Bext}^2(G, T) = 0$ for all torsion-free groups G and torsion groups T. In view of [11] (see also Theorem 6.1) this implies that CH must hold. From Corollary 4.4 we infer that G admits an $\aleph_0$-prebalanced chain. □

Under CH, the next corollary is essentially due to Rangaswamy [21].

Corollary 4.6. *Suppose that $0 \to H \to C \to G \to 0$ is a balanced-exact sequence where C is a B_2-group and H, G are B_1-groups. If one of H and G is a B_2-group, then so is the other.*

Proof. First, suppose C is completely decomposable. If G is a B_2-group, then Theorem 4.1 and Corollary 2.4 imply that H is likewise a B_2-group. If H is a B_2-group, then by Corollary 2.4 G admits an $\aleph_0$-prebalanced chain, so by Theorem 4.1 G is a B_2-group.

If C is an arbitrary B_2-group, then using a balanced-projective resolution $0 \to X \to A \to C \to 0$ of C in the middle column (with A completely decomposable) and defining K as the kernel of the composite map $A \to C \to G$, we can form the followings commutative diagram with balanced-exact rows and columns:

$$\begin{array}{ccccccccc}
 & & X & = & X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & H & \longrightarrow & C & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

Observe that Corollary 2.4 and Theorem 4.1 imply that X is a B_2-group, since C is a B_2-group.

If H is a B_2-group, then K is also a B_2-group as a balanced extension of a B_2-group by a B_2-group. Again by Corollary 2.4, G has then an $\aleph_0$-prebalanced chain, so Theorem 4.1 guarantees that the B_1-group G is a B_2-group.

On the other hand, if G is a B_2-group, then by Corollary 2.4 K is a B_2-group. What has been proved in the preceding paragraph implies that H too is a B_2-group. □

The next corollary which follows from Corollary 4.6 at once shows that, fortunately, the terminologies for B_1- and B_2-groups have been chosen in the correct way.

Corollary 4.7 (Rangaswamy [21], CH). *A torsion-free group B is a B_2-group if and only if both* $\mathrm{Bext}^1(B, T) = 0$ *and* $\mathrm{Bext}^2(B, T) = 0$ *hold for all torsion groups T.*

The sufficiency part of the last corollary can be generalized as we shall see in Theorem 5.2.

By a *long balanced-projective resolution* of the torsion-free group G we mean a long exact sequence

$$0 \to K_n \to C_n \to \cdots \to C_1 \to C_0 \to G \to 0$$

in which the groups C_i are completely decomposable and in each C_i the image of the preceding map is a balanced subgroup; in other words, the long sequence is balanced-exact. From Theorem 2.2 it is easy to conclude by induction that if G admits an $\aleph_n$-prebalanced chain for some integer $n \geq 0$, then the kernel K_n is a B_2-group.

Lemma 4.8. *If a torsion-free group B has an $\aleph_k$-prebalanced chain, and if*

$$\mathrm{Bext}^i(B, T) = 0 \textit{ for all torsion groups } T \textit{ and for } 1 \leqslant i \leqslant k+1.$$

then B is a B_2-group.

Proof. Consider a long balanced-projective resolution of B, $0 \to K_i \to C_i \to \cdots \to C_1 \to C_0 \to B \to 0$ where the groups C_i are completely decomposable. By virtue of Theorem 2.2, hypothesis implies that the kernel K_0 has an $\aleph_{k-1}$-prebalanced chain, and a straightforward induction shows that the kernel K_{k-1} admits an $\aleph_0$-prebalanced chain. As $\text{Bext}^1(K_i, T) = \text{Bext}^{i+2}(B, T) = 0$, it follows that B and all the kernels K_i $(i = 0, \ldots, k-1)$ are B_1-groups. From Theorem 4.1 we conclude that K_{k-1} is a B_2-group. In view of Corollary 4.6, in the balanced-exact sequence $0 \to K_i \to C_i \to K_{i-1} \to 0$ of B_1-groups $(i = k-1, \ldots, 0)$ K_{i-1} is a B_2-group if so is K_i. Therefore K_0, and hence B itself, is a B_2-group, indeed. □

Another corollary which is worth while recording for reference is as follows.

Corollary 4.9. *If B is a B_2-group, then* $\text{Bext}^i(B, T) = 0$ *for all $i \geq 1$ and for all torsion groups T.*

5. A homological characterization of B_2-group

We now take up the problem of characterizing B_2-groups in terms of Bext. Our main objective is to generalize Rangaswamy's theorem [21] (see Corollary 4.7) by getting rid of CH. In case the continuum is $\aleph_n$ for some integer $n \geq 1$, a most satisfactory result can be established.

Theorem 5.1. *Suppose that $2^{\aleph_0} = \aleph_n$ for an integer $n \geq 1$. Then a torsion-free group G has an $\aleph_0$-prebalanced chain if and only if*

$$\text{Bext}^2(G, T) = \cdots = \text{Bext}^{n+1}(G, T) = 0 \quad \textit{for all torsion groups } T.$$

Equivalently, if and only if $\text{Bext}^i(G, T) = 0$ *for all $i \geq 2$ and all torsion groups T.*

Proof. The equivalence of the two stated conditions will be evident from Theorem 7.3.

Consider a long balanced-projective resolution of G, $0 \to K_i \to C_i \to \cdots \to C_1 \to C_0 \to G \to 0$, where the groups C_i are completely decomposable.

First assume G has an $\aleph_0$-prebalanced chain. Then by Corollary 2.4 the kernel K_0 is a B_2-group. But once a kernel K_i is a B_2-group, then Corollary 4.6 implies, in view of the balanced-exact sequence $0 \to K_{i+1} \to C_{i+1} \to K_i \to 0$, that all subsequent kernels K_j $(j > i)$ are B_2-groups. Hence necessity is immediate.

In order to verify the converse, assume that G satisfies $\text{Bext}^i(G, T) = 0$ for $i = 2, \ldots, n+1$. Noting that if $\aleph_n$ is the continuum, then every torsion-free group has an $\aleph_n$-prebalanced chain, we can argue as in the proof of Lemma 4.8 to show by induction that all of K_i $(i = n-1, \ldots, 0)$ are B_2-groups. Hence sufficiency follows at once from Corollary 2.4. □

The next result is a generalization of Rangaswamy's theorem [21].

Theorem 5.2. *Let $2^{\aleph_0} = \aleph_n$ for an integer $n \geq 1$. Then a torsion-free group B is a B_2-group if and only if*

$$\mathrm{Bext}^1(B, T) = \mathrm{Bext}^2(B, T) = \cdots = \mathrm{Bext}^{n+1}(B, T) = 0 \quad \text{for all torsion groups } T.$$

Equivalently, if and only if $\mathrm{Bext}^i(B, T) = 0$ *for all $i \geq 1$ and all torsion groups T.*

Proof. This follows at once from Theorem 5.1 by referring to Theorem 4.1 again. □

Note that the proof shows that if a torsion-free group G satisfies $\mathrm{Bext}^i(G, T) = 0$ for all $i \geq 1$ and all torsion groups T, then either G and all the kernels in its long balanced-projective resolution are B_2-groups, or none of them is a B_2-group. It is an open problem whether or not the second alternative can occur. (Of course, then we must have $2^{\aleph_0} \geq \aleph_\omega$ and none of the kernels is a B_2-group.)

A proof similar to the one above applies to yield the following proposition.

Proposition 5.3. *A torsion-free group B of cardinality $\aleph_n$ (for an integer $n \geq 1$) is a B_2-group if and only if*

$$\mathrm{Bext}^1(B, T) = \mathrm{Bext}^2(B, T) = \cdots = \mathrm{Bext}^{n+1}(B, T) = 0 \quad \text{for all torsion groups } T.$$

6. Groups without $\aleph_i$-prebalanced chains

We owe the reader examples of groups which fail to admit $\aleph_i$-prebalanced chains for $i \geq 0$.

Let $\mathscr{S} = \{S_\alpha | \alpha < \Omega\}$ be a set of almost disjoint countable subsets S_α of the set of all prime numbers; i.e. $S_\alpha \cap S_\beta$ is finite for different $\alpha, \beta < \Omega$; here Ω stands for the initial ordinal of the power of the continuum. (For the existence of such an $\mathscr{S}$ see e.g. Jech [20, p. 242, Lemma 23.9].) For each $\alpha < \Omega$, let $\mathbf{t}_\alpha$ be the type represented by the characteristic $(n_2, n_3, \ldots, n_p, \ldots)$ where $n_p = 1$ or 0 according as $p \in S_\alpha$ or $p \notin S_\alpha$. It is easily seen that $\mathbf{t}_\alpha \leq \mathbf{t}_{\alpha_1} \cup \cdots \cup \mathbf{t}_{\alpha_k}$ holds if and only if $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$. Consequently, in the lattice of types, the ideal generated by the $\mathbf{t}_\alpha$ cannot be countably generated.

With the indicated choice of types, let A be the direct sum of rational groups A_α with $\mathbf{t}(A_\alpha) = \mathbf{t}_\alpha$ for $\alpha < \Omega$, and G a corank 1 pure subgroup of A that does not contain any A_α. This group G is homogeneous of type $\mathbb{Z}$, but by Lemma 1.1 it is not a B_2-group, so it is not free [3].

Dugas and Thomé [11] proved that there exist two collections of the power of the continuum consisting of almost disjoint sets, $\mathscr{S}$ and $\mathscr{S}'$, as above, with the additional property that the intersection $S_\alpha \cap S'_\beta$ is infinite for all $S_\alpha \in \mathscr{S}$, $S'_\beta \in \mathscr{S}'$. (A moment's reflection shows that this simply means that, for each α, $\{S_\alpha \cap S'_\beta | \beta < \Omega\}$ is a collection of almost disjoint sets.)

The following is a simplified version, and at the same time a generalization, of the main result of [11].

Theorem 6.1. *Assume $2^{\aleph_0} \geq \aleph_n$ for some integer $n \geq 1$. Choose an index set I of cardinality ω_n and form the completely decomposable group $A = \bigoplus_{\alpha < \omega_n} A_\alpha$ where the* *set of types of the rank 1 summands A_α is the union of the sets of types defined by the almost disjoint systems $\mathscr{S}$ and $\mathscr{S}'$ of cardinality $\aleph_n$. Then no corank 1 subgroup G of A that contains none of the A_α admits an $\aleph_{n-2}$-prebalanced chain.*

Proof. Let G be as stated. It is obvious that we can find, for each $\alpha < \omega_n$, an $a_\alpha \in A_\alpha$ such that $\phi(a_\alpha) = m_\alpha \in \mathbb{N}$. If $\beta < \alpha < \omega_n$, let $A_{\beta\alpha}$ denote the purification of the subgroup $\langle m_\beta a_\alpha - m_\alpha a_\beta \rangle$ in A. For $\nu < \omega_n$, we set

$$A(\nu) = \bigoplus_{\alpha<\nu} A_\alpha \quad \text{and} \quad G(\nu) = G \cap A(\nu).$$

Evidently, $A_{\beta\alpha}$ has type $\mathbf{t}(A_\beta) \cap \mathbf{t}(A_\alpha)$ in G, and G is generated by all the $A_{\beta\alpha}$.

We interrupt the argument to prove an easy lemma.

Lemma 6.2. *For any ν, $G(\nu)$ is κ-prebalanced in $G(\nu+1)$ if and only if there is a set I of indices $\mu_\alpha < \nu$, such that I has cardinality $\leq \kappa$, and the types $\mathbf{t}(A_{\nu\mu_\alpha})$ with $\mu_\alpha \in I$ generate the ideal generated by all types $\mathbf{t}(A_{\nu\mu})$ with $\mu < \nu$.*

Proof. Since the ideal generated by the types of the rank 1 pure subgroups in $G(\nu+1) \setminus G(\nu)$ is generated by the types $\mathbf{t}(A_{\nu\mu})$ for $\mu < \nu$, the condition is evidently necessary and sufficient for $G(\nu)$ to be κ-prebalanced in $G(\nu+1)$. □

Proof of Theorem 6.1 (conclusion). Resuming the proof of Theorem 6.1, suppose that the well-ordering of types is done alternately from the two systems $\mathscr{S}$ and $\mathscr{S}'$. Consider the chain $\{G(\nu)\}$ for $\nu < \omega_n$.

If $\nu > \omega_{n-1}$, then $G(\nu)$ is not $\aleph_{n-2}$-prebalanced in G, as a matter of fact, not even in $G(\nu+1)$. This is an immediate consequence of Lemma 6.2. In fact, the lattice ideal generated by the types $\mathbf{t}(A_\alpha) \cap \mathbf{t}(A_\nu)$ ($\alpha < \nu$) ($\aleph_{n-1}$ of them) is $\aleph_{n-1}$-generated. For, if the types $\mathbf{t}_\nu$ and the $\mathbf{t}_\alpha$'s are chosen from different systems $\mathscr{S}$ and $\mathscr{S}'$, then $\mathbf{t}_\nu \cap \mathbf{t}_\alpha \leq (\mathbf{t}_\nu \cap \mathbf{t}_{\alpha_1}) \cup \cdots \cup (\mathbf{t}_\nu \cap \mathbf{t}_{\alpha_k})$ can hold only if $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$, while adjoining intersections of types from the same system as $\mathbf{t}_\nu$ belongs to does not change the type of the union at all.

Now, if G had an $\aleph_{n-2}$-prebalanced chain, then – the cardinal $\aleph_n$ being regular – there would be a cub E in ω_n such that $G(\nu)$ is $\aleph_{n-2}$-prebalanced in G for all $\nu \in E$. This is absurd, therefore G cannot have an $\aleph_{n-2}$-prebalanced chain. □

Corollary 6.3. *If for a model of ZFC there is an integer $n \geq 1$, such that every torsion-free group admits an $\aleph_n$-prebalanced chain, then $2^{\aleph_0} \leq \aleph_{n+1}$ in this model.*

Let us formulate another noteworthy consequence of our study.

Theorem 6.4. *In any model of ZFC, the following conditions are equivalent:*

(i) $\mathrm{Bext}^2(G, T) = 0$ *for all torsion-free groups G and torsion groups T.*

(ii) *Every torsion-free group admits an $\aleph_0$-prebalanced chain.*

(iii) CH *holds and balanced subgroups of completely decomposable groups are B_2-groups.*

Proof. The equivalence of (i) and (ii) has been proved in Theorem 4.5. By Lemma 6.2, (i) implies CH, and Lemma 4.3 shows that, in the presence of CH, a balanced subgroup of a completely decomposable group is a B_2-group if and only if it is a B_1-group. (iii) trivially implies (i), while the converse follows from Theorem 4.5. □

It is easy to find examples showing that $\aleph_0$-prebalanced subgroups are not necessarily separative, but it is not so obvious that it is possible for a group to admit an $\aleph_0$-prebalanced chain but no separative chains. We show this in models of ZFC in which CH fails.

Example 6.5. ($\neg$CH). There is a torsion-free group G of cardinality $\aleph_2$ which admits an $\aleph_0$-prebalanced chain but no separative chains.

As above, choose two collections, $\mathscr{S}$ and $\mathscr{S}'$, both of cardinality $\aleph_2$, consisting of almost disjoint sets such that the intersections $S_\alpha \cap S'_\beta$ are infinite for all $S_\alpha \in \mathscr{S}$, $S'_\beta \in \mathscr{S}'$. Define the types $\mathbf{t}_\alpha$ and $\mathbf{t}'_\beta$ as above. Let $A = \bigoplus_{\alpha < \omega_2} A_\nu$ be a completely decomposable group where the rational groups A_ν with indices $\nu > \omega$ have different types $\mathbf{t}_\alpha$ corresponding to $S_\alpha \in \mathscr{S}$, while A_ν with $\nu \geq \omega$ have different types $\mathbf{t}_\beta{}'$ corresponding to $S'_\beta \in \mathscr{S}'$. If G is a corank 1 subgroup of A which is disjoint from all A_ν, then in the notation of Theorem 6.1 we have that $G(\nu)$ is an $\aleph_0$-prebalanced subgroup. However, it is not separative in $G(\nu + 1)$, since the generating types $\neq \mathbf{t}(\mathbb{Z})$ are pairwise incomparable, so there is no cofinal subset in the set of characteristics. As in Lemma 6.2 we argue that G cannot have a separative chain.

7. The groups Bext^n

Albrecht and Hill [1] proved that $\mathrm{Bext}^2(G, T) = 0$ for all torsion-free groups G of cardinality $\leq \aleph_1$ and all torsion groups T, and that CH implies $\mathrm{Bext}^3(G, T) = 0$ for all torsion-free groups G. We extend these results to groups of higher cardinalities by showing that, for all torsion groups T, $\mathrm{Bext}^{i+1}(G, T) = 0$ for torsion-free groups G of cardinality $\leq \aleph_i$ $(i \geq 1)$, and $\mathrm{Bext}^{n+2}(G, T) = 0$ holds for all torsion-free groups G whenever $\aleph_n$ (for some integer $n \geq 1$) is the continuum.

We shall require an analogue of Auslander's well-known lemma that gives an upper estimate of the projective dimension of the union of a chain. Since the proof of Lemma 7.1 runs parallel to the proof of the original lemma [2], it suffices to state it without any comment.

Lemma 7.1. *If the group G is the union of a continuous well-ordered ascending chain $0 = G_0 < G_1 < \cdots < G_\alpha < \cdots (\alpha < \tau)$ of balanced subgroups G_α, for some ordinal τ, and if there is an integer $n \geq 1$ such that* $\mathrm{Bext}^n(G_{\alpha+1}/G_\alpha, T) = 0$ *for all torsion groups T and for all $\alpha < \tau$, then also* $\mathrm{Bext}^n(G, T) = 0$ *for all torsion groups T.*

We are now able to verify the following lemma.

Lemma 7.2. *For all torsion-free groups G of cardinality $\leq \aleph_i$ $(i \geq 1)$ and for all torsion groups T, we have*

$$\mathrm{Bext}^{i+k}(G, T) = 0 \quad \textit{for } k = 1, 2, \ldots .$$

Proof. We induct on i. Let $0 \to B \to C \to^{\gamma} G \to 0$ be a balanced-exact sequence where C is a completely decomposable group of cardinality $\aleph_i$. Our claim is equivalent to the assertion that $\mathrm{Bext}^i(B, T) = 0$ for all T.

If $\kappa = \aleph_1$, then the claim is obvious, since balanced subgroups of completely decomposable groups of cardinality $\leq \aleph_1$ are B_2-groups [1]. Let G have cardinality $\aleph_i$ $(i > 1)$, and assume that the claim holds for $i - 1$. Because of Lemma 1.4, B admits an $\aleph_{i-1}$-filtration with balanced subgroups, say, $0 = B_0 < B_1 < \cdots < B_\alpha < \cdots$ $(\alpha < \omega_i)$. The factors $B_{\alpha+1}/B_\alpha$ have cardinality $\leq \aleph_{i-1}$, so by induction hypothesis, $\mathrm{Bext}^i(B_{\alpha+1}/B_\alpha, T) = 0$ for all α. Hence Lemma 7.1 implies $\mathrm{Bext}^i(B, T) = 0$ – this is exactly what we wanted to prove. □

We can derive the following conclusion.

Theorem 7.3. *If $2^{\aleph_0} = \aleph_n$ for an integer $n \geq 1$, then*

$$\mathrm{Bext}^{n+k+l}(G, T) = 0 \quad (k = 1, 2, \ldots)$$

for all torsion-free groups G and torsion groups T.

Proof. Refer to Lemma 7.2 to conclude that $\mathrm{Bext}^{n+k}(G, T) = 0$ $(k \geq 1)$ for all groups G whose cardinality does not exceed the continuum. If $|G| = \kappa > \aleph_n$, then we apply Lemma 1.3 to a balancd-projective resolution $0 \to B \to C \to^{\gamma} G \to 0$ of G where C is a completely decomposable group. Thus, B has an $\aleph_n$-filtration with balanced subgroups $0 = B_0 < B_1 < \cdots < B_\alpha < \cdots$ $(\alpha < \kappa)$. As the factors $B_{\alpha+1}/B_\alpha$ have cardinality $\leq \aleph_n$, Lemma 7.2 guarantees that $\mathrm{Bext}^{n+1}(B_{\alpha+1}/B_\alpha, T) = 0$ for all α. An appeal to Lemma 7.1 completes the proof that $\mathrm{Bext}^{n+1}(B, T) = 0$. □

We conjecture that if $2^{\aleph_0} = \aleph_n$ for an integer $n \geq 1$, then $\mathrm{Bext}^n(G, T) \neq 0$ for some torsion-free group G and some torsion group T. This is certainly true for $n = 1$ and $n = 2$.

8. Subgroups of B_2-groups

We now turn our attention to the problem as to when a pure subgroup of a B_2-group is again a B_2-group. While Hill and Megibben [19] proved that a completely decomposable pure subgroup of a torsion-free group ought to be separative, this question for B_2-groups has been neglected in the literature. Only very recently have Bican and Fuchs [5] shown that if CH is assumed, then $\aleph_0$-prebalancedness is a necessary condition. Here we can do much better by establishing necessary and sufficient conditons; we do not even rely on additional set-theoretical hypotheses.

Theorem 8.1. *For a pure subgroup A of a B_2-group G the following conditions are equivalent*:

(i) *A is a B_2-group*;

(ii) *there is an $\aleph_0$-prebalanced chain from A to G*;

(iii) *there is a continuous well-ordered ascending chain of B_2-subgroups from A to G with rank 1 factors.*

Proof (i) $\Leftrightarrow$ (ii) Let $0 \to B \to A \oplus C \xrightarrow{\varphi} G \to 0$ be a balanced-projective resolution of G relative to A where C is completely decomposable and ϕ induces the identity on A. This gives rise to the exact sequence

$$0 = \mathrm{Bext}^1(G, T) \to \mathrm{Bext}^1(A, T) \to \mathrm{Bext}^1(B, T) \to \mathrm{Bext}^2(G, T) = 0$$

for every torsion group T where the terms involving G vanish on account of the assumption on G; see Corollary 4.9. It follows that B is a B_1-group exactly if A is a B_1-group.

Now, if A is a B_2-group, then so is $A \oplus C$, and Corollary 4.6 implies that B is a B_2-group. (ii) follows at once from Theorem 2.3. Conversely, if (ii) holds, then B is a B_2-group, so $A \oplus C$ is likewise a B_2-group. Since summands of B_2-groups are B_2-groups [5], we obtain (i).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) If (ii) holds, then from the equivalence of (i) and (ii) it follows that all the members in the $\aleph_0$-prebalanced chain are B_2-groups. In order to show that (iii) entails (ii), it suffices to ascertain that a corank 1 pure B_2-subgroup A of a B_2-group G is $\aleph_0$-prebalanced. The implication (iii) $\Rightarrow$ (i) is trivial. □

Since there is always an $\aleph_0$-prebalanced chain from an $\aleph_0$-prebalanced subgroup of index $\leq \aleph_1$ to the group itself, we conclude the following.

Corollary 8.2. *In any B_2-group, a pure subgroup of index $\leq \aleph_1$ is a B_2-group exactly if it is an $\aleph_0$-prebalanced subgroup.*

It is worth while mentioning the following result generalizing Theorem 4.1.

Corollary 8.3. *If the B_2-group A is a pure subgroup of a B_1-group G such that there is an $\aleph_0$-prebalanced chain from A to G, then G is a B_2-group, too.*

Proof. In a balanced-projective resolution $0 \to B \to A \oplus C \xrightarrow{\varphi} G \to 0$ of G relative to A, the groups B and $A \oplus C$ are now B_2-groups, while G is a B_1-group. The claim follows immediately from Corollary 4.6. □

Let us point out that implication (i) $\Rightarrow$ (iii) in Theorem 8.1 can be proved directly as follows. Suppose A is a pure B_2-subgroup of the B_2-group G. They admit $G(\aleph_0)$-families $\mathscr{A}$, $\mathscr{G}$ of decent, TEP-subgroups. Using standard back-and-forth arguments, one can easily construct continuous well-ordered ascending chains $\{A_\alpha\}$ in A and $\{G_\alpha\}$ in G such that $G_\alpha \in \mathscr{A}$, $G_\alpha \cap A = A_\alpha \in \mathscr{G}$ and $A + G_\alpha$ is pure in G. Then $(A + G_\alpha)/G_\alpha \cong A/A_\alpha$ is a B_2-group, so $A + G_\alpha$ is a B_2-group as an extension of a decent B_2-subgroup by a B_2-group. This yields a chain of pure B_2-subgroups with countable factors.

Our final corollary is an analogue of Theorem 5 by Dugas and Rangaswamy [10].

Corollary 8.4. *A homogeneous pure subgroup A of a B_2-group B is completely decomposable exactly if there is an $\aleph_0$-prebalanced chain from A to B.*

Proof. All what we have to note is that a homogeneous B_2-group is completely decomposable. □

References

[1] U. Albrecht and P. Hill, Butler groups of infinite rank and Axiom 3, Czechoslovak Math. J. 37 (1987) 293–309.

[2] L. Auslander, On the dimension of modules and algebras. III, Nagoya Math. J. 9 (1955) 67–77.

[3] L. Bican, Pure subgroups of Butler groups, in: Abelian Groups and Modules, CISM Courses and Lectures, Vol. 287 (Springer, Vienna, 1984) 203–213.

[4] L. Bican and L. Fuchs, On abelian groups by which balanced extensions of a rational group split, J. Pure Appl. Algebra 78 (1992) 221–238.

[5] L. Bican and L. Fuchs, Subgroups of Butler groups, Comm. Algebra 22 (1994) 1037–1047.

[6] L. Bican and L. Salce, Butler groups of infinite rank, in: Abelian Group Theory, Lecture Notes in Mathematics, Vol. 1006 (Springer, Berlin, 1983) 171–189.

[7] M.C.R. Butler, A class of torsion-free abelian groups of finite rank, Proc. London Math. Soc. 15 (1965) 680–698.

[8] M. Dugas, P. Hill and K.M. Rangaswamy, Infinite rank Butler groups. II, Trans. Amer. Math. Soc. 320 (1990) 643–664.

[9] M. Dugas and K.M. Rangaswamy, Infinite rank Butler groups. Trans. Amer. Math. Soc. 305 (1988) 129–142.

[10] M. Dugas and K.M. Rangaswamy, Separable pure subgroups of completely decomposable torsion-free abelian groups. Arch. Math. 58 (1992) 332–337.

[11] M. Dugas and R. Thomé, The functor Bext under the negation of CH, Forum Math. 3 (1991) 23–33.

[12] L. Fuchs, Infinite Abelian Groups, Vol. 2 (Academic Press, New York, 1973).

[13] L. Fuchs and M. Magidor, Butler groups of arbitrary cardinality, Israel J. Math. 84 (1993) 239–263.

[14] L. Fuchs and C. Metelli, Countable Butler groups, Contemp. Math. 130 (1992) 133–143.

[15] L. Fuchs, C. Metelli and K.M. Rangaswamy, Corank one subgroups of completely decomposable abelian groups, Comm. Algebra 22 (1994) 1031–1036.

[16] L. Fuchs and K.M. Rangaswamy, Butler groups that are unions of subgroups with countable typesets, Archiv d. Math. 61 (1993) 105–110.

[17] P. Hill, Isotype subgroups of totally projective groups, in: Lecture Notes in Mathematics, Vol. 874 (Springer, Berlin, 1981) 305–321.

[18] P. Hill, The third axiom of countability for abelian groups, Proc. Amer. Math. Soc. 82 (1981) 347–350.

[19] P. Hill and C. Megibben, Pure subgroups of torsion-free groups, Trans. Amer. Math. Soc. 303 (1987) 765–778.

[20] T. Jech, Set Theory (Academic Press, New York, 1973).

[21] K.M. Rangaswamy, A homological characterization of abelian B_2-groups, Proc. Amer. Math. Soc. 121 (1994) 409–415.