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Note

On graphic and 3-hypergraphic sequences

S.A. Choudum

School of Mathematical Sciences, Madurai Kamaraj University, Madurai—625 021, India

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Abstract

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In this paper we give a necessary condition for a sequence π of integers to be 3-hypergraphic. This necessary condition is on the lines of Erdős and Gallai conditions for graphic sequences and depends on a function M_r defined on π.

1. Introduction

Throughout this paper π: $d_1, d_2, \ldots, d_p$ denotes a non-increasing sequence of non-negative integers. A r-graph is a loopless undirected graph with at most r edges joining a pair of vertices. As usual V and E respectively denote the vertex set and the edge set of a graph under consideration and $\deg(v)$ denotes the number of edges incident with a vertex v. Let $G_r(\pi) = \{G$: G is a r-graph on p vertices, say $v_1, v_2, \ldots, v_p$ such that $\deg_G(v_i) \leq d_i$, for every i, $1 \leq i \leq p\}$. Let $M_r(\pi) = \max\{|E(G)|: G \in G_r(\pi)\}$.

In the literature, π is said to be r-graphic if there is a r-graph whose degree sequence is π. Clearly, $M_r(\pi) \leq \frac{1}{2}(\sum_{i=1}^{p} d_i)$ and moreover, π is r-graphic iff $\sum_{i=1}^{p} d_i$ is even and $M_r(\pi) = \frac{1}{2}(\sum_{i=1}^{p} d_i)$.

A n-hypergraph H is a pair (V, E) where V is a non-empty set and E is a family of n-subsets (that is, subsets with exactly n elements) of V; the elements of E need not be distinct. The elements of V are called vertices and the elements of E are called edges. A n-hypergraph $H = (V, E)$ is called simple if all the elements in E are distinct. Thus a 1-graph and a simple 2-hypergraph are the same objects and usually they are referred to as simple graphs. The degree $\deg_H(v)$ of a vertex v in a n-hypergraph H is the number of edges containing the vertex v. π is said to be n-hypergraphic, if there is a simple n-hypergraph H on p vertices, say v_1, $v_2, \ldots, v_p$, such that $\deg_H(v_i) = d_i$, for every i, $1 \leq i \leq p$. For general terminology we refer to [1]. The sequence $a_1^{r_1}, a_2^{r_2}, \ldots, a_n^{r_n}$ denotes the sequence in which a_i is repeated r_i times, $i = 1, 2, \ldots, n$.

A well-known theorem (see Theorem 6, Chapter 6 in [1]) due to Erdős and Gallai characterizes 2-hypergraphic (that is, graphic) sequences as follows:

Theorem A: *π is 2-hypergraphic iff $\sum_{i=1}^{p} d_i$ is even and*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{p} \min(d_j, k), \quad \text{for } k = 1, \ldots, p. \tag{1}$$

Its generalization to n-hypergraphic ($n \geq 3$) sequences is unknown; see [2, 4–6]. The problem seems to be difficult even for $n = 3$. In fact, in [4], the authors report that they were neither able to give a polynomial time algorithm to test the 3-hypergraphicness nor able to prove that the problem is NP-complete, though they settle several related problems.

In this note we first give a necessary condition for a sequence π to be 3-hypergraphic and then derive a formula for $M_r(\pi)$ to show that our necessary condition can be checked in polynomial time.

2. A necessary condition for 3-hypergraphic sequences

As usual for a real number x let $\lfloor x \rfloor$ denote the greatest integer not greater than x and let x^+ denote $\max(0, x)$. We denote $(x_1^+, x_2^+, \ldots, x_p^+)$ by $(x_1, x_2, \ldots, x_p)^+$.

Theorem 1. *If a sequence π: $d_1 \geq d_2 \geq \cdots \geq d_p$ is 3-hypergraphic, then*

$$\sum_{i=1}^{p} d_i \equiv 0 \pmod 3; \tag{2}$$

$$\sum_{i=1}^{k} d_i \leq k\binom{k-1}{2} + \sum_{j=k+1}^{p} 2\min\left(d_j, \binom{k}{2}\right) + M_k\left(d_{k+1} - \binom{k}{2}, \ldots, d_p - \binom{k}{2}\right)^+, \quad \textit{for } k = 1, 2, \ldots, p. \tag{3}$$

Proof. (2) is obvious. To prove (3), let $H = (V, E)$ be a simple 3-hypergraph on the vertex set $V = \{v_1, v_2, \ldots, v_p\}$ such that $\deg(v_i) = d_i$, for every i, $1 \leq i \leq p$. Let $A = \{v_1, v_2, \ldots, v_k\}$ and $B = \{v_{k+1}, v_{k+2}, \ldots, v_p\}$. For an integer j, where $k + 1 \leq j \leq p$, let $E_j = \{e \in E$: $e = \{v_j, v_x, v_y\}$, where $1 \leq x \leq k$, $1 \leq y \leq k\}$; $|E_j| = d_{j,1}$; $E_j' = \{e \in E$: $e = \{v_j, v_x, v_y\}$ where $k + 1 \leq x \leq p$ and $1 \leq y \leq k\}$; $|E_j'| = d_{j,2}$.

Then

$$\sum_{i=1}^{k} d_i \leq 3\left|\binom{A}{3}\right| + \sum_{j=k+1}^{p} 2|E_j| + \left|\bigcup_{j=k+1}^{p} E'_j\right|, \tag{4}$$

since an edge $e \in E \cap \binom{A}{3}$ contributes 3, an edge $e \in E_j$ contributes 2, an edge $e \in E'_j$ contributes 1 and every other edge contributes 0 to the sum $d_1 + d_2 + \cdots + d_k$.

$$d_j \geq d_{j,1} + d_{j,2}; \tag{5}$$

and

$$d_{j,1} \leq \min\left(d_j, \binom{k}{2}\right); \tag{6}$$

let $d_{j,1} = \min(d_j, \binom{k}{2}) - s_j$.

Since, the edges in H are all distinct it follows that the pair $(B, \{e - A: e \in \bigcup_{j=k+1}^{p} E'_j\})$ is a k-graph with degree sequence $d_{k+1,2}, d_{k+2,2}, \ldots, d_{p,2}$. So

$$\left|\bigcup_{j=k+1}^{p} E'_j\right| \leq M_k(d_{k+1,2}, d_{k+2,2}, \ldots, d_{p,2}). \tag{7}$$

Now, substituting (6) and (7) in (4), we obtain

$$\sum_{i=1}^{k} d_i \leq k\binom{k-1}{2} + \sum_{j=k+1}^{p} 2\left(\min\left(d_j, \binom{k}{2}\right) - s_j\right) + M_k(d_{k+1,2}, \ldots, d_{p,2}). \tag{8}$$

But for $j = k+1, k+2, \ldots, p$, we have

$$d_{j,2} \leq d_j - d_{j,1} = d_j - \min\left(d, \binom{k}{2}\right) + s_j = \left(d_j - \binom{k}{2}\right)^+ + s_j.$$

So,

$$M_k(d_{k+1,2}, \ldots, d_{p,2}) \leq M_k\left(\left(d_{k+1} - \binom{k}{2}\right)^+ + s_{k+1}, \ldots, \left(d_p - \binom{k}{2}\right)^+ + s_p\right)$$

$$\leq M_k\left(d_{k+1} - \binom{k}{2}, \ldots, d_p - \binom{k}{2}\right)^+ + \sum_{j=k+1}^{p} s_j,$$

since $M_k(b_1 + c_1, \ldots, b_p + c_p) \leq M_k(b_1, \ldots, b_p) + \sum_{j=1}^{p} c_j$, where b_i and c_j are nonnegative integers. Hence (3) follows from (8). □

It easily follows that the conditions (2) and (3) can be checked in polynomial time provided M_r can be evaluated in polynomial time. Our next theorem shows that $M_r(\pi)$ can be indeed evaluated in polynomial time.

3. A formula for $M_r(\pi)$

Let

$$\delta_r(\pi, k) = \sum_{i=1}^{k} d_i - rk(k-1) - \sum_{j=k+1}^{p} \min(d_j, rk), \tag{9}$$

$$\delta_r(\pi) = \max\{\delta_r(\pi, k): 1 \leq k \leq p\}. \tag{10}$$

Theorem B (Chungphaisan [3]). *π is r-graphic iff $\sum_{i=1}^{p} d_i$ is even, and*

$$\delta_r(\pi, k) \leq 0, \quad \text{for } k = 1, 2, \ldots, p. \tag{11}$$

We use this theorem to derive a formula for $M_r(\pi)$.

Theorem 2. $M_r(\pi) = \lfloor \frac{1}{2}(\sum_{i=1}^{p} d_i - \delta_r^+(\pi)) \rfloor$.

Proof. Without loss of generality we assume that $r \geq 1$, $p \geq 2$ and $d_p \geq 1$.

$$M_r(\pi) \leq \left\lfloor \tfrac{1}{2}\left(\sum_{i=1}^{p} d_i - \delta_r^+(\pi)\right)\right\rfloor. \tag{12}$$

To prove this inequality, let $G \in G_r(\pi)$ be a r-graph with $|E(G)| = M_r(\pi)$. Let π': $d_1' \geq d_2' \geq \cdots \geq d_p'$ be its degree sequence; so $d_i' \leq d_i$, for $i = 1, 2, \ldots, p$. Let t be such that $\delta_r(\pi) = \delta_r(\pi, t)$. Then

$$\begin{aligned} 2M_r(\pi) = 2|E(G)| &= \sum_{i=1}^{t} d_i' + \sum_{i=t+1}^{p} d_i' \\ &\leq rt(t-1) + \sum_{i=t+1}^{p} \min(d_i', rt) + \sum_{i=t+1}^{p} d_i' \\ &\leq rt(t-1) + \sum_{i=t+1}^{p} \min(d_i, rt) + \sum_{i=t+1}^{p} d_i \\ &= \sum_{i=1}^{p} d_i - \delta_r(\pi), \quad \text{by (9) and (10).} \end{aligned}$$

Since, $M_r(\pi) \leq \frac{1}{2}(\sum_{i=1}^{p} d_i)$ and it is an integer, we obtain

$$\begin{aligned} M_r(\pi) &\leq \min\left\{\left\lfloor \tfrac{1}{2}\left(\sum_{i=1}^{p} d_i\right)\right\rfloor, \left\lfloor \tfrac{1}{2}\left(\sum_{i=1}^{p} d_i - \delta_r(\pi)\right)\right\rfloor\right\} \\ &= \left\lfloor \tfrac{1}{2}\left(\sum_{i=1}^{p} d_i - \delta_r^+(\pi)\right)\right\rfloor. \end{aligned}$$

$$M_r(\pi) \geq \left\lfloor \tfrac{1}{2}\left(\sum_{i=1}^{p} d_i - \delta_r^+(\pi)\right\rfloor\right). \tag{13}$$

To prove this inequality, let π^1 denote the sequence $d_1, d_2, \ldots, d_p, 1^n$ where

$$n = \begin{cases} \delta_r^+(\pi) & \text{if } \sum_{i=1}^{p} d_i \equiv \delta_r^+(\pi) \pmod 2, \\ \delta_r^+(\pi) + 1 & \text{otherwise.} \end{cases}$$

Clearly, the sum of the terms in π^1 is even. Next one can routinely show that $\delta_r(\pi^1, k) \leq 0$, for $k = 1, 2, \ldots, p+n$. So, π^1 is r-graphic, by Theorem B. Let G^1 be a realization of π^1 and let $u_1, u_2, \ldots, u_n$ be n vertices of degree 1 in G^1. Then $G = G^1 - \{u_1, u_2, \ldots, u_n\} \in G_r(\pi)$. So,

$$2M_r(\pi) \geq 2|E(G)| \geq 2(|E(G^1)| - n = \sum_{i=1}^{p} d_i - n. \qquad \square$$

4. Remarks

(1) If a sequence π: $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$ satisfies the conditions (2) and (3), then there is a 3-hypergraph H (not necessarily simple) on p vertices, say $v_1, v_2, \ldots, v_p$ such that $\deg_H(v_i) = d_i$, for $i = 1, 2, \ldots, p$. This assertion follows easily from Theorem 1.1 on p. 63 in [7]. It is easy to construct sequences which realize 3-hypergraphs but do not satisfy condition (3). For example, the sequence π: $(2p+1)^3\, 3^{p-3}$ (where $p \geqslant 4$) realizes 3-hypergraphs but does not satisfy (3) for $k = 2$.

(2) We feel that our conditions (2) and (3) are not sufficient for a sequence to be 3-hypergraphic. The upper bound in (3) is probably larger than what it should be. So, the straightforward generalization of Theorem 1 for n-hypergraphic sequences ($n \geqslant 4$) does not seem to be worthwhile.

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