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Uniquely Colorable Graphs

BÉLA BOLLOBÁS

Department of Mathematics, University of Cambridge, Cambridge, U.K.

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A graph is called uniquely k-colorable if there is only one partition of its vertex set into k color classes. The first result of this note is that if a k-colorable graph G of order n is such that its minimal degree, $\delta(G)$, is greater than $(3k - 5)/(3k - 2)\,n$ then it is uniquely k-colorable. This result can be strengthened considerably if one considers only graphs having an obvious property of k-colorable graphs. More precisely, the main result of the note states the following. If G is a graph of order n that has a k-coloring in which the subgraph induced by the union of any two color classes is connected then $\delta(G) > (1 - (1/(k - 1)))\,n$ implies that G is uniquely k-colorable. Both these results are best possible.

A *coloring* of a graph G with vertex set V is the partitioning of V into so called *color classes* in such a way that no two vertices of the same class are adjacent. A k-coloring contains exactly k color classes. We shall think of a k-coloring of G as a map $\psi: V \to \{1, 2,..., k\}$ such that $\psi^{-1}(i)$, $i = 1, 2,..., k$, are the color classes of G. Naturally two maps, ψ_1 and ψ_2, represent the same k-coloring if and only if $\psi_1 = \psi_2 \circ \pi$ for some permutation π of $\{1, 2,..., k\}$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimal k for which G has a k-coloring. A graph with exactly one k-coloring is called *uniquely k-colorable*. It is obvious that if G is uniquely k-colorable then $\chi(G) = k$ or n, so we shall say simply that G is *uniquely colorable* if it is uniquely $\chi(G)$-colorable.

As in the book [1], denote by K^p the complete graph of order p and by $K^{p,p}$ the complete bipartite graph with p vertices in each class. $K_r(p)$ denotes the complete r-partite graph with p vertices in each class. The degree of a vertex x of G is denoted by $\deg x$ or $\deg_G x$. The minimal degree of a vertex of a graph G is denoted by $\delta(G)$. The join of G and H is denoted by $G + H$.

Uniquely colorable graphs have been investigated by Cartwright and Harary [2], Chartrand and Geller [3], Harary *et al.* [4], and Osterweil [5]. In this note we give best possible sufficient conditions involving $\delta(G)$ for a graph G to be uniquely colorable.

Given $k - 2$ and $l - 1$ let $G_2 = 2K^{l,l}$ and $G_a = G_2 + K_{k-2}(3l)$ (Fig. 1).

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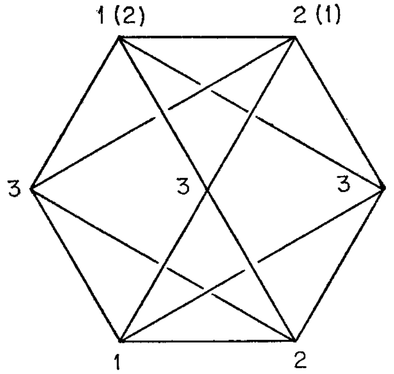

FIG. 1. G_3 for $l = 1$.

Then G_a is a graph of order $n = (3k-2)l$ and clearly $\chi(G_a) = k$ and $\delta(G_a) = (3k-5)l = ((3k-5)/(3k-2))n$. Furthermore, G_2 is clearly not uniquely 2-colorable so G_a is not uniquely k-colorable.

As our first result we show that this graph G_a has the largest minimal degree among all nonuniquely k-colorable graphs of order n.

THEOREM 1. *Let G be a k-colorable* ($k \geqslant 2$) *graph of order n such that*

$$\delta(G) > ((3k-5)/(3k-2))n.$$

Then G is uniquely colorable.

Proof. We prove the result by induction on k. Suppose first that $k = 2$. If G is not connected, let H be a component of G of order $m \leqslant n/2$. In H every vertex has degree $>m/2$ so H contains a triangle. As this is impossible, we can conclude that G is connected and so it is uniquely 2-colorable.

Let now $k \geqslant 3$ and suppose the result holds for smaller values of k. If x is a vertex of G, denote by G_x the subgraph of G spanned by the vertices adjacent to x. Denote the order of G_x by n_x. Then

$$n_x > ((3k-5)/(3k-2))n$$

and the degree of a vertex y of G_x (in G_x) is at least

$$\frac{3k-5}{3k-2}\,n - (n - n_x) = n_x - \frac{3}{3k-2}\,n > \frac{3(k-1)-5}{3(k-1)-2}\,n_x .$$

Therefore by the induction hypothesis G_x is uniquely $(k-1)$-colorable.

Let now u_1 and u_2 be vertices of G. As

$$\deg u_i \geqslant \delta(G) > \frac{3k-5}{3k-2}\,n \geqslant \frac{4}{7}\,n > \frac{1}{2}\,n,$$

there is a vertex x adjacent to both u_1 and u_2. In other words, u_1 and u_2 belong to G_x. Now a k-coloring of G always gives a $(k-1)$-coloring of G_x. As this $(k-1)$-coloring is unique, either u_1 and u_2 get the same color or they get different colors, independently of the k-coloring of G. Thus G is uniquely colorable and so the proof is complete.

Cartwright and Harary [2] pointed out that if a graph is uniquely k-colorable $(k \geq 2)$ then

(*) the subgraph induced by the union of any two color classes of the k-coloring is connected.

If $k = 2$ then (*) says simply that G is connected and then, naturally, G is uniquely 2-colorable if it is 2-colorable. However, if $k \geq 3$ then it is easily seen that a k-colorable graph with a k-coloring satisfying (*) is not necessarily uniquely k-colorable. Thus it is natural to ask how large $\delta(G)$ has got to be to ensure that a graph G with a k-coloring satisfying (*) is uniquely k-colorable. As before, we start with an example of a nonuniquely k-colorable graph G that satisfies (*) and for which $\delta(G)$ is large and then we prove our main result, essentially stating that the example is best possible.

Let $k \geq 3$ and $l \geq 1$ be integers. Let H_3 be the graph obtained from the graph of the triangular prism (Fig. 2) by replacing each vertex by a cloud of l vertices. Thus two vertices of H_3 are joined if and only if they belong to different clouds that were adjacent in the graph of the prism. Put $H_a = H_3 + K_{k-3}(3l)$. Then H_k is a graph of order $n = 3(k-1)l$,

$$\chi(H_k) = k \qquad \text{and} \qquad \delta(H_k) = 3(k-2)\,l = \frac{k-2}{k-1}\,n.$$

Even more, H_a has two different k-colorings, corresponding to the two 3-colorings of the prism, shown in Fig. 2, and both of these k-colorings satisfy (*).

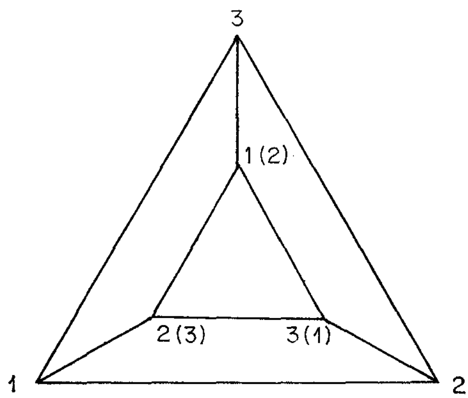

FIG. 2. Two 3-colourings of the prism.

THEOREM 2. *Let G be a graph of order n — k having a k-coloring* ($k \geqslant 2$) *satisfying* (*). *If*

$$\delta(G) > (1 - (1/(k-1)))n$$

then G is uniquely colorable.

Proof. Note first that the degree condition in the theorem is exactly the one that ensures that G is *not* $(k-1)$-colorable and so $\chi(G) = k$. In fact, the condition $\delta(G) > (1 - (1/(k-1)))n$ is exactly the condition of Zarankiewicz [6] ensuring that G contains a K^k. Thus G has got a uniquely k-colorable subgraph.

We prove the theorem by induction on k. For $k = 2$ the result is trivial: A connected 2-colorable graph is uniquely colorable. Suppose $k \geqslant 3$ and the theorem holds for smaller values of k.

Denote by V the vertex set of G and let ψ_1 be a k-coloring satisfying (*). Denote by $G(i,j)$ the connected subgraph induced by the classes of colors i and j. *Suppose that, contrary to the assertion, G has another k-coloring, say* ψ_2. We may suppose without loss of generality that ψ_2 gives the same colors to a uniquely k-colorable subgraph. This implies that

$$V(i) = V(i,i) = \{x \in V: \psi_1(x) = \psi_2(x) = i\} \neq \varnothing, \qquad i = 1, 2,..., k.$$

Put also

$$V(i,j) = \{x \in V: \psi_1(x) = i, \psi_2(x) = j\},$$

where $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant k$ and $i \neq j$.

Put furthermore

$$n(i) = |V(i)|,$$
$$n(i,j) = |V(i,j)|,$$

where $|X|$ denotes the number of elements in a set X.

We may suppose without loss of generality that G is the maximal graph having these two colorings ψ_1 and ψ_2, i.e., if $x_l \in V(i_l, j_l)$, $l = 1, 2$, then x_1x_2 is an edge of G if and only if $i_1 \neq i_2$ and $j_1 \neq j_2$.

As in the sequel we shall use the connectedness of $G(i,j)$ a number of times. Note that the structure of $G(1, 2)$ is given in Fig. 3 in the following sense: xy is an edge of $G(1, 2)$ if and only if the vertex classes $(V(i), V(k, l))$ containing them are joined by an edge. Note, e.g., that if $V(1, 2) \neq \varnothing$ and $V(2, 1) = \varnothing$ then $V(2, l) \neq \varnothing$ for some $l > 2$.

Let H_i be the subgraph of G induced by the vertices adjacent to a vertex $x_i \in V(i)$. If H_i has order m_i and y_i is any vertex in H_i, then

$$\deg_{H_i} y_i \geqslant \deg_G y_i - (n - m_i) > m_i - \frac{n}{k-1} > \left(1 - \frac{1}{k-2}\right) m_i. \quad (1)$$

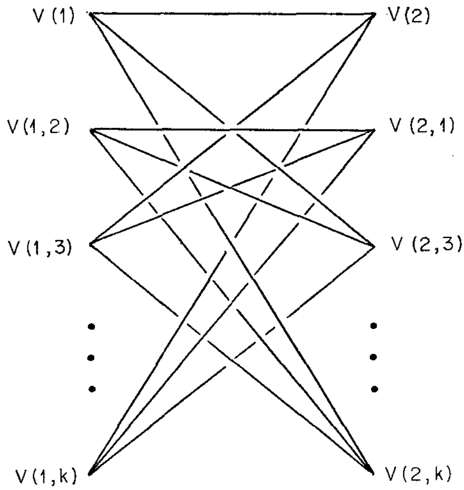

FIG. 3. The structure of $G(1, 2)$.

Suppose the coloring ψ_1 restricted to a subgraph H_i, say to H_k, is a $(k-1)$-coloring satisfying (*).

Then, as (1) holds, the induction hypothesis can be applied to H_k and so $\psi_1 = \psi_2$ on H_k, i.e., $V(i,j) = \varnothing$ unless $i = k$ or $j = k$. Note now that if $V(l, k) = \varnothing$ for some $l < k$ then the connectedness of $G(l, k)$ implies $V(k, l) = \varnothing$. In that case ψ_1 is a $(k-1)$-coloring of H_l that satisfies (*) so $V(i,j) = \varnothing$ unless $i = l$ or $j = l$. Consequently $V(i,j) = \varnothing$ whenever $i \neq j$, contradicting $\psi_1 \neq \psi_2$. Therefore $V(l, k) \neq \varnothing$, $l = 1, 2,..., k-1$, and so $V(k, i) \neq \varnothing$ for at least two values of i, say $V(k, i) \neq \varnothing$ for $i \leqslant j\,(-2)$ and $V(k, i) = \varnothing$ for $j < i < k$.

Denote by $d(i)$ the degree of a vertex in $V(i)$ and by $d(i,j)$ the degree of a vertex in $V(i,j)$. (Recall that if $x, y \in V(i,j)$ then x and y are joined to exactly the same vertices.) Put

$$S = \sum_{1}^{k-1} d(i) + \sum_{i=1}^{k-1} d(i, k) + \sum_{i=j+1}^{k-1} d(i, k) + \sum_{i=1}^{j} d(k, i).$$

As $\delta(G) > (1 - (1/(k-1))n$, and S is the sum of $3(k-1)$ degrees,

$$S > 3(k-1)\left(1 - \frac{1}{k-1}\right) n = 3(k-2)\, n.$$

However, this is impossible, since when expanding S as a linear combination of the $n(i)$'s and $n(i,j)$'s, no coefficient is larger than $3(k-2)$. This contradiction shows that the coloring ψ_1 restricted to a graph H_i $(1 \leqslant i \leqslant k)$ *does* *not satisfy* (*). In other words, for every i, $1 \leqslant i \leqslant k$, there is a pair (j, l), $1 \leqslant j < l \leqslant k, j \neq i \neq l$, such that the subgraph of G spanend by

$$\bigcup_{m \neq i} V(j, m) \cup \bigcup_{m \neq i} V(l, m)$$

is disconnected. Call this unordered pair (j, l) the *edge belonging to the color i*. It is easily checked that, as $G(j, l)$ is *connected*,

(**) *one of the following three assertions holds.*

(1) $V(j, l) \neq \varnothing$, $V(l, j) \neq \varnothing$, $V(j, m) = V(l, m) = \varnothing$ if $m \neq i$ and $V(j, i) \cup V(l, i) \neq \varnothing$.

(2) $V(j, l) \neq \varnothing$, $V(l, j) \neq \varnothing$, $V(l, m) = \varnothing$ if $m \neq i$ and $V(l, i) \neq \varnothing$.

(3) $V(j, l) = \varnothing$, $V(l, j) \neq \varnothing$, $V(j, m) = \varnothing$ if $m \neq i$ and $V(j, i) \neq \varnothing$.

(Note that (3) is obtained from (2) by interchanging j and l.)

It is easily checked that by (**) an "edge" can not belong to two different colors. There is an "edge" belonging to each color so there must be a *cycle* formed by such edges.

To simplify the notation we shall consider colors 1, 2,... instead of i_1, i_2,...; naturally we can do this without loss of generality.

Suppose $1\ 2\ 3 \cdots m$ $(m \geqslant 3)$ *is a cycle* in the following sense: $(i, i + 1)$ belongs to a color c_i $(1 \leqslant i < m)$ and $(m, 1)$ belongs to c_m.

(a) *Suppose furthermore, that* (1) *of* (*)) *holds for the edge* (1, 2) belonging to the color c_1 and, say, $V(2, c_1) \neq \varnothing$. Then (**) applied to (2, 3) gives that $c_1 = 3$. By repeated applications of (**) one can show that $c_2 = 1$, $c_3 = 2$ and $V(1, l)$, $V(2, l)$, $V(3, l)$ are empty for all $l > 3$. It is easily checked that the notation can be chosen in such a way that $V(1, 2)$, $V(2, 3)$ and $V(3, 1)$ are not empty.

(b) *Suppose now that* (1) *of* (**) *does not hold for any edge of the cycle* $1\ 2 \cdots m$. We may suppose without loss of generality that $V(1, 2) \neq \varnothing$, $V(2, m) = \varnothing$ if $m \neq c_1$ and $V(2, c_1) \neq \varnothing$. Applying (**) to the "edge" (2, 3) belonging to c_2 ($\neq c_1$, 2, and 3) we see that $c_1 = 3$ and, as (1) does not hold for (2, 3), $V(3, m) = \varnothing$ if $m \neq c_2$ and $V(3, c_2) \neq \varnothing$. The application of (**) to (3. 4) gives $c_2 = 4$, etc. Therefore we obtain the following: $\{c_1, \ldots, c_k\} = \{1, 2, \ldots, m\}$, the sets $V(1, 2)$, $V(2, 3)$,..., $V(m - 1, m)$, $V(m, 1)$ are nonempty and all other sets of the form $V(i, j)$ are empty where $1 \leqslant i \leqslant m$, $j \neq i$.

We have shown in particular, that the k "edges" belonging to the colors 1, 2,..., k form a 2-factor with vertex set $\{1, 2, \ldots, k\}$.

We are now ready to prove the theorem by arriving at a contradiction in the situation above.

For each color i we take a vertex $x_i \in V(i)$ and another vertex x_i' of color i in the coloring ψ_1 and add the degrees of these $2k$ vertices. These vertices x_i' are chosen as follows. (As before, instead of "i" we use the color "1".)

If 123 is a triangle in (a), then choose the notation in such a way that $V(1, 2)$, $V(2, 3)$, and $V(3, 1)$ are nonempty and let $x_1' \in V(1, 2)$, $x_2' \in V(2, 3)$, and $x_3' \in V(3, 1)$.

If $1\ 2 \cdots m$ is a cycle in (b) then let $x_1' \in V(1, 2)$, $x_2' \in V(2, 3), \ldots, x_m' \in V(m, 1)$.

Then

$$S = \sum_{i=1}^{k} \deg x_2' + \sum_{i=1}^{k} \deg x_i' > 2k\left(1 - \frac{1}{k-1}\right) n. \tag{2}$$

Note now that when expanding $\deg x_i$, $\deg x_i'$ in terms of $n(i)$ and $n(i, j)$ then each $n(i)$ and $n(i, j)$ is missing at least three times. As the sum of all $n(i)$ and $n(i, j)$'s is exactly n, this expansion gives

$$S \leqslant (2k - 3)n.$$

Comparing this with (2) we obtain

$$2k(1 - (1/(k-1)))n < (2k - 3)n, \qquad k < 3.$$

This contradiction completes the proof of the theorem.

Finally let me mention an open problem connected to the ones discussed here. What is the minimal number of edges of a uniquely k-colorable ($k \geqslant 3$) graph of order n? Denoting this minimal number by $m(n, k)$ it is easily seen that

$$\frac{kn}{2} \leqslant m(n, k) \leqslant (k-1)\, n - \binom{k}{2},$$

but it does not seem to be trivial to disprove either of the following relations:

$$\liminf_{n\to\infty} \frac{m(n, k)}{n} = \frac{k}{2}, \qquad \limsup_{n\to\infty} \frac{m(n, k)}{n} = k - 1.$$

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