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A spectral sequence for string cohomology

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Abstract

Let X be a 1-connected space with free-loop space ΛX. We introduce two spectral sequences converging towards $H^*(\Lambda X;\mathbb{Z}/p)$ and $H^*((\Lambda X)_{h\mathbb{T}};\mathbb{Z}/p)$. The E_2-terms are certain non-Abelian-derived functors applied to $H^*(X;\mathbb{Z}/p)$. When $H^*(X;\mathbb{Z}/p)$ is a polynomial algebra, the spectral sequences collapse for more or less trivial reasons. If X is a sphere it is a surprising fact that the spectral sequences collapse for $p=2$.

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1. Introduction

Let X be a space and let ΛX denote its free-loop space. The circle group $\mathbb{T}$ acts on ΛX by rotation of loops. The associated homotopy orbit space $\Lambda X_{h\mathbb{T}}$ is sometimes called the string space.

For a manifold X, the free-loop space has numerous geometric applications. The most basic one is via the Morse theory approach to the study of geodesic curves on X [12]. But later the free-loop space has also been used to study diffeomorphisms. The main connection is through Waldhausen's algebraic K-theory of spaces, the so-called A-theory [23]. One of the more refined versions of this connection relates pseudo-isotopies at a prime p to the p-local spectrum $TC(X,p)$.

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One can define $TC(X, p)$ as the following homotopy pullback [2]:

$$\begin{array}{ccc} TC(X,p) & \longrightarrow & \Sigma^{\infty}(\Lambda X_{+})^{\wedge}_{p} \\ \downarrow & & \downarrow{\scriptstyle 1-\Delta_p} \\ \Sigma^{\infty}(\Sigma(\Lambda X_{h\mathbb{T}})_{+})^{\wedge}_{p} & \xrightarrow{Trf} & \Sigma^{\infty}(\Lambda X_{+})^{\wedge}_{p} \end{array}$$

where Δ_p is the map which winds a loop p times around itself and Trf is the S^1-transfer map. The spectrum $TC(X, p)$ is an approximation to Waldhausen's $A(X)$, which in turn gives a hold on the stable pseudoisomorphism space of X. Even if the application we have in mind is for differentiable manifolds, it does not matter to TC that X is a manifold, as opposed to just a homotopy type.

There is a third train of thoughts, inspired by analogies with mathematical physics, especially with quantum field theory and string theory. In [9] Sullivan and Chas introduced algebraic structures relating $H_*(\Lambda X; \mathbb{Z})$ and $H_*(\Lambda X_{h\mathbb{T}}; \mathbb{Z})$. These algebraic structures use that X is a closed manifold. More precisely, the Thom class of the tangent bundle of X plays an essential role.

The approach of this paper is homotopy theoretical. We start with a homotopy-type X, and try to recover the modulo p cohomology of ΛX and of the Borel construction $\Lambda X_{h\mathbb{T}}$ by homotopy theoretical spectral sequences. We return to the construction of these spectral sequences later in the introduction. The most essential properties are

- It is derived from a cosimplicial space similar to the cosimplicial space used to define the Adams spectral sequence for X.
- The E_2 page of the spectral sequence is computed by non-Abelian homological algebra in the sense of André–Quillen homology.

One competing homotopy theoretical approach to the cohomology of the Borel construction is the following three-step method. Let us call it the fibration method.

- Compute the cohomology of ΩX using a spectral sequence (Serre or Eilenberg–Moore) belonging to the fibration $\Omega X \to PX \to X$.
- Compute the cohomology of ΛX using the previous result and the fibration $\Omega X \to \Lambda X \to X$.
- Compute the cohomology of the $\Lambda X_{h\mathbb{T}}$ using the previous result and the fibration $\Lambda X \to \Lambda X_{h\mathbb{T}} \to B\mathbb{T}$.

The Eilenberg–Moore spectral sequence can be thought of as the spectral sequence for the cohomology of a cosimplicial space, just like our spectral sequence. But the particular cosimplicial object is entirely different from ours. We use a Postnikov decomposition of X, which is not visible in the Eilenberg–Moore situation.

We feed two types of information into the machine, which are not used by the fibration approach. Firstly, we use that one can explicitly compute the cohomology in the case when X is an Eilenberg–MacLane space. Secondly, we use non-Abelian homological algebra to keep track of how the pieces of our resolution fit together. In particular, a large part of the information about the $\mathbb{T}$-action on ΛX is internal to the machine of non-Abelian homological algebra. So this part is taken care of already in the E_2 page.

This does not necessarily say that our method is better than the fibration method, however it does suggest that out method is different. So there might be reasons to use both methods simultaneously.

One possible drawback of our method is that non-Abelian-derived functors are hard to calculate. We have little use for a spectral sequence with an incalculable E_2 page.

In order to show that this is not so bad, we do a few comparatively simple computations at the end of this paper. We show that at least for the sphere S^n we can solve the non-Abelian homological algebra, and that the spectral sequence we get seems different from whatever comes out of the fibration method. We find it quite surprising and encouraging that in this case the spectral sequence converging to $H^*(\Lambda X_{h\mathbb{T}}; \mathbb{F}_2)$ collapses. We do not actually prove this collapsing by methods internal to our spectral sequence. We quote the result which is known from other methods, and check by a counting argument that there is no room for differentials.

Because of the homotopy theoretical nature of our work, it seems likely that we can use it to study $TC(X)$ for spaces X with pleasant cohomology. We intend to study this closer, but have not yet done so.

It seems more difficult, but potentially very profitable to compare our computations with the Sullivan–Chas theory. We have not done this either yet.

What we have done [4], is that we have studied the spectral sequence converging to $H^*(\Lambda X; \mathbb{F}_2)$ in the very special case where $H^*(X; \mathbb{F}_2)$ is a truncated polynomial algebra on one generator. We can compute the relevant E_2 page, and this makes it possible to compute the Steenrod algebra action on $H^*(\Lambda X; \mathbb{F}_2)$ when X is one of the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$ or the Cayley projective plane CaP^2. The results led us to conjecture a stable splitting of ΛX for these spaces. We have later proved this splitting by unrelated methods.

The rest of the introduction is a more detailed description of our method. Consider the cohomology $H^*(X; \mathbb{F}_p)$ as given. The purpose of this paper is to study the cohomology of the free-loop space and of its homotopy orbit space.

In some cases, it is relatively easy to compute this cohomology. For instance, suppose that X is an Eilenberg–MacLane space. Then there is a homotopy splitting $\Lambda X \simeq X \times \Omega X$. The space ΩX is also a Eilenberg–MacLane space, so that the cohomology of ΛX is known.

The cohomology of the homotopy orbits $\Lambda X_{h\mathbb{T}}$ is more difficult to compute. However, this is achieved in [3,20].

The main idea of the present paper is to use these computations to study the case of a general X. In essence, this application is done using a Postnikov decomposition of X. From our point of view, the simplest case is when X is a product of Eilenberg–MacLane spaces, and correspondingly, the more k-invariants a space X has, the more complicated it appears. In particular, the spheres are very complicated spaces for this approach.

Formally, we will study two spectral sequences converging towards the cohomology groups $H^*(\Lambda X; \mathbb{F}_p)$ and $H^*(\Lambda X_{h\mathbb{T}}; \mathbb{F}_p)$. Both spectral sequences have origin in the Bousfield homology spectral sequence [5].

This is a remarkable spectral sequence that under fortunate circumstances converges to the homology of the total space of a cosimplicial space.

Let X be a simply connected space. We re-write its Postnikov tower as a cosimplicial space, whose total space is the p-completion of X. This cosimplicial space is the *cosimplicial resolution* $\mathbf{R}X$ of X with $R = \mathbb{F}_p$. Given this, we can form two cosimplicial spaces $\Lambda\mathbf{R}X$ and $(\Lambda\mathbf{R}X)_{h\mathbb{T}}$ by applying the functors $\Lambda(-)$ and $\Lambda(-)_{h\mathbb{T}}$ in each codegree. The total space of these new cosimplicial spaces are the completions of ΛX, respectively $(\Lambda X)_{h\mathbb{T}}$. These cosimplicial spaces have associated Bousfield homology spectral sequences $\{\hat{E}^r\}$ and $\{E^r\}$, respectively.

For 1-connected X it is well known that $\{\hat{E}^r\}$ converges strongly towards $H_*(\Lambda X; \mathbb{F}_p)$. We show that $\{E^r\}$ converges strongly towards $H_*(\Lambda X_{h\mathbb{T}}; \mathbb{F}_p)$ under the additional assumption that $H_*(X; \mathbb{F}_p)$ is of finite type.

For the dual cohomology spectral sequences, $\{\hat{E}_r\}$ and $\{E_r\}$, we give an interpretation of the E_2 page. The idea is that the E_1 page are given by the cohomology of the respective functors (from spaces to spaces) applied to the Eilenberg–MacLane spaces. This cohomology can, according to [3,19,20] be written as certain functors $\overline{\Omega}$, respectively, ℓ (from algebras with a certain extra structure to algebras), applied to the cohomology of the Eilenberg–MacLane spaces.

This means that the E_2 page is the homology of a chain complex, where the chains are given by these functors applied to the cohomology of Eilenberg–MacLane spaces. Since the cohomology of an Eilenberg–MacLane space turns out to be a free object, we can compute the E_2 pages as derived functors.

To be precise, they are the non-Abelian derived functor of $\overline{\Omega}$ applied to $H^*(X; \mathbb{F}_p)$, respectively, the non-Abelian derived functor of ℓ applied to $H^*(X; \mathbb{F}_p)$. When $H^*(X; \mathbb{F}_p)$ is a polynomial algebra the higher derived functors vanish so the spectral sequences collapse at the E_2 page.

So far, the results are of a theoretical nature. As a concrete example, we finally study the case $X = S^n$ and $p = 2$. We develop homological algebra sufficient for computing the relevant E_2 pages.

For these spaces, there are other methods for computing $H^*(\Lambda X; \mathbb{F}_p)$ and $H^*(\Lambda X_{h\mathbb{T}}; \mathbb{F}_p)$.

Comparing our E_2 pages with these results, we show that for $X = S^n$ with $n \geqslant 2$ and $p = 2$ the spectral sequences collapse at the E_2 pages.

We emphasize that this collapsing is not something to be expected a priori. Since spheres have complicated Postnikov systems, from the point of view of our spectral sequences, one would naively expect that these spectral sequence could have many non-trivial differentials. So maybe the collapsing happens for a larger class of spaces?

Finally, we want to thank the referee of Bökstedt and Ottosen [3] for suggesting that we look at the Bousfield spectral sequence in this connection.

2. Cosimplicial spaces with group actions

In this section, the category of simplicial sets is denoted $\mathscr{S}$ and the category of cosimplicial spaces $c\mathscr{S}$. For $A, B \in \mathscr{S}$ we let $\mathrm{map}(A, B) = B^A$ denote the simplicial mapping space. We write $\mathbf{c}A$ for the constant cosimplicial space with $(\mathbf{c}A)^n = A$ for each n.

The category $c\mathscr{S}$ is a model category with weak equivalences, cofibrations and fibrations as described in [7, Chapter X, Section 4]. The fibrations are here defined in terms of matching spaces. By this definition it is clear that if $f : A \to B$ is a fibration in $\mathscr{S}$ then $\mathbf{c}(f) : \mathbf{c}A \to \mathbf{c}B$ is a fibration in $c\mathscr{S}$.

The category $c\mathscr{S}$ is in fact a simplicial model category in the sense of Quillen [21] with $\mathbf{X} \otimes K \in c\mathscr{S}$, $\mathbf{X}^K \in c\mathscr{S}$ and $\mathrm{Map}(\mathbf{X}, \mathbf{Y}) \in \mathscr{S}$ defined as follows for $K \in \mathscr{S}$ and $\mathbf{X}, \mathbf{Y} \in c\mathscr{S}$:

$$(\mathbf{X} \otimes K)(\alpha) = \mathbf{X}(\alpha) \times K,$$
$$(\mathbf{X}^K)(\alpha) = \mathbf{X}(\alpha)^K,$$
$$\mathrm{Map}(\mathbf{X}, \mathbf{Y})_n = \mathrm{Hom}_{c\mathscr{S}}(\mathbf{X} \otimes \Delta^n, \mathbf{Y}),$$

where α is a morphism in the simplicial category and $\Delta^n = \Delta[n] \in \mathscr{S}$ denotes the standard n-simplex. In case K is a simplicial group, this notation potentially clashes with the usual notation for fixed points. In this paper, we are not going to consider fixed points.

Let Δ be the cosimplicial space which in codegree n equals Δ^n. We write $\Delta^{[m]}$ for the simplicial m-skeleton and put $\Delta^{[\infty]} = \Delta$. By Bousfield and Kan [7, Chapter X, Section 4.3] we have that $\Delta^{[m]}$ is a cofibrant cosimplicial space for each $0 \leqslant m \leqslant \infty$.

The total space of a cosimplicial space $\mathbf{X}$ is defined as $\operatorname{Tot} \mathbf{X} = \operatorname{Map}(\Delta, \mathbf{X})$. If $\mathbf{X}$ is not fibrant, the total space might not give you the "right" homotopy type. In this case, we have to choose a fibrant replacement $\mathbf{Z}$ of $\mathbf{X}$, that is a weekly equivalent, fibrant cosimplicial space, and define $\overline{\operatorname{Tot}}\, \mathbf{X} = \operatorname{Tot} \mathbf{Z}$.

When the cosimplicial space has a group action one can choose an equivariant fibrant replacement in the following sense:

Lemma 2.1. *Let G be a simplicial group and $\mathbf{X}$ a cosimplicial G-space. Assume that $\mathbf{X}^n$ is a fibrant simplicial set for each $n \geqslant 0$. Then there is a cosimplicial G-space $E(\mathbf{X})$ such that both $E(\mathbf{X})$ and $E(\mathbf{X})/G$ are fibrant cosimplicial spaces and such that the following diagram commutes*:

$$\begin{array}{ccc} EG \times \mathbf{X} & \xrightarrow{\sim} & E(\mathbf{X}) \\ \downarrow & & \downarrow \\ EG \times_G \mathbf{X} & \xrightarrow{\sim} & E(\mathbf{X})/G. \end{array} \tag{1}$$

Here the vertical maps are the obvious quotient maps, and the horizontal maps are weak equivalences. The map $E(\mathbf{X}) \to E(\mathbf{X})/G$ is the pullback of the principal G-fibration $\mathbf{c}EG \to \mathbf{c}BG$ over a fibration $E(\mathbf{X})/G \to \mathbf{c}BG$.

Proof. By the model category properties we can factor the projection map $EG \times_G \mathbf{X} \to \mathbf{c}BG$ as a composite $p \circ i$ where $i : EG \times_G \mathbf{X} \to \mathbf{Y}$ is a cofibration which is simultaneously a weak equivalence, and $p : \mathbf{Y} \to \mathbf{c}BG$ is a fibration. BG is a fibrant space by Goerss and Jardine [11, Lemma I.3.5.] so $\mathbf{c}BG$ is a fibrant cosimplicial space. Thus $\mathbf{Y}$ is fibrant.

We form the codegree wise pullback of $\pi : \mathbf{c}EG \to \mathbf{c}BG$ over p.

$$\begin{array}{ccccc} E(\mathbf{X}) & = & E(\mathbf{X}) & \xrightarrow{\bar{p}} & \mathbf{c}EG \\ \downarrow & & \downarrow{\scriptstyle \pi^p} & & \downarrow{\scriptstyle \pi} \\ E(\mathbf{X})/G & \xrightarrow{\cong} & \mathbf{Y} & \xrightarrow{p} & \mathbf{c}BG. \end{array}$$

The principal G-action (in the sense of May [16]) of G on EG gives a principal G-action on $E(\mathbf{X})^n$ for each n and an isomorphism of cosimplicial spaces $E(\mathbf{X})/G \cong \mathbf{Y}$ as written in the diagram. By Bousfield [5, Lemma 7.1] it follows that π^p is a fibration so $E(\mathbf{X})$ is fibrant.

By the pullback property we can lift the map i to a map $EG \times \mathbf{X} \to E(\mathbf{X})$. This constructs the missing map in the statement of the lemma. In each codegree (1) is a map of fibrations over BG and we conclude that the lifting is also a weak equivalence. □

Theorem 2.2. *Let $\mathbf{X}$ be a fibrant cosimplicial space and G a simplicial group. Then $\mathbf{X}^G$ is a cosimplicial G-space and we can form its equivariant fibrant replacement $E(\mathbf{X}^G)$. There is a natural map of fibrations of simplicial sets for each m with $0 \leqslant m \leqslant \infty$:*

$$\begin{array}{ccccc}
(\mathrm{Tot}_m \mathbf{X})^G & \longrightarrow & EG \times_G (\mathrm{Tot}_m \mathbf{X})^G & \longrightarrow & BG \\
\downarrow \sim & & \downarrow \sim & & \downarrow \cong \\
\mathrm{Tot}_m(E(\mathbf{X}^G)) & \longrightarrow & \mathrm{Tot}_m(E(\mathbf{X}^G)/G) & \longrightarrow & \mathrm{Tot}_m(\mathbf{c}BG).
\end{array}$$

The first and middle vertical maps are weak equivalences and the right vertical map is an isomorphisms of simplicial sets.

Proof. Since $\mathbf{X}$ is fibrant each $\mathbf{X}^n$ is fibrant such that $(\mathbf{X}^G)^n = (\mathbf{X}^n)^G$ is fibrant by May [16, Theorem 6.9.]. Hence we can form $E(\mathbf{X}^G)$.

By May [16, Definition 20.3 and Theorem 20.5], we have that the top vertical line in the diagram is a fiber bundle. By Bousfield and Kan [7, Chapter X, Section 5] SM7 and the fact that $\Delta^{[m]} \in c\mathscr{S}$ is cofibrant we see that if $p : \mathbf{A} \to \mathbf{B}$ is a fibration in $c\mathscr{S}$ then $\mathrm{Tot}_m(p) : \mathrm{Tot}_m \mathbf{A} \to \mathrm{Tot}_m \mathbf{B}$ is a fibration in $\mathscr{S}$. In particular $\mathrm{Tot}_m \mathbf{X}$ is fibrant since $\mathbf{X}$ is fibrant and by May [16, Theorem 6.9] we have that $(\mathrm{Tot}_m \mathbf{X})^G$ is fibrant. Thus the top vertical line is a Kan fiber bundle and hence a fibration by May [16, Lemma 11.9]. The lower vertical line is Tot_m of a fibration and hence a fibration.

There is a commutative diagram as follows:

$$\begin{array}{ccccc}
(\mathrm{Tot}_m \mathbf{X})^G & \longrightarrow & EG \times_G (\mathrm{Tot}_m \mathbf{X})^G & \longrightarrow & BG \\
\downarrow \cong & & \downarrow f_m & & \downarrow \cong \\
\mathrm{Tot}_m(\mathbf{X}^G) & \longrightarrow & \mathrm{Tot}_m(EG \times_G \mathbf{X}^G) & \longrightarrow & \mathrm{Tot}_m(\mathbf{c}BG) \\
\downarrow \sim & & \downarrow & & \| \\
\mathrm{Tot}_m(E(\mathbf{X}^G)) & \longrightarrow & \mathrm{Tot}_m(E(\mathbf{X}^G)/G) & \longrightarrow & \mathrm{Tot}_m(\mathbf{c}BG).
\end{array}$$

The isomorphism $(\mathrm{Tot}_m \mathbf{X})^G \cong \mathrm{Tot}_m(\mathbf{X}^G)$ is one of the axiomatic isomorphisms in a simplicial model category. We examine it closer in order to define f_m. A cosimplicial space is a diagram in $\mathscr{S}$ and the axiomatic isomorphism comes from the corresponding isomorphism in the simplicial model category $\mathscr{S}$. For $A, B, C \in \mathscr{S}$ this isomorphism is the composite

$$F : (A^B)^C \cong A^{B \times C} \cong A^{C \times B} \cong (A^C)^B.$$

The following commutative diagram shows that F is equivariant with respect to actions of the monoid C^C:

$$\begin{array}{ccccc}
C^C \times (A^B)^C & & \xrightarrow{\quad\quad\circ\quad\quad} & & (A^B)^C \\
\downarrow & & & & \downarrow \\
C^C \times A^{B\times C} & \xrightarrow{i_2\times 1} & (B\times C)^{B\times C} \times A^{B\times C} & \xrightarrow{\circ} & A^{B\times C} \\
\downarrow & & & & \downarrow \\
C^C \times A^{C\times B} & \xrightarrow{i_1\times 1} & (C\times B)^{C\times B} \times A^{C\times B} & \xrightarrow{\circ} & A^{C\times B} \\
\downarrow & & & & \downarrow \\
C^C \times (A^C)^B & \xrightarrow{i\times 1} & (C^C)^B \times (A^C)^B & \xrightarrow{(\circ)^B} & (A^C)^B.
\end{array}$$

For $Z \in \mathscr{S}$ the action of G on the mapping space Z^G is defined by

$$G \times Z^G \xrightarrow{\quad ad(\mu)\times 1 \quad} G^G \times Z^G \xrightarrow{\circ} Z^G,$$

where $ad(\mu)$ denotes the adjoint of the product $\mu : G \times G \to G$. So taking $C = G$ in the above we see that F is G-equivariant such that we have a map

$$1\times_G F : EG\times_G (A^B)^G \to EG\times_G (A^G)^B.$$

The composite

$$EG \times (A^G)^B \xrightarrow{i\times 1} (EG \times A^G)^B \longrightarrow (EG\times_G A^G)^B$$

factors through $EG\times_G (A^G)^B$ and we compose with $1\times_G F$ to get a map

$$EG\times_G (A^B)^G \to (EG\times_G A^G)^B.$$

The morphism f_m in the theorem is codegree wise given by this map.

The lower part of the diagram is induced by (1). The functor $(-)^K : c\mathscr{S} \to c\mathscr{S}$ where $K \in \mathscr{S}$ preserves fibrations as one sees from the right lifting property by taking adjoints. Hence $\mathbf{X}^G$ is fibrant since $\mathbf{X}$ is fibrant. By Bousfield and Kan [7, Chapter X, Section 5.2] we get a weak equivalence when applying Tot_m to a weak equivalence between fibrant cosimplicial spaces. Thus, the left vertical map is a weak equivalence. The result follows. □

3. Bousfield homology spectral sequences

Let $\mathbf{X}$ be a fibrant cosimplicial space and let A be an Abelian group. In [5], Bousfield constructs a spectral sequence $\{E^r(\mathbf{X}; A)\}$ with the homology of the total space $H_*(\mathrm{Tot}\,\mathbf{X}; A)$ as expected target.

The precise convergence statement is as follows. Recall that there is a tower of fibrations

$$\cdots \to \mathrm{Tot}_m \mathbf{X} \to \mathrm{Tot}_{m-1} \mathbf{X} \to \cdots \to \mathrm{Tot}_0 \mathbf{X}$$

with inverse limit Tot $\mathbf{X}$. Hence for each $n \geqslant 0$ there is a tower map

$$P_n(\mathbf{X}) : \{H_n(\operatorname{Tot}\mathbf{X}; A)\}_{m\geqslant 0} \to \{H_n(\operatorname{Tot}_m \mathbf{X}; A)\}_{m\geqslant 0},$$

where the domain tower is constant. Let $A \otimes \mathbf{X}$ denote the cosimplicial simplicial Abelian group with $(A \otimes \mathbf{X})^m_t = A \otimes \mathbf{X}^m_t$ where $A \otimes S = \oplus_{x \in S} A$ for a set S. Bousfield forms the double normalized complex and let $T(A \otimes \mathbf{X})$ denote its total complex. It is filtered by subcomplexes $F^m T(A \otimes \mathbf{X})$ and the quotient complex $T(A \otimes \mathbf{X})/F^{m+1}T(A \otimes \mathbf{X})$ is denoted $T_m(A \otimes \mathbf{X})$. A comparison map is defined

$$\Phi_n(\mathbf{X}) : \{H_n(\operatorname{Tot}_m \mathbf{X}; A)\}_{m\geqslant 0} \to \{H_n T_m(A \otimes \mathbf{X})\}_{m\geqslant 0}$$

and the following result is proved:

Lemma 3.1. *$\{E^r(\mathbf{X}; A)\}$ converges strongly to $H_*(\operatorname{Tot}\mathbf{X}; A)$ if and only if the tower map $\Phi_n(\mathbf{X}) \circ P_n(\mathbf{X})$ is a pro-isomorphism for each n.*

If $\Phi_n(\mathbf{X})$ is a pro-isomorphism for each n then $\mathbf{X}$ is called an A-pro-convergent cosimplicial space and $\{E^r(\mathbf{X}; A)\}$ is called pro-convergent.

We are interested in two special cases of this spectral sequence. Let $R = \mathbb{F}_p$ be the field on p elements where p is a fixed prime. For a space X we let $\mathbf{R}X$ denote the cosimplicial resolution of X in the sense of Bousfield and Kan [7]. Note that $(\mathbf{R}X)^n = R^{n+1}X$. The free-loop space on X is by definition the simplicial mapping space $\Lambda X = \operatorname{map}(\mathbb{T}, X)$ where we take $\mathbb{T} = B\mathbb{Z}$. By applying Λ codegree wise we get a cosimplicial space $\Lambda\mathbf{R}X$. We can also form the $\mathbb{T}$ homotopy orbit space codegree wise and get the cosimplicial space $(\Lambda\mathbf{R}X)_{h\mathbb{T}}$. We are interested in the Bousfield homology spectral sequences for these two spaces. As a corollary of Bousfield [6, Proposition 9.7] we have

Proposition 3.2. *If X is a 1-connected and fibrant space then $P_n(\Lambda\mathbf{R}X)$ and $\Phi_n(\Lambda\mathbf{R}X)$ are pro-isomorphisms for each n and the spectral sequence $\{E^r(\Lambda\mathbf{R}X; \mathbb{F}_p)\}$ converges strongly to $H_*(\Lambda(X^\wedge_p); \mathbb{F}_p) \cong H_*(\Lambda X; \mathbb{F}_p)$.*

4. Strong convergence

In this section, we discuss convergence of the Bousfield homology spectral sequence associated with $(\Lambda\mathbf{R}X)_{h\mathbb{T}}$ where $R = \mathbb{F}_p$, the field on p elements. We use $\mathbb{F}_p$ coefficients everywhere unless stated otherwise.

Proposition 4.1. *If X is a 1-connected space then ΛX and $\Lambda X_{h\mathbb{T}}$ are nilpotent spaces. In fact we have $\pi_1(\Lambda X)$—respectively, $\pi_1(\Lambda X_{h\mathbb{T}})$—central series as follows for each $i \geqslant 1$:*

$$\pi_i(\Lambda X) \supseteq \pi_i(\Omega X) \supseteq 0, \tag{2}$$

$$\pi_i(\Lambda X_{h\mathbb{T}}) \supseteq \pi_i(\Lambda X) \supseteq \pi_i(\Omega X) \supseteq 0. \tag{3}$$

Proof. (2) The fibration $\Omega X \to \Lambda X \to X$ splits by the constant loop inclusion $X \to \Lambda X$.So we have $\pi_i(\Lambda X) \cong \pi_i(\Omega X) \oplus \pi_i(X)$ for $i \geqslant 1$. Since the action of the fundamental group is natural there is a commutative diagram

$$\begin{array}{ccc}
\pi_1(\Omega X) \times \pi_i(\Omega X) & \longrightarrow & \pi_i(\Omega X) \\
\downarrow & & \downarrow \\
\pi_1(\Lambda X) \times \pi_i(\Lambda X) & \longrightarrow & \pi_i(\Lambda X) \\
\downarrow & & \downarrow \\
\pi_1(X) \times \pi_i(X) & \longrightarrow & \pi_i(X).
\end{array}$$

We have $\pi_1(\Lambda X) \cong \pi_1(\Omega X)$ since X is simply connected. Further, $\pi_1(\Omega X)$ acts trivially on $\pi_i(\Omega X)$ since ΩX is an H-space. From the upper square we see that the filtration (2) is $\pi_1(\Lambda X)$-stable and that the action on $\pi_i(\Omega X)$ is trivial. Since $\pi_1(X) = 0$ the lower square shows that the action on the quotient $\pi_i(\Lambda X)/\pi_i(\Omega X)$ is trivial.

(3) The fibration $\Lambda X \to \Lambda X_{h\mathbb{T}} \to B\mathbb{T}$ splits by a map constructed from a constant loop.So for $i \geqslant 1$ we have $\pi_i(\Lambda X_{h\mathbb{T}}) \cong \pi_i(\Lambda X) \oplus \pi_i(B\mathbb{T})$. Especially $\pi_1(\Lambda X_{h\mathbb{T}}) \cong \pi_1(\Lambda X)$. By naturality there is a commutative diagram

$$\begin{array}{ccc}
\pi_1(\Lambda X) \times \pi_i(\Lambda X) & \longrightarrow & \pi_i(\Lambda X) \\
\downarrow & & \downarrow \\
\pi_1(\Lambda X_{h\mathbb{T}}) \times \pi_i(\Lambda X_{h\mathbb{T}}) & \longrightarrow & \pi_i(\Lambda X_{h\mathbb{T}}) \\
\downarrow & & \downarrow \\
\pi_1(B\mathbb{T}) \times \pi_i(B\mathbb{T}) & \longrightarrow & \pi_i(B\mathbb{T}).
\end{array}$$

From the upper square we see that the inclusion $\pi_i(\Lambda X_{h\mathbb{T}}) \supseteq \pi_i(\Lambda X)$ is $\pi_1(\Lambda X_{h\mathbb{T}})$-stable. The lower square shows that the action on the quotient $\pi_i(\Lambda X_{h\mathbb{T}})/\pi_i(\Lambda X)$ is trivial. The rest of the sequence (3) has the desired properties since (2) is a $\pi_1(\Lambda X)$-central series. □

Proposition 4.2. *If X is a* 1*-connected space then the cosimplicial space* $E(\Lambda \mathbf{R} X)/\mathbb{T}$ *is R-pro-convergent.*

Proof. This is a consequence of Bousfield [5, Section 3.3]. Via the weak equivalences from Lemma 2.1 we can use the filtrations from Proposition 4.1 in each codegree. Then the quotients are $\pi_i(\mathbf{c}B\mathbb{T})$, $\pi_i(\mathbf{R}X)$ and $\pi_{i+1}(\mathbf{R}X)$. Hence, it suffices to show that when $n \leqslant 0$ the following holds for all $m \geqslant 0$:

$$\pi^m \pi_{m+n}(\mathbf{c}B\mathbb{T}) = 0, \quad \pi^m \pi_{m+n}(\mathbf{R}X) = 0, \quad \pi^m \pi_{m+n+1}(\mathbf{R}X) = 0. \tag{4}$$

Clearly $\pi^m \pi_{m+n}(\mathbf{c}B\mathbb{T}) = 0$ unless $m+n=2$ and $\pi^{2-n}\pi_2(\mathbf{c}B\mathbb{T}) = 0$ since the differentials in the complex $\pi_2(\mathbf{c}B\mathbb{T})$ are alternating zeros and ones.

By the proof of 6.1 in [7, Chapter I, and Proposition 6.3 in Chapter X], the following holds for any space Y: If $\widetilde{H}_i(Y; R) = 0$ for $i \leqslant k$ then $\pi^j \pi_i(\mathbf{R}Y) = 0$ for $i \leqslant k + j$. So the last two groups in (4) are also zero. □

Lemma 4.3. *Let X be a 1-connected space with H_*X of finite type. Then R_sX is 1-connected and H_*R_sX is of finite type for each $0 \leqslant s < \infty$.*

Proof. By Bousfield and Kan [7, Chapter I, Section 6.1] we have that R_sX is 1-connected for each s. Recall that $R(Y)$ is weakly equivalent to $\prod_{n=0}^{\infty} K(\tilde{H}_n(Y), n)$ for any space Y. So if H_*Y is of finite type then $H_*R(Y)$ is also of finite type and $\pi_i R(Y) = \tilde{H}_i Y$ is finite for each i. Hence $\pi_i((\mathbf{R}X)^m)$ is finite for each i, m. From Shipley [22, Lemma 2.6] we see that $\pi_i(R_sX)$ is finite for each i, s. By the Postnikov tower for R_sX we conclude that H_*R_sX is of finite type for each s. □

Lemma 4.4. *Let $\cdots \to C_*(2) \to C_*(1) \to C_*(0)$ be a sequence of maps of chain complexes. If for all n and m the group $C_n(m)$ is finite, then there is an isomorphism $H_n(\lim C_*(m)) \cong \lim H_n(C_*(m))$ for all n.*

Proof. This is a consequence of the $\lim^1$-sequence which can be found in, e.g. [15, Appendix A.5.] □

Proposition 4.5. *Let G be a simplicial group such that $H_n(BG)$ is finite for all n. Let $\{Z_m\}$ be a tower of G-spaces and put $Z_\infty = \lim Z_m$. Assume that $\{H_*(Z_\infty)\}_{m\geqslant 0} \to \{H_*(Z_m)\}_{m\geqslant 0}$ is a pro-isomorphism and that $H_n(Z_m)$ is finite for all integers n, m. Then $\{H_*((Z_\infty)_{hG})\}_{m\geqslant 0} \to \{H_*((Z_m)_{hG})\}_{m\geqslant 0}$ is also a pro-isomorphism.*

Proof. We have Leray–Serre spectral sequences for $0 \leqslant m \leqslant \infty$ as follows:

$$E^2(m) = H_*(BG; H_*(Z_m)) \Rightarrow H_*((Z_m)_{hG}).$$

The tower map $\{E^2_{i,j}(\infty)\}_{m\geqslant 0} \to \{E^2_{i,j}(m)\}_{m\geqslant 0}$ is a pro-isomorphism for all i and j by the pro-isomorphism in the assumption, so $E^2_{i,j}(\infty) \cong \lim E^2_{i,j}(m)$. By the assumptions on the homology of BG and Z_m, the groups $E^2_{i,j}(m)$ with $m < \infty$ are all finite so by Lemma 4.4 we have $E^3_{i,j}(\infty) \cong \lim E^3_{i,j}(m)$. By induction $E^r_{i,j}(\infty) = \lim E^r_{i,j}(m)$ for each r and since we have only finite filtrations $E^\infty_{i,j}(\infty) \cong \lim E^\infty_{i,j}(m)$. Since $E^\infty_{i,j}(m)$ is finite for all i, j, m it follows that $\{E^\infty(\infty)\}_{m\geqslant 0} \to \{E^\infty(m)\}_{m\geqslant 0}$ is a pro-isomorphism. The result follows by the five lemma [7, Chapter III, Section 2.7]. □

Theorem 4.6. *If X is a 1-connected fibrant space with $H_*(X; \mathbb{F}_p)$ of finite type, then the Bousfield spectral sequence $\{E^r(\Lambda\mathbf{R}X_{h\mathbb{T}}; \mathbb{F}_p)\}$ converges strongly to $H_*(\Lambda(X^\wedge_p)_{h\mathbb{T}}; \mathbb{F}_p) \cong H_*(\Lambda X_{h\mathbb{T}}; \mathbb{F}_p)$.*

Proof. Let $\mathbf{Y} = \Lambda\mathbf{R}X$. The spectral sequence abuts to the homology of the total space of a fibrant replacement of $\mathbf{Y}_{h\mathbb{T}}$. We choose the fibrant replacement $E(\mathbf{Y})/\mathbb{T}$ from Lemma 2.1. The total space of this fibrant replacement is weakly equivalent to $\Lambda(X^\wedge_p)_{h\mathbb{T}}$ by Theorem 2.2. Thus, the spectral sequence converges to the stated result if it converges. (A Leray–Serre spectral sequence argument shows that we can remove the p-completion inside the homology group.)

We have shown in Proposition 4.2 that the spectral sequence is pro-convergent. Hence, it suffices to show that $P_n(E(\mathbf{Y})/\mathbb{T})$ or equivalently $P_n(\mathbf{Y}_{h\mathbb{T}})$ is a pro-isomorphism. By the Eilenberg–Moore spectral sequence and Lemma 4.3 we see that $H_*(\mathrm{Tot}_s\,\mathbf{Y}) \cong H_*(\Lambda R_sX)$ is of finite type for each $0 \leqslant s < \infty$. By Propositions 4.5 and 3.2 the result follows. □

We now change to cohomology. The dual of Proposition 3.2 and Theorem 4.6 is as follows:

Theorem 4.7. *If X is a 1-connected and fibrant space with H_*X of finite type then we have strongly convergent Bousfield cohomology spectral sequences*

$$\hat{E}_r \Rightarrow H^*(\Lambda X), \qquad \hat{E}_2^{-m,t} = (\pi_m H^*(\Lambda \mathbf{R} X))^t,$$
$$E_r \Rightarrow H^*((\Lambda X)_{h\mathbb{T}}), \quad E_2^{-m,t} = (\pi_m H^*((\Lambda \mathbf{R} X)_{h\mathbb{T}}))^t.$$

We are going to give a description of the E_2-terms as certain non-Abelian derived functors evaluated at H^*X. In the next section we set up categories relevant for this purpose.

5. The category $\mathscr{F}$ and the simplicial model category $s\mathscr{F}$

For a fixed prime p we let $\mathscr{A}$ denote the mod p Steenrod algebra and $\mathscr{K}$ the category of unstable $\mathscr{A}$-algebras. The category of non-negatively graded unital $\mathbb{F}_p$-algebras with the property that A^0 is a p-Boolean algebra (i.e. $x = x^p$ for all $x \in A^0$) is denoted $\mathscr{A}lg$. In [19,20], we defined a category $\mathscr{F}$ with forgetful functors $\mathscr{K} \to \mathscr{F} \to \mathscr{A}lg$ as follows:

Definition 5.1. An object in $\mathscr{F}$ consists of an object A in $\mathscr{A}lg$ which is equipped with an $\mathbb{F}_p$-linear map $\lambda : A \to A$ with the following properties:

- $|\lambda x| = p(|x| - 1) + 1$ for all $x \in A$.
- $\lambda x = x$ when $|x| = 1$ and if p is odd and $|x|$ is even then $\lambda x = 0$.
- $\lambda(xy) = \lambda(x)y^p + x^p\lambda(y)$ for all $x, y \in A$.

Furthermore A is equipped with an $\mathbb{F}_p$-linear map $\beta : A \to A$ with the following properties:

- $|\beta x| = |x| + 1$ for all $x \in A$.
- $\beta \circ \beta = 0$ and if $|x| = 0$ then $\beta x = 0$.
- $\beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y)$ for all $x, y \in A$.

If $p = 2$ we require that $\beta = 0$. A morphism $f : A \to A'$ in $\mathscr{F}$ is an algebra homomorphism such that $f(\lambda x) = \lambda' f(x)$ and $f(\beta x) = \beta' f(x)$.

Remark 5.2. For an object $K \in \mathscr{K}$ the map $\lambda : K \to K$ is defined by $\lambda x = Sq^{|x|-1}x$ when $p = 2$ and $\lambda x = P^{(|x|-1)/2}x$ when p is odd and $|x|$ is odd. The map β is the Bockstein operation when p is odd.

There is an obvious product on $\mathscr{F}$. There is also a coproduct. For two objects A and A' in $\mathscr{F}$ the coproduct $A \otimes A'$ is the tensor product of the underlying objects in $\mathscr{A}lg$ equipped with maps $\lambda * \lambda'$ and $\beta * \beta'$ as follows:

$$\lambda * \lambda'(x \otimes y) = \lambda(x) \otimes y^p + x^p \otimes \lambda'(y),$$

$$\beta * \beta'(x \otimes y) = \beta(x) \otimes y + (-1)^{|x|}x \otimes \beta'(y).$$

In appendices in [19,20] we showed that $\mathscr{F}$ is complete and cocomplete. It is well known that $\mathscr{K}$ and $\mathscr{A}lg$ also possess these properties.

In the following $\mathscr{R}$ denotes any one of the categories $\mathscr{K}$, $\mathscr{F}$ or $\mathscr{A}lg$. Let $n\mathbb{F}_p$ denote the category of non-negatively graded $\mathbb{F}_p$-vector spaces. The free functor $S_{\mathscr{R}} : n\mathbb{F}_p \to \mathscr{R}$ is by definition the left adjoint of the forgetful functor $\mathscr{R} \to n\mathbb{F}_p$. If X is a non-negatively graded set we put $S_{\mathscr{R}}(X) = S_{\mathscr{R}}(\mathbb{F}_p \otimes X)$ where $\mathbb{F}_p \otimes X$ is the free graded $\mathbb{F}_p$-vector space with basis X. In particular we have free objects $S_{\mathscr{R}}(x_n)$ on one generator x_n of degree n.

Remark 5.3. Note that $S_{\mathscr{F}}(V) = S_{\mathscr{A}lg}(\bar{V})$ where

$$\bar{V} = V \oplus \bigoplus_{i \geqslant 1} \lambda^i V^{* \geqslant 2}, \qquad p = 2,$$
$$\bar{V} = V \oplus \beta V^{* \geqslant 1} \oplus \bigoplus_{i \geqslant 1, v \in \{0,1\}} \beta^v \lambda^i (\beta V^{even, * \geqslant 2} \oplus V^{odd, * \geqslant 2}), \quad p > 2.$$

In the following, we use [21, Chapter II, Section 4, Theorem 4] to see that the category $s\mathscr{R}$ of simplicial objects in $\mathscr{R}$ is a simplicial model category. The arguments are standard but we have included them anyhow.

We start by verifying that $\mathscr{R}$ has enough projectives. Recall that a morphism $f : X \to Y$ in a category $\mathscr{D}$ is called an *effective epimorphism* if for any object T and morphism $\alpha : X \to T$ there is a unique $\beta : Y \to T$ with $\beta \circ f = \alpha$ provided α satisfies the necessary condition that $\alpha \circ u = \alpha \circ v$ whenever $u, v : S \rightrightarrows X$ are maps such that $f \circ u = f \circ v$ [21, Chapter II, Section 4, proof of Proposition 2].

Proposition 5.4. *Let f be an effective epimorphism in a category $\mathscr{D}$. Then f is an epimorphism. Furthermore if f can be factored as $f = i \circ p$ where i is a monomorphism then i is an isomorphism.*

Proof. Assume that f is an effective epimorphism. Let r, s be two parallel arrows such that $r \circ f = s \circ f$. Then for $\alpha = r \circ f$ we have $\beta \circ f = \alpha$ both for $\beta = r$ and $\beta = s$. So by uniqueness $r = s$. Thus f is an epimorphism.

Assume that $f = i \circ p$ where i is a monomorphism. If $f \circ u = f \circ v$ for two parallel arrows u, v then $i \circ p \circ u = i \circ p \circ v$ and $p \circ u = p \circ v$ since i is a monomorphism. Hence there exists an arrow j such that $p = j \circ f$. Now, $i \circ j \circ f = i \circ p = f$ which implies that $i \circ j = id$ since f is an epimorphism. Furthermore $i \circ j \circ i = id \circ i = i$ which implies that $j \circ i = id$ since i is a monomorphism. □

Proposition 5.5. *A morphism in $\mathscr{R}$ is an effective epimorphism if and only if it is a surjection on underlying graded sets.*

Proof. Any morphism $f : X \to Y$ may be factored as $X \to f(X) \to Y$ where the last map is clearly a monomorphism. So by the previous proposition we see that an effective epimorphism is surjective.

Assume that $f : X \to Y$ is a surjection and let $\beta : X \to T$ be a map which satisfies $\beta \circ u = \beta \circ v$ whenever $f \circ u = f \circ v$. For a given $x \in \ker f$ let $n = |x|$ and define $u, v : S_{\mathscr{R}}(x_n) \rightrightarrows X$ by $u(x_n) = x$ and $v(x_n) = 0$. Then $x \in \ker \beta$ so we have $\ker f \subseteq \ker \beta$. Now $\alpha : Y \to T$ with $\alpha(f(a)) = \beta(a)$ is well defined and has $\alpha \circ f = \beta$. □

Recall that in [21] an object P in a category $\mathscr{D}$ is called *projective* if $\mathrm{Hom}_D(P, -)$ sends any effective epimorphism to an surjection of hom-sets.

Proposition 5.6. *The following statements hold in the category $\mathscr{R}$:*

1. *$S_{\mathscr{R}}(V)$ is projective for any object V in $n\mathbb{F}_p$.*

2. *$\mathscr{R}$ has enough projectives.*
3. $\{S_{\mathscr{R}}(x_n) | n \geqslant 0\}$ *is a set of small projective generators.*

Proof. (1) By taking adjoints and applying the previous proposition we see that $S_{\mathscr{R}}(V)$ is projective. (2) Let $U : \mathscr{R} \to n\mathbb{F}_p$ denote the forgetful functor and let X be an object in $\mathscr{R}$. The adjoint $\eta : S_{\mathscr{R}}(U(X)) \to X$ of $id_{U(X)}$ is surjective and hence an epimorphism. Thus $\mathscr{R}$ has enough projectives.

(3) The object $S_{\mathscr{R}}(x_n)$ is projective by (1). Since $\mathrm{Hom}_{\mathscr{R}}(S_{\mathscr{R}}(x_n), X) = X^n$ we have that $\mathrm{Hom}_{\mathscr{R}}(S_{\mathscr{R}}(x_n), -)$ commutes with filtered colimits so $S_{\mathscr{R}}(x_n)$ is small. Finally, for two different morphisms $f, g : X \rightrightarrows Y$ there exist an $x \in X$ such that $f(x) \neq g(x)$. Hence, the map $S_{\mathscr{R}}(x_n) \to X$ with $x_n \mapsto x$ where $n = |x|$ separates f and g such that we have a set of generators as stated. □

We now turn to the category $s\mathscr{R}$ of simplicial objects in $\mathscr{R}$. The homotopy groups of an object R in $\mathscr{R}$ is defined as the homology $\pi_* R = H_*(R, \partial)$ where ∂ is the differential given by the alternating sums

$$\partial = \sum_{i=0}^{n} (-1)^i d_i : R_n \to R_{n-1}.$$

Especially $\pi_0(R) = R/(d_0 - d_1)R$ and we have a morphism $\varepsilon : R \to \pi_0(R)$ in $s\mathscr{R}$ given by projection where we view $\pi_0(R)$ as a constant simplicial object.

If $f : X \to Y$ is a morphism in $\mathscr{R}$ we can form the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon} & \pi_0 X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_0 f} \\ Y & \xrightarrow{\varepsilon} & \pi_0 Y. \end{array}$$

One says that f is *surjective on components* if the map from X into the pullback $(f, \varepsilon) : X \to Y \times_{\pi_0 Y} \pi_0 X$ is a surjection. Note that if $\pi_0(f)$ is an isomorphism then f is surjective on components if and only if f is surjective.

Proposition 5.7. *There is a simplicial model category structure on $s\mathscr{R}$ as follows*:

- $f : X \to Y$ *is a weak equivalence if* $\pi_* f : \pi_* X \to \pi_* Y$ *is an isomorphism.*
- $f : X \to Y$ *is a fibration if it is surjective on components and an acyclic fibration if it is both a fibration and a weak equivalence.*
- $f : X \to Y$ *is a cofibration if for any commutative diagram*

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & B, \end{array} \qquad (5)$$

where p is an acyclic fibration, there exist a map $Y \to A$ making both triangles commute.

The solution to the arrow diagram (5) *is unique up to simplicial homotopy under X and over B.*

Proof. This is a special case of Quillen [21, Chapter II, Section 4, Theorem 4.]. The uniqueness part follows from [21, Chapter II, Section 2, Proposition 4]. □

Note that the cofibrations are described in an indirect way. The concept of an almost free map make up for this weakness. See [21, Chapter 2, p. 4.11, Remark 4] and the main source [17, Section 3], [18, Section 2] or [10].

Definition 5.8. Let $\widetilde{\Delta}$ denote the subcategory of the simplicial category Δ with objects $[n]=\{0, 1, \ldots, n\}$ for $n \geqslant 0$ and morphisms the order preserving maps which sends 0 to 0. An almost simplicial object in a category $\mathscr{C}$ is a functor from $\widetilde{\Delta}^{op}$ to $\mathscr{C}$.

Definition 5.9. A morphism $f : X \to Y$ in $s\mathscr{R}$ is called *almost free* if there is an almost simplicial subvector space V of Y such that for each $n \geqslant 0$, the natural map $X_n \otimes S_{\mathscr{R}}(V_n) \to Y_n$ is an isomorphism.

Proposition 5.10. (1) *Almost free morphisms are cofibrations in $s\mathscr{R}$.*

(2) *Any morphism $A \to B$ may be factored canonically and functorially as $A \to X \to B$ where the first map is almost free and the second is an acyclic fibration.*

(3) *Any cofibration is a retract for an almost free map.*

Proof. Similar to the one given in [17]. See also [10]. □

Definition 5.11. A simplicial resolution of an object $A \in s\mathscr{R}$ is an acyclic fibration $P \to A$ in $s\mathscr{R}$ with P cofibrant. An almost free resolution of A is an acyclic fibration $Q \to A$ such that $\mathbb{F}_p \to Q$ is almost free.

Note that an almost free resolution is a resolution and that almost free resolutions always exist by the above proposition.

Theorem 5.12. *Let X be a space with mod p homology of finite type. Let $\mathbf{R}X$ be the cosimplicial resolution of X. Then $H^*(\mathbf{R}X)$ is an almost free resolution of H^*X in each of the categories $\mathscr{K}$, $\mathscr{F}$ and $\mathscr{A}lg$.*

Proof. This is a reformulation of well-known results. We use [11, Chapter VII, Example 4.1] as a reference. Let RX denote the simplicial R-module defined by applying the free R-module functor on each simplicial degree. Let $\eta : X \to RX$ be the map defined by $x \mapsto 1x$.

As in [11] one gets a cosimplicial space $\mathbf{R}X$ with $(\mathbf{R}X)^n = R^{n+1}X$ and augmentation $\eta : cX \to \mathbf{R}X$. Note that $\mathbf{R}X$ is a version of the Bousfield–Kan R-resolution of X [7]. The homology $H_*(\eta; R) = \pi_*(R\eta)$ is computed in [11] and taking the dual of this, we find that η^* induces an isomorphism as follows:

$$\pi_s H^*(\mathbf{R}X) \cong \begin{cases} H^*X, & s=0, \\ 0, & s>0. \end{cases}$$

Thus η^* is surjective and hence a fibration. Furthermore η^* is a weak equivalence.

In order to show that $\mathbb{F}_p \to H^*(\mathbf{R}X)$ is almost free, we must find an almost simplicial subvector space V of $H^*(\mathbf{R}X)$ such that $S_{\mathscr{R}}(V_n) \to H^*(\mathbf{R}X)$ is an isomorphism for each $n \geqslant 0$.

As remarked in [11] the cosimplicial maps d^i for $i \geqslant 1$ and s^i for $i \geqslant 0$ for $\mathbf{R}X$ are all morphisms of simplicial R-modules. Thus, it suffices to show that $H^*(\mathbf{R}X)^n$ is a free object in $\mathscr{R}$ for each $n \geqslant 0$.

But it is well known that RX is homotopy equivalent to a product of Eilenberg–MacLane spaces of the type $K(\mathbb{F}_p, m)$. The cohomology of such a product is a free object in $\mathscr{K}$ and hence also in $\mathscr{F}$ and $\mathscr{A}lg$. $\square$

6. Derived functors

In this section $\mathscr{R}$ denotes any of the categories $\mathscr{K}$, $\mathscr{F}$ or $\mathscr{A}lg$. We use the following notation for non-Abelian-derived functors:

Definition 6.1. The homology of an object R in $s\mathscr{R}$ with coefficients in a functor $E : \mathscr{R} \to \mathscr{A}lg$ is defined by

$$H_*(R; E) = \pi_* E(P),$$

where $P \to R$ is a simplicial resolution of R. By the uniqueness statement in Proposition 5.7 this homology theory is well defined and functorial in R.

For an object $R \in \mathscr{R}$ we also write R for the corresponding constant simplicial object in $s\mathscr{R}$. We are mainly interested in $H_*(R; E)$ when $R \in \mathscr{R}$. These homology groups have certain properties which we now describe.

Let E, F and G be functors from $\mathscr{R}$ to $\mathscr{A}lg$ with natural transformations $E \to F \to G$. Let $V : \mathscr{A}lg \to n\mathbb{F}_p$ denote the forgetful functor to graded $\mathbb{F}_p$-vector spaces. If $0 \to VE \to VF \to VG \to 0$ is short exact when evaluated on any free object in $\mathscr{R}$ then we get a long exact sequence

$$\cdots \leftarrow H_i(R; E) \leftarrow H_i(R; F) \leftarrow H_i(R; G) \leftarrow H_{i+1}(R; E) \leftarrow \cdots.$$

The 0th homology group is sometimes given by the following result:

Lemma 6.2. *Define the category $\mathscr{R}'$ as we defined the category $\mathscr{R}$ except that we do no longer require that objects are unital. Let $F : \mathscr{R}' \to \mathscr{A}lg'$ be a functor. Assume that for every surjective morphism $f : A \to B$ in $\mathscr{R}'$ the following two conditions hold*:

1. $F(f) : F(A) \to F(B)$ *is surjective,*
2. $F(\ker f) \to F(A) \to F(B)$ *is exact,*

then $H_0(C; F) \cong F(C)$ for all objects C in $\mathscr{R}$.

Proof. Let $P \to C$ be a simplicial resolution of C. From the normalized chain complex $N_*F(P)$ we see that $H_0(C; F) = F(P_0)/F(d_1)(\ker F(d_0))$.

The maps $d_0, d_1 : P_1 \to P_0$ are surjective by the simplicial identities. Let $i : \ker d_0 \to P_1$ denote the inclusion. By Condition 2 we have that $\ker F(d_0) = F(i)(F(\ker d_0))$. Thus

$$F(d_1)(\ker F(d_0)) = F(d_1) \circ F(i)(F(\ker d_0)).$$

There is a commutative diagram

$$\begin{array}{ccc} \ker d_0 & \xrightarrow{d_1'} & d_1(\ker d_0) \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\ P_1 & \xrightarrow{d_1} & P_0, \end{array}$$

where d_1' denotes the restriction of d_1 and j is the inclusion. By this diagram $F(d_1)\circ F(i) = F(j)\circ F(d_1')$. Furthermore $F(d_1')(F(\ker d_0)) = F(d_1(\ker d_0))$ by Condition 1. So we have

$$F(d_1)(\ker F(d_0)) = F(j) \circ F(d_1')(F(\ker d_0)) = F(j)(F(d_1(\ker d_0)))$$

and $H_0(C; F) = F(P_0)/F(j)(F(d_1(\ker d_0)))$.

Using Conditions 1 and 2 on the projection map $P_0 \to P_0/d_1(\ker d_0)$ we see that $H_0(C; F) \cong F(P_0/d_1(\ker d_0)) \cong F(C)$. □

The following result can sometimes be used to compute derived functors of pushouts. We denote the pushout of a diagram $A' \leftarrow A \rightarrow A''$ in $s\mathscr{R}$ or $\mathscr{R}$ by $A' \otimes_A A''$.

Proposition 6.3. *Let $E : \mathscr{R} \to \mathscr{A}lg$ be a functor.*

(1) *If there is a natural isomorphism $E(A' \otimes A'') \cong E(A') \otimes E(A'')$ for objects A', A'' in $\mathscr{R}$ then there is an isomorphism*

$$H_*(B' \otimes B''; E) \cong H_*(B'; E) \otimes H_*(B''; E) \text{ for } B', B'' \in \mathscr{R}.$$

(2) *Assume that there is a natural isomorphism*

$$E(A' \otimes_A A'') \cong E(A') \otimes_{E(A)} E(A'')$$

for diagrams $A' \leftarrow A \rightarrow A''$ in $\mathscr{R}$. Assume further that $B' \leftarrow B \rightarrow B''$ is a diagram in $\mathscr{R}$ such that $\mathrm{Tor}_i^B(B', B'') = 0$ for $i > 0$. Then there is a first quadrant spectral sequence as follows:

$$E^2_{i,j} = \mathrm{Tor}_i^{H_*(B;E)}(H_*(B'; E), H_*(B''; E))_j \Rightarrow H_{i+j}(B' \otimes_B B''; E).$$

Proof. Let $P \to B$ be a simplicial resolution of B. By the factorization axiom we get a diagram

$$\begin{array}{ccccc} P' & \leftarrowtail & P & \rightarrowtail & P'' \\ \downarrow\!\!\!\downarrow\sim & & \downarrow\!\!\!\downarrow\sim & & \downarrow\!\!\!\downarrow\sim \\ B' & \longleftarrow & B & \longrightarrow & B'', \end{array}$$

where the vertical maps are acyclic fibrations and the upper horizontal maps are cofibrations as indicated. Since $\mathbb{F}_p \to P$ is a cofibration and cofibrations are stable under composition we see that $P' \to B'$ and $P'' \to B''$ are simplicial resolutions.

Now form the map of pushouts $f : P' \otimes_P P'' \to B' \otimes_B B''$ and consider the corresponding map of derived tensor products in the sense of Quillen [21, Chapter II, Section 6]:

$$Lf : P' \otimes_P P'' = P' \otimes^L_P P'' \to B' \otimes^L_B B''.$$

By Quillen [21, Chapter II, Section 6, Theorem 6], there are second quadrant spectral sequences

$$\mathrm{Tor}_i^{\pi_* P}(\pi_* P', \pi_* P'')_j \Rightarrow \pi_{i+j}(P' \otimes_P P''),$$

$$\mathrm{Tor}_i^{\pi_* B}(\pi_* B', \pi_* B'')_j \Rightarrow \pi_{i+j}(B' \otimes_B^L B'').$$

The above diagram gives a map of spectral sequences which is an isomorphism at the E_2-terms. Hence Lf is a weak equivalence. By the Corollary following Quillen's Theorem 6 we have that $B' \otimes_B^L B'' \to B' \otimes_B B''$ is a weak equivalence. Thus f is itself a weak equivalence.

Since f is surjective it is a fibration. Since the pushout of a cofibration is a cofibration $P' \to P' \otimes_P P''$ is a cofibration and thus the domain of f is cofibrant. So f is a simplicial resolution.

For the proof of (1) take $B = \mathbb{F}_p$ and apply E codegree wise. The result follows by the Eilenberg–Zilber theorem. For the proof of (2) apply E codegree wise. The result follows by Quillen's Theorem 6. □

If one knows that the higher derived functors vanish on a certain class of objects, they can be used to compute derived functors by the following result.

Theorem 6.4. *Let $E : \mathscr{R} \to \mathscr{A}lg$ be a functor and let $A \in \mathscr{R}$. Assume that $Q \xrightarrow{\sim} A$ is an acyclic fibration in $s\mathscr{R}$ and that*

$$H_i(Q_j; E) = \begin{cases} E(Q_j) & i = 0, \\ 0 & i > 0. \end{cases}$$

Then $H_*(A; E) \cong \pi_* E(Q)$.

Proof. We have shown that $s\mathscr{R}$ is a simplicial model category. So $ss\mathscr{R}$ is a simplicial model category by the Reedy structure [11, Chapter VII, Section 2.13]. A fibration in $ss\mathscr{R}$ is especially a level fibration and a cofibration is especially a level cofibration by Goerss and Jardine [11, Chapter VII, Section 2.6]. A weak equivalence is a level weak equivalence by definition.

We use a dot to denote a simplicial direction in the following. Let $cQ_{\bullet\bullet}$ denote the object in $ss\mathscr{R}$ defined by $(cQ)_{ij} = Q_j$ for all i. Let $P_{\bullet\bullet}$ be a resolution of $cQ_{\bullet\bullet}$ i.e. $(\mathbb{F}_p)_{\bullet\bullet} \rightarrowtail P_{\bullet\bullet} \xrightarrow{\sim} cQ_{\bullet\bullet}$.

We have that $(\mathbb{F}_p)_\bullet \rightarrowtail P_{i\bullet} \xrightarrow{\sim} Q_\bullet$ for each i by the above. By composition with the acyclic fibration $Q_\bullet \xrightarrow{\sim} A$ we see that $P_{i\bullet}$ is a resolution of A.

So the horizontal homotopy of $E(P_{\bullet\bullet})$ is given by $\pi_j^h E(P_{i\bullet}) = H_j(A; E)$. We apply vertical homotopy on this and obtain

$$\pi_i^v \pi_j^h E(P_{\bullet\bullet}) \cong \begin{cases} H_j(A; E) & i = 0 \\ 0 & i > 0. \end{cases}$$

We also have that $P_{\bullet j}$ is a resolution of Q_j for each j. So $\pi_i^v E(P_{\bullet j}) \cong H_i(Q_j; E)$ which equals $E(Q_j)$ for $i = 0$ and equals 0 for $i > 0$. We apply horizontal homotopy on this and obtain

$$\pi_j^h \pi_i^v E(P_{\bullet\bullet}) \cong \begin{cases} \pi_j E(Q_\bullet) & i = 0, \\ 0 & i > 0. \end{cases}$$

Thus both spectral sequences associated with $E(P_{\bullet\bullet})$ collapse and the result follows. □

7. The E_2 terms seen as derived functors

In [3,19,20] we introduced a functor $\overline{\Omega} : \mathscr{F} \to \mathscr{A}lg$ as follows:

Definition 7.1. $\overline{\Omega}(R)$ is the quotient of the free graded commutative and unital R-algebra on generators $\{dx | x \in R\}$ of degree $|dx| = |x| - 1$, modulo the ideal generated by the elements

$$\begin{array}{ll} d(x+y) - dx - dy, & d(xy) - d(x)y - (-1)^{|x|} x d(y), \\ d(\lambda x) - (dx)^p, & d(\beta\lambda x). \end{array}$$

There is a differential $d : \overline{\Omega}(R) \to \overline{\Omega}(R)$ given by $d(x) = dx$ for $x \in R$.

Note that for $p = 2$ the Bockstein is trivial so here the functor $\overline{\Omega}$ is the same as the functor which we originally denoted Ω_λ.

It was shown that there is a lift to a functor $\overline{\Omega} : \mathscr{K} \to \mathscr{K}$ and that this lift is nothing but Lannes' division functor $(- : H^*(\mathbb{T}))_{\mathscr{K}}$. In particular there is a morphism $\overline{\Omega}(H^*X) \to H^*(\Lambda X)$ for any space X which is an isomorphism when H^*X is a free object in $\mathscr{K}$.

An other functor $\ell : \mathscr{F} \to \mathscr{A}lg$ was also introduced in [3,19,20] as follows:

Definition 7.2. Let $p = 2$ and let A be an object in $\mathscr{F}$. The $\mathbb{F}_2$-algebra $\ell(A)$ is the quotient of the free graded commutative $\mathbb{F}_2$-algebra on generators

$$\phi(x), q(y), \delta(z), u \text{ for } x, y, z \in A$$

of degrees $|\phi(x)| = 2|x|$, $|q(x)| = 2|x| - 1$, $|\delta(x)| = |x| - 1$ and $|u| = 2$, by the ideal generated by the elements

$$\begin{array}{l} \phi(a+b) + \phi(a) + \phi(b), \\ \delta(a+b) + \delta(a) + \delta(b), \\ q(a+b) + q(a) + q(b) + \delta(ab), \\ \delta(xy)\delta(z) + \delta(yz)\delta(x) + \delta(zx)\delta(y), \\ \phi(xy) + \phi(x)\phi(y) + uq(x)q(y), \\ q(xy) + q(x)\phi(y) + \phi(x)q(y), \\ \delta(x)^2 + \delta(\lambda x), \\ q(x)^2 + \phi(\lambda x) + \delta(x^2\lambda x), \\ \delta(x)\phi(y) + \delta(xy^2), \\ \delta(x)q(y) + \delta(x\lambda y) + \delta(xy)\delta(y), \\ \delta(x)u, \\ \phi(1)u + u, \end{array}$$

where a, b, x, y, z are homogeneous elements in A with $|a| = |b|$.

Definition 7.3. Let p be an odd prime and let A be an object in $\mathscr{F}$. The $\mathbb{F}_p$-algebra $\ell(A)$ is the quotient of the free graded commutative $\mathbb{F}_p$-algebra on generators

$$\phi(x), q(y), \delta(z), u \text{ for } x, y, z \in A$$

of degrees $|\phi(x)| = p|x| - \sigma(x)(p-1)$, $|q(y)| = p|y| - 1 - \sigma(y)(p-3)$, $|\delta(z)| = |z| - 1$ and $|u| = 2$ by the ideal generated by the elements

$$\begin{aligned}
&\phi(a+b) - \phi(a) - \phi(b) + \sigma(a)\sum_{i=0}^{p-2}(-1)^i \delta(a)^i \delta(b)^{p-2-i}\delta(ab),\\
&\delta(a+b) - \delta(a) - \delta(b),\\
&q(a+b) - q(a) - q(b) + \hat{\sigma}(a)\sum_{i=1}^{p-1}(-1)^i \frac{1}{i}\delta(a^i b^{p-i}),\\
&(-1)^{\sigma(x)\hat{\sigma}(z)}\delta(x)\delta(yz) + (-1)^{\sigma(y)\hat{\sigma}(x)}\delta(y)\delta(zx) + (-1)^{\sigma(z)\hat{\sigma}(y)}\delta(z)\delta(xy),\\
&\phi(xy) - (-u^{p-1})^{\sigma(x)\sigma(y)}\phi(x)\phi(y),\\
&q(xy) - (-u^{p-1})^{\sigma(x)\sigma(y)}(u^{\sigma(y)}q(x)\phi(y) + (-u)^{\sigma(x)}\phi(x)q(y)),\\
&q(x)^p - u^{p-1}q(\lambda x) - \phi(\beta\lambda x),\\
&\delta(x)\phi(y) - \delta(xy^p) - \delta(x\lambda y) + \delta(xy)\delta(y)^{p-1},\\
&\delta(x)q(y) - \delta(xy^{p-1})\delta(y) - \delta(x\beta\lambda y),\\
&\delta(x)u,\\
&\phi(1)u - u,\\
&q(\beta\lambda x),\\
&\delta(x^p),
\end{aligned}$$

where $a, b, x, y, z \in A$ with $|a| = |b|$. Furthermore, $\sigma(x) = 1$ for $|x|$ odd, $\sigma(x) = 0$ for $|x|$ even and $\hat{\sigma}(x) = 1 - \sigma(x)$.

The functor ℓ also lifts to an endofunctor on $\mathscr{K}$ and there is a natural morphism $\ell(H^*X) \to H^*(\Lambda X_{h\mathbb{T}})$ which is an isomorphism if H^*X is a free object in $\mathscr{K}$. For details on this see [3,19,20].

Via Theorem 5.12, we can now restate Theorem 4.7 in an appropriate form.

Theorem 7.4. *If X is a 1-connected and fibrant space with H_*X of finite type then we have strongly convergent Bousfield cohomology spectral sequences $\hat{E}_r \Rightarrow H^*(\Lambda X)$ and $E_r \Rightarrow H^*(\Lambda X_{h\mathbb{T}})$ with the following E_2 terms*:

$$\hat{E}_2^{-m,t} \cong H_m(H^*(X); \overline{\Omega})^t \quad \textit{and} \quad E_2^{-m,t} \cong H_m(H^*(X); \ell)^t.$$

We now introduce other functors in order to study the derived functors of ℓ. Recall that the functors $\mathscr{L}$ and $\widetilde{\Omega}$ from $\mathscr{F}$ to $\mathscr{A}lg$ are defined by $\mathscr{L}(R) = \ell(R)/(u)$ and $\widetilde{\Omega}(R) = \mathscr{L}(R)/(\delta(x)|x \in R)$.

Proposition 7.5. *For each object $R \in \mathscr{F}$ there are isomorphisms as follows*: $H_0(R; \overline{\Omega}) \cong \overline{\Omega}(R)$, $H_0(R; \widetilde{\Omega}) \cong \widetilde{\Omega}(R)$ *and* $H_0(R; \mathscr{L}) \cong \mathscr{L}(R)$.

Proof. We use Lemma 6.2 to prove this. By their definitions we may consider $\overline{\Omega}$, $\widetilde{\Omega}$ and $\mathscr{L}$ as functors from $\mathscr{F}'$ to $\mathscr{A}lg'$.

Let A be an object in $\mathscr{F}'$ and let $I \subseteq A$ be an ideal. We must verify Conditions 1 and 2 in Lemma 6.2 for these functors where $f : A \to A/I$ is the natural projection. We do this for the functor $\mathscr{L}$. The verification for the other functors is similar but easier.

The map $\mathscr{L}(f)$ is surjective with kernel

$$J = (\phi(x) - \phi(y), q(x) - q(y), \delta(x) - \delta(y) | x - y \in I) \subseteq \mathscr{L}(A),$$

so $\mathscr{L}(A)/J \cong \mathscr{L}(A/I)$. We must check that $\mathscr{L}(I) = J$.

The inclusion $\mathscr{L}(I) \subseteq J$ holds since $\phi(0) = q(0) = \delta(0) = 0$.

For the inclusion $\mathscr{L}(I) \supseteq J$ assume first that $p = 2$. Since ϕ and q are additive we have that $\delta(x) - \delta(y)$ and $\phi(x) - \phi(y)$ lie in $\mathscr{L}(I)$. Further $q(x) - q(y) = q(x - y) + \delta(xy)$ but $\delta(xy) = \delta(x(y - x))$ so also $q(x) - q(y) \in \mathscr{L}(I)$. Thus the inclusion holds.

For p odd δ is additive, ϕ is additive on elements of even degree and q is additive on elements of odd degree. For $|x| = |y|$ odd we have

$$\phi(x) - \phi(y) = \phi(x - y) + \sum_{i=0}^{p-2} \delta(x)^i \delta(y)^{p-2-i} \delta(xy)$$

and again $\delta(x(y - x)) = \delta(xy)$ such that this lies in $\mathscr{L}(I)$. For $|x| = |y|$ even we have

$$q(x) - q(y) = q(x - y) - \delta\left(\sum_{i=1}^{p-1} \frac{1}{i} x^i y^{p-i}\right)$$

so it suffices to see that $y - x$ divides the sum inside the $\delta(-)$. The following equation in $\mathbb{F}_p[x, y]$ shows that this is the case:

$$xy(y - x) \sum_{k=0}^{p-3} a_k x^k y^{p-3-k} = \sum_{i=1}^{p-1} \frac{1}{i} x^i y^{p-i} \quad \text{where } a_k = \sum_{j=0}^{k} \frac{1}{j+1}.$$

The equation holds since by Euler's sum formula $\sum_{n=1}^{p-1} n = 0$ modulo p. □

Definition 7.6. Let $\mathscr{Z}, \mathscr{B}, \mathscr{H} : \mathscr{F} \to \mathscr{A}lg$ denote the functors given by

$$\mathscr{Z}(R) = \ker(d), \quad \mathscr{B}(R) = \operatorname{im}(d), \quad \mathscr{H}(R) = \mathscr{Z}(R)/\mathscr{B}(R),$$

where d is the differential on $\overline{\Omega}(R)$.

Recall from [3,19,20] that there are natural transformations of functors $\Phi : \widetilde{\Omega} \to \mathscr{H}$ and $Q : \mathscr{L} \to \mathscr{Z}$. It was shown that if $A \in \mathscr{F}$ is a free object, or its underlying algebra is polynomial, then Φ_A and Q_A are isomorphisms. We can now give a nice interpretation of the functor $\mathscr{L}$, which was originally defined by generators and complicated relations.

Theorem 7.7. *For any $R \in \mathscr{F}$ one has $\mathscr{L}(R) \cong H_0(R; \mathscr{Z})$.*

Proof. The induced map $Q_* : H_*(R; \mathscr{L}) \to H_*(R; \mathscr{Z})$ is an isomorphism since Q is an isomorphism on free objects. The 0th derived functor of $\mathscr{L}$ was computed in Proposition 7.5. □

For any functor $E : \mathscr{F} \to \mathscr{A}lg$ we have that $H_i(A; F) = 0$ for $i > 0$ when A is a free object since we can use the trivial almost free resolution to compute the derived functors. For polynomial algebras we also have nice results.

Theorem 7.8. *Assume that the underlying algebra of $A \in \mathscr{F}$ is a polynomial algebra. Then one has $H_i(A; \overline{\Omega}) = 0$, $H_i(A; \widetilde{\Omega}) = 0$, $H_i(A; \mathscr{L}) = 0$ and $H_i(A; \ell) = 0$ for each $i > 0$.*

Proof. We first prove the statements for $\overline{\Omega}$ and $\widetilde{\Omega}$. Let $\Omega : \mathscr{A}lg \to \mathscr{A}lg$ denote the usual de Rham complex functor. Pick an almost free resolution $P \in s\mathscr{F}$ of A. The forgetful functor $U : \mathscr{F} \to \mathscr{A}lg$ takes free objects to free objects. So we can apply U to P and get an almost free resolution of $U(A)$ in $s\mathscr{A}lg$. Thus there is an isomorphism

$$H_i^{\mathscr{F}}(A; \Omega U) \cong H_i^{\mathscr{A}lg}(U(A); \Omega)$$

and the last group is trivial for $i > 0$ since $U(A)$ is a free object in $\mathscr{A}lg$.

There is a linear map $\Omega U(A) \to \overline{\Omega}(A); x_0 dx_1 \dots dx_n \mapsto x_0 dx_1 \dots dx_n$. The map is not multiplicative and it does not commute with the de Rham differential, but it is an isomorphism of graded vector spaces. Thus $H_i(A; \overline{\Omega})$ is additively isomorphic to $H_i(A; \Omega U)$ which is trivial for $i > 0$. A similar isomorphism gives the result for the functor $\widetilde{\Omega}$.

Next we consider the functor $\mathscr{L}$. The short exact sequence $0 \to \mathscr{Z} \to \overline{\Omega} \to \mathscr{B} \to 0$ gives a long exact sequence of derived functors. By the above this sequence breaks up into the exact sequence

$$0 \to H_1(A; \mathscr{B}) \to H_0(A; \mathscr{Z}) \to H_0(A; \overline{\Omega}) \to H_0(A; \mathscr{B}) \to 0$$

together with the isomorphisms $H_i(A; \mathscr{Z}) \cong H_{i+1}(A; \mathscr{B})$ for $i \geqslant 1$.

There is also a short exact sequence $0 \to \mathscr{B} \to \mathscr{Z} \to \mathscr{H} \to 0$ with corresponding long exact sequence of derived functors. Since Φ is an isomorphism on free objects we have a natural isomorphism $\Phi_* : H_*(-; \widetilde{\Omega}) \cong H_*(-; \mathscr{H})$. By the above vanishing result for $H_*(A; \widetilde{\Omega})$ the long exact sequence breaks up into the short exact sequence

$$0 \to H_0(A; \mathscr{B}) \to H_0(A; \mathscr{Z}) \to H_0(A; \mathscr{H}) \to 0$$

and the isomorphisms $H_i(A; \mathscr{B}) \cong H_i(A; \mathscr{Z})$ for $i \geqslant 1$.

Using Proposition 7.5 and Theorem 7.7 we can rewrite the exact sequences involving 0th derived functors as

$$0 \to H_1(A; \mathscr{B}) \to \mathscr{L}(A) \to \overline{\Omega}(A) \to H_0(A; \mathscr{B}) \to 0,$$

$$0 \to H_0(A; \mathscr{B}) \to \mathscr{L}(A) \to \mathscr{H}(A) \to 0.$$

Since $Q : \mathscr{L}(A) \to \mathscr{Z}(A)$ is an isomorphism we see that $H_1(A; \mathscr{B}) = 0$. By the isomorphisms $H_1(A; \mathscr{B}) \cong H_1(A; \mathscr{Z}) \cong H_2(A; \mathscr{B}) \cong \dots$ we conclude that $H_i(A; \mathscr{Z})$ is trivial for $i > 0$. But $H_*(-; \mathscr{L})$ is isomorphic to $H_*(-; \mathscr{Z})$ so we are done.

Finally we consider the functor ℓ. By definition of $\mathscr{L}$ the sequence $0 \to u\ell \to \ell \to \mathscr{L} \to 0$ is short exact. From the corresponding long exact sequence of derived functors we find that $H_i(A; u\ell) \cong H_i(A; \ell)$ for $i > 0$. By Theorem A.3 from the appendix and Proposition 4.4 from [20] there is a short exact sequence

$$0 \to u^{j+1}\ell(B) \to u^j \ell(B) \to u^j \otimes \widetilde{\Omega}(B) \to 0, \quad j > 0, \tag{6}$$

when B is a free object in $\mathscr{F}$ or when the underlying algebra of B is a polynomial algebra. The corresponding long exact sequence of derived functors shows that $H_i(A; u^j\ell) \cong H_i(A; u^{j+1}\ell)$ so we have $H_i(A; \ell) \cong H_i(A; u^j\ell)$ for all $j \geqslant 0$. But $(u^j\ell)^k = 0$ for $k < 2j$ so $H_i(A; u^j\ell)^k = 0$ for $k < 2j$ and the result follows. □

Proposition 7.9. *If the underlying algebra of an object $A \in \mathscr{F}$ is a polynomial algebra, then $H_0(A; \ell) \cong \ell(A)$.*

Proof. The short exact sequence $0 \to u\ell \to \ell \to \mathscr{L} \to 0$ gives a short exact sequence of 0th derived functors since $H_1(A; \mathscr{L}) = 0$. Furthermore, there is a natural map $H_0(-; F) \to F$ for any functor $F : \mathscr{F} \to \mathscr{A}lg$. So we have a commutative diagram with exact rows as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(A; u\ell) & \longrightarrow & H_0(A; \ell) & \longrightarrow & H_0(A; \mathscr{L}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & u\ell(A) & \longrightarrow & \ell(A) & \longrightarrow & \mathscr{L}(A) & \longrightarrow & 0. \end{array}$$

The right vertical map is an isomorphism so it suffices to show that the left vertical map is also an isomorphism.

Since $H_1(A; \widetilde{\Omega}) = 0$ the short exact sequence (6) gives a commutative diagram as follows for $j > 0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_0(A; u^{j+1}\ell) & \longrightarrow & H_0(A; u^j\ell) & \longrightarrow & H_0(A; u^j \otimes \widetilde{\Omega}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & u^{j+1}\ell(A) & \longrightarrow & u^j\ell(A) & \longrightarrow & u^j \otimes \widetilde{\Omega}(A) & \longrightarrow & 0, \end{array}$$

where the right vertical map is an isomorphism. Fix a degree n. For $j+1 > n/2$ the map $H_0(A; u^{j+1}\ell)^n \to (u^{j+1}\ell(A))^n$ is an isomorphism since both domain and target space are zero. The result follows by induction. □

Corollary 7.10. *Let X be a 1-connected space such that H_*X is of finite type and H^*X is a polynomial algebra. Then then the spectral sequences of Theorem 7.4 collapses at the E_2 terms. So there are isomorphisms*

$$H_*(H^*(X); \overline{\Omega})^* \cong H^*(\Lambda X) \quad \textit{and} \quad H_*(H^*(X); \ell)^* \cong H^*((\Lambda X)_{h\mathbb{T}}).$$

8. The derived functors of an exterior algebra

In the rest of this paper we take $p = 2$. Let $\Lambda = \Lambda(\sigma) \in \mathscr{F}$ be an exterior algebra on one generator of degree $|\sigma| \geqslant 2$. Note that $\lambda\sigma = 0$ for dimensional reasons. We intend to compute the higher derived functors of the various functors we have been considering for this algebra.

Proposition 8.1. *There are isomorphisms*

$$H_*(\Lambda; \overline{\Omega}) \cong \overline{\Omega}(\Lambda) \otimes \Gamma[\omega], \quad H_*(\Lambda; \widetilde{\Omega}) \cong \widetilde{\Omega}(\Lambda) \otimes \Gamma[\widetilde{\omega}].$$

The inner degrees are $|\gamma_i(\omega)| = i(2|\sigma| - 1)$, $|\gamma_i(\widetilde{\omega})| = i(4|\sigma| - 1)$ *and the grading of the homology groups are given by*

$$H_i(\Lambda; \overline{\Omega}) \cong \overline{\Omega}(\Lambda) \otimes \gamma_i(\omega), \quad H_i(\Lambda; \widetilde{\Omega}) \cong \widetilde{\Omega}(\Lambda) \otimes \gamma_i(\widetilde{\omega}).$$

Proof. The algebra Λ is the pushout of $\mathbb{F}_2 \leftarrow \mathbb{F}_2[y] \rightarrow \mathbb{F}_2[x]$ where $y \mapsto x^2$. Put $\lambda x = 0$ and $\lambda y = 0$. By Proposition 6.3 we find

$$H_i(\Lambda; E) \cong \mathrm{Tor}_i^{E(\mathbb{F}_2[y])}(\mathbb{F}_2, E(\mathbb{F}_2[x])) \quad \text{for} \quad E = \overline{\Omega}, \widetilde{\Omega}.$$

The result follows by standard computations. □

In order to compute derived functors of the other functors we need an explicit simplicial resolution of Λ. By Theorems 6.4, 7.8, Propositions 7.5 and 7.9 we may use an almost free resolution of Λ in $s\mathscr{A}lg$ and equip it with $\lambda = 0$.

Proposition 8.2. *There is an almost free resolution* $R_\bullet \in s\mathscr{A}lg$ *of the algebra* Λ *with* $R_n = \mathbb{F}_2[x, y_1, y_2, \ldots, y_n]$ *for* $n \geqslant 0$. *The structure maps* $d_i : R_n \rightarrow R_{n-1}$ *and* $s_i : R_n \rightarrow R_{n+1}$ *are given by*

$$s_i(x) = x,$$
$$s_i(y_j) = \begin{cases} y_j & j \leqslant i, \\ y_{j+1} & j > i, \end{cases}$$
$$d_i(x) = x,$$
$$d_i(y_j) = \begin{cases} x^2 & i = 0, \ j = 1, \\ y_{j-1} & i < j, \ j > 1, \\ y_j & i \geqslant j, \ j < n, \\ 0 & i = n, \ j = n. \end{cases}$$

The degrees of the generators are $|x| = |\sigma|$ *and* $|y_i| = 2|\sigma|$ *for all i.*

Proof. We first give a description of the simplicial set $\Delta^1_\bullet = \mathrm{Hom}_\Delta(-, [1])$ suited for our purpose.

Define the elements $y_j \in \Delta^1_n$ for $n \geqslant 0$ and $0 \leqslant j \leqslant n+1$ by $y_j(i) = 0$ if $i < j$ and $y_j(i) = 1$ if $i \geqslant j$. We have $\Delta^1_n = \{y_0, \ldots, y_{n+1}\}$. The structure maps are as follows:

$$d_i y_j = \begin{cases} y_{j-1} & i < j \\ y_j & i \geqslant j \end{cases} \quad \text{and} \quad s_i y_j = \begin{cases} y_{j+1} & i < j, \\ y_j & i \geqslant j. \end{cases}$$

Let $\mathbb{F}_2[-]$ denote the functor which takes a graded set to the polynomial algebra generated by that set. Let $\mathbb{F}_2[\Delta^1_\bullet, *]$ denote the pushout of $\mathbb{F}_2 \leftarrow \mathbb{F}_2[a] \rightarrow \mathbb{F}_2[\Delta^1_\bullet]$ where $\mathbb{F}_2$ and $\mathbb{F}_2[a]$ are constant simplicial algebras. In degree n the maps are as follows: $a \mapsto 0 \in \mathbb{F}_2$ and $a \mapsto y_{n+1} \in \mathbb{F}_2[\Delta^1_\bullet]$. Note that $\mathbb{F}_2[\Delta^1_\bullet] \simeq \mathbb{F}_2[*]$ by the simplicial contraction of $\Delta^1_\bullet$. The spectral sequence [21, Chapter II, Section 7, Theorem 6] gives that $\pi_i(\mathbb{F}_2[\Delta^1_\bullet, *]) \cong \mathbb{F}_2$ for $i = 0$ and 0 otherwise.

Define $R_\bullet$ as the pushout of $\mathbb{F}_2[x] \leftarrow \mathbb{F}_2[z] \rightarrow \mathbb{F}_2[\Delta^1_\bullet, *]$ where in degree n the maps are $z \mapsto x^2$ and $z \mapsto y_0$. For this pushout Quillen's spectral sequence gives that

$$\pi_i(R_\bullet) \cong \mathrm{Tor}_i^{\mathbb{F}_2[z]}(\mathbb{F}_2, \mathbb{F}_2[x]) = \begin{cases} \Lambda & i = 0, \\ 0 & i > 0. \end{cases}$$

Thus $R_\bullet$ has the right homotopy groups. Further R_n is as stated and the structure maps are as stated. Note that $R_\bullet$ is almost free. The degrees are correct since the structure maps must be degree preserving. □

Lemma 8.3. *$H_n(\Lambda; \Omega)$ has the following $\mathbb{F}_2$-basis*:

$$dy_1 \dots dy_n,\ x dy_1 \dots dy_n,\ dx dy_1 \dots dy_n,\ x dx dy_1 \dots dy_n.$$

Proof. Using the formulas in Proposition 8.2 it is easy to check that the four given classes are in the kernel of d_i for all i. To check linear independency, we introduce two gradings of Ω.

Firstly, the wedge grading on $\Omega(R_n)$ is defined as the number of wedge factors, i.e. the number of d's in a homogeneous element. Secondly, the polynomial grading is defined as follows: Give x grading 1, y_j grading 2 and each dx or dy_j grading 0 and extend multiplicatively. Note that the maps d_i preserve both gradings. We write $\Omega^{q,t}(R_n)$ for the elements in $\Omega(R_n)$ of wedge degree q and polynomial degree t. Thus, there is a direct sum decomposition

$$H_n(\Lambda; \Omega) = \bigoplus_{q,t \geqslant 0} H_n(\Lambda; \Omega^{q,t}).$$

The classes we consider sit in different bigradings, so we only have to check that they individually do not represent the trivial class.

We have the following bases for $\Omega^{n,0}(R_{n+1})$, $\Omega^{n,0}(R_n)$ and $\Omega^{n,0}(R_{n-1})$ respectively:

$$\{dy_1 \dots \widehat{dy_j} \dots dy_{n+1}\} \cup \{dx dy_1 \dots \widehat{dy_j} \dots \widehat{dy_k} \dots dy_{n+1}\},$$
$$\{dx dy_1 \dots \widehat{dy_j} \dots dy_n\} \cup \{dy_1 \dots dy_n\},$$
$$\{dx dy_1 \dots dy_{n-1}\}.$$

We use the normalized complex consisting of $\cap_{i>0} \ker(d_i)$ with differential d_0 to compute the homology. For this normalized complex we have the respective bases $\emptyset$, $\{dy_1 \dots dy_n\}$, $\{dx dy_1 \dots dy_{n-1}\}$. Taking homology and using that $\Omega^{n,0}(R_{n-2}) = 0$ we see that the classes $dx dy_1 \cdots dy_{n-1}$ and $dy_1 \dots dy_n$ do not represent zero.

Similarly, $x dx dy_1 \dots dy_{n-1}$ and $x dy_1 \dots dy_n$ do not represent zero. Keeping track of degrees and dimensions, the result follows. □

Lemma 8.4. *$H_n(\Lambda; \mathscr{H})$ has an $\mathbb{F}_2$-basis as follows*:

$$y_1 \dots y_n dy_1 \dots dy_n,\ x^2 y_1 \dots y_n dy_1 \dots dy_n,$$
$$x dx y_1 \dots y_n dy_1 \dots dy_n,\ x^3 dx y_1 \dots y_n dy_1 \dots dy_n.$$

Proof. This follows from Lemma 8.3 and the Cartier isomorphism. □

Corollary 8.5. *The long exact sequence associated to $\mathscr{B} \to \mathscr{Z} \to \mathscr{H}$ splits into short exact sequences when evaluated at Λ*

$$0 \to H_*(\Lambda; \mathscr{B}) \to H_*(\Lambda; \mathscr{Z}) \to H_*(\Lambda; \mathscr{H}) \to 0.$$

Proof. The generators from Lemma 8.4 are visibly in the image of $H_n(\Lambda; \mathscr{Z}) \to H_n(\Lambda; \mathscr{H})$. □

Definition 8.6. Let $\gamma_{i,j} \in \Omega(R_j)$ denote the following elements:

$$\gamma_{i,j} = y_1 y_2 \dots y_i dy_1 dy_2 \dots dy_j \quad \text{for } i, j > 0,$$
$$\gamma_{0,j} = dy_1 dy_2 \dots dy_j \quad \text{for } j > 0 \quad \text{and} \quad \gamma_{0,0} = 1.$$

These elements are actually in $\mathscr{Z}(R_j)$ if $i \leqslant j$, and in $\mathscr{B}(R_j)$ in case $i < j$. They are even in the normalized chain complex, so they define classes in $H_*(\Lambda; \mathscr{Z})$, respectively, $H_*(\Lambda; \mathscr{B})$.

Theorem 8.7. *For $n \geqslant 0$ the homology group $H_{2n}(\Lambda; \mathscr{B})$ has $\mathbb{F}_2$-basis*

$$\{dx\gamma_{0,2n}\} \cup \{\gamma_{2i,2n}\ x^2\gamma_{2i,2n},\ xdx\gamma_{2i,2n}\ x^3dx\gamma_{2i,2n} | 0 \leqslant i < n\}$$

and the homology group $H_{2n+1}(\Lambda; \mathscr{B})$ has $\mathbb{F}_2$-basis

$$\{\gamma_{0,2n+1},\ dx\gamma_{0,2n+1},\ xdx\gamma_{0,2n+1}\}$$
$$\cup \{\gamma_{2i+1,2n+1},\ x^2\gamma_{2i+1,2n+1},\ xdx\gamma_{2i+1,2n+1},\ x^3dx\gamma_{2i+1,2n+1} | 0 \leqslant i < n\}.$$

Similarly, $H_{2n}(\Lambda; \mathscr{Z})$ has $\mathbb{F}_2$-basis

$$\{dx\gamma_{0,2n}\} \cup \{\gamma_{2i,2n},\ x^2\gamma_{2i,2n},\ xdx\gamma_{2i,2n},\ x^3dx\gamma_{2i,2n} | 0 \leqslant i \leqslant n\}$$

and $H_{2n+1}(\Lambda; \mathscr{Z})$ has $\mathbb{F}_2$-basis

$$\{\gamma_{0,2n+1},\ dx\gamma_{0,2n+1},\ xdx\gamma_{0,2n+1}\}$$
$$\cup \{\gamma_{2i+1,2n+1},\ x^2\gamma_{2i+1,2n+1},\ xdx\gamma_{2i+1,2n+1},\ x^3dx\gamma_{2i+1,2n+1} | 0 \leqslant i \leqslant n\}.$$

Proof. Recall that $H_m(\Lambda; \Omega) \cong \Omega(\Lambda) \otimes \gamma_{0,m}$ and $H_m(\Lambda; \mathscr{H}) \cong \widetilde{\Omega}(\Lambda) \otimes \gamma_{m,m}$. From the splitting $H_m(\Lambda; \mathscr{Z}) \cong H_m(\Lambda; \mathscr{B}) \oplus H_m(\Lambda; \mathscr{H})$ together with the computation of $H_m(\Lambda; \mathscr{H})$ in Lemma 8.4, it follows that the statements about $H_m(\Lambda; \mathscr{Z})$ and $H_m(\Lambda; \mathscr{B})$ are equivalent.

Recall the long exact sequence

$$H_{m+1}(\Lambda; \mathscr{B}) \xrightarrow{b} H_m(\Lambda; \mathscr{Z}) \xrightarrow{i_*} H_m(\Lambda; \Omega) \xrightarrow{d_*} H_m(\Lambda; \mathscr{B}) \tag{7}$$

Claim. *The image of d_* is a one-dimensional vector space, generated by a class, which in the normalized chain complex is represented by $dx\gamma_{0,m}$.*

Note that three out of four of the generators of $H_m(\Lambda; \Omega)$ are annihilated by d_*. So the image of d_* is spanned by the single class, represented in the normalized complex by $dx\gamma_{0,m} \in \mathscr{B}(\Lambda)$. This element actually represents a non-zero homology class, since the map $H_m(\Lambda; \mathscr{B}) \to H_m(\Lambda; \Omega)$ induced by inclusion, send it to the element represented by $dxdy_1 \cdots dy_m$. But this is non-trivial according to Lemma 8.3, and the claim follows.

We will now prove the theorem by induction on m. We first treat the case $m = 0$. Here, we see that the map $d_0 : H_0(\Lambda; \Omega) \to H_0(\Lambda; \mathscr{B})$ is surjective by the long exact sequence (7), and the statement follows directly from the claim we just proved.

Assume that the theorem holds for m. The long exact sequence (7) gives us a short exact sequence

$$0 \to \operatorname{im}(d_{m+1}) \to H_{m+1}(\Lambda; \mathscr{B}) \xrightarrow{b} \ker(i_m) \to 0.$$

So, in order to prove the theorem, we have to show that b maps the classes given in the statement of the theorem *with* $dx\gamma_{0,m+1}$ *removed* to a basis for the kernel of i_m.

By induction we have a basis for $H_m(\Lambda;\mathscr{Z})$. From this we see that the kernel of the map $i_{2n+1} : H_{2n+1}(\Lambda;\mathscr{Z}) \to H_{2n+1}(\Lambda;\Omega)$ has basis

$$\{\gamma_{2i+1,2n+1},\ x^2\gamma_{2i+1,2n+1},\ xdx\gamma_{2i+1,2n+1},\ x^3dx\gamma_{2i+1,2n+1} | 0\leqslant i\leqslant n\}$$

and that the kernel of the map $i_{2n} : H_{2n}(\Lambda;\mathscr{Z}) \to H_{2n}(\Lambda;\Omega)$ has basis

$$\{x^2\gamma_{0,2n},\ x^3dx\gamma_{0,2n}\} \cup \{\gamma_{2i,2n},\ x^2\gamma_{2i,2n},\ xdx\gamma_{2i,2n},\ x^3dx\gamma_{2i,2n} | 1\leqslant i\leqslant n\}.$$

It is convenient now to pass from the normalized to the unnormalized complex. The normalized complex is a subcomplex of the unnormalized one, and in the notation we do not distinguish between a class in the subcomplex and its image in the unnormalized one.

Let us first consider the odd case, $m = 2n+1$. The element $\gamma_{2i,2n+2} \in \mathscr{B}(R_{2n+2})$ is a cycle with respect to the boundary $\partial = \sum d_k$. We compute its image under the map b.

Let $\alpha_r \in \Omega(R_{2n+2})$ denote the following element:

$$\alpha_r = y_1 y_2 \ldots y_{2i} y_r dy_1 dy_2 \ldots \widehat{dy_r} \ldots dy_{2n+2}, \quad 2i+1 \leqslant r \leqslant 2n+2,$$

where the hat means that the factor is left out. Put $\beta = \sum_{r=2i+2}^{2n+2} \alpha_r$. We have $d\alpha_r = \gamma_{2i,2n+2}$ for each r, so that also $d\beta = \gamma_{2i,2n+2}$. This means that $b(\gamma_{2i,2n+2})$ is represented by

$$\partial\beta = \sum_{r=2i+2}^{2n+2} (y_{r-1} + y_r)\gamma_{2i,2n+1} + y_{2n+1}\gamma_{2i,2n+1} = \gamma_{2i+1,2n+1}.$$

Since b is linear with respect to multiplication by x^2, xdx, x^3dx this gives the desired result for $H_{2n+2}(\Lambda;\mathscr{B})$.

In the even case $m = 2n$, a similar argument shows that $b(\gamma_{2i+1,2n+1})$ is represented by $\gamma_{2i+2,2n}$.

Checking with the lists of classes above, we see that we are left to prove that $b(\gamma_{0,2n+1}) = x^2\gamma_{0,2n}$. The argument is very similar. Let

$$\alpha_r = y_r dy_1 dy_2 \ldots \widehat{dy_r} \ldots dy_{2n+1} \quad \text{and} \quad \beta = \alpha_1 + \alpha_2 + \cdots + \alpha_{2n+1}.$$

Then $d\alpha_r = \gamma_{0,2n+1}$ so also $d\beta_r = \gamma_{0,2n+1}$. Thus $b(\gamma_{0,2n+1})$ is represented by $\partial\beta = x^2\gamma_{0,2n}$. □

Proposition 8.8. *There are short exact sequences for* $i \geqslant 0$, $t \geqslant 1$ *as follows*:

$$0 \longrightarrow H_i(\Lambda; u\ell) \longrightarrow H_i(\Lambda;\ell) \longrightarrow H_i(\Lambda;\mathscr{L}) \longrightarrow 0,$$
$$0 \longrightarrow H_i(\Lambda; u^{t+1}\ell) \longrightarrow H_i(\Lambda; u^t\ell) \longrightarrow H_i(\Lambda; u^t\widetilde{\Omega}) \longrightarrow 0.$$

Proof. The first short exact sequence follows if we can prove that the connecting homomorphism $b : H_{i+1}(\Lambda;\mathscr{L}) \to H_i(\Lambda; u\ell)$ is trivial. So consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & u\ell(R_{i+1}) & \longrightarrow & \ell(R_{i+1}) & \longrightarrow & \mathscr{L}(R_{i+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & u\ell(R_i) & \longrightarrow & \ell(R_i) & \longrightarrow & \mathscr{L}(R_i) & \longrightarrow & 0.
\end{array}$$

The element $q(y_1)\dots q(y_j)\delta(y_{j+1})\dots\delta(y_{i+1}) \in \ell(R_{i+1})$ maps to the element $\gamma_{j,i+1} \in \mathscr{L}(R_{i+1})$. By the relations $\delta(a)^2 = \delta(\lambda a)$, $q(a)^2 = \phi(\lambda a) + \delta(a^2\lambda a)$ and $\delta(a)q(b) = \delta(a\lambda b) + \delta(ab)\delta(b)$ we see that this element maps down to zero in $\ell(R_i)$. So the connecting homomorphism b is trivial.

The proof for the second short exact sequence is similar. □

Corollary 8.9. *For each $i \geqslant 0$ there are isomorphisms of $\mathbb{F}_2$-vector spaces*

$$H_i(\Lambda; \ell) \cong H_i(\Lambda; \mathscr{L}) \oplus \left(\bigoplus_{t\geqslant 1} u^t \otimes H_i(\Lambda; \widetilde{\Omega}) \right) \cong H_i(\Lambda; \mathscr{B}) \oplus (\mathbb{F}_2[u] \otimes H_i(\Lambda; \widetilde{\Omega})).$$

Let $F : \mathscr{F} \to \mathscr{A}lg$ be a functor. We define the total degree of a class in $H_i(R; F)^n$ to be $n - i$. For $F = \ell, \overline{\Omega}$ this corresponds through the spectral sequences of Theorem 7.4 to the grading of cohomology groups. We write $P_F(t)$ for the Poincaré series corresponding to the total degree of $H_*(\Lambda; F)^*$.

Theorem 8.10. *Let s denote the degree of the class $\sigma \in \Lambda$. Then we have the following Poincaré series*:

$$P_{\overline{\Omega}}(t) = (1 + t^s)(1 - t^{s-1})^{-1},$$
$$P_{\widetilde{\Omega}}(t) = (1 + t^{2s})(1 - t^{2s-1})^{-1},$$
$$P_{\mathscr{B}}(t) = t^{s-1}(1 + t^{s-1} - t^{2s-1})(1 - t^{2s-1})^{-1}(1 - t^{2s-2})^{-1},$$
$$P_{\ell}(t) = (1 + t^{s-1} - t^{s+1} + t^{2s-1})(1 - t^2)^{-1}(1 - t^{2s-2})^{-1}.$$

Proof. The first two formulas follow from Proposition 8.1: the total degree of $\gamma_i(\omega)$ is $|\gamma_i(\omega)| - i = (2s-2)i$ and $\overline{\Omega}(\Lambda) = \Lambda \otimes \Lambda(d\sigma)$ so

$$P_{\overline{\Omega}}(t) = (1 + t^s)(1 + t^{s-1})(1 - t^{2s-2})^{-1} = (1 + t^s)(1 - t^{s-1})^{-1}.$$

A similar argument gives $P_{\widetilde{\Omega}}(t)$.

To determine $P_{\mathscr{B}}(t)$ we must count the classes given in Theorem 8.7, according to the total degree.

We divide these classes into three groups. The first group are those of the form $dx\gamma_{0,m}$. The Poincaré series of the subspace generated by those classes is $t^{s-1}/(1 - t^{2s-2})$. The second group are those of the type $\gamma_{0,2n+1}$ or $xdx\gamma_{0,2n+1}$. These have Poincaré series $t^{2s-2}(1 + t^{2s-1})/(1 - t^{4s-4})$.

The third group is the remaining classes. They span a free $\widetilde{\Omega}(\Lambda)$-module with basis $X = \{\gamma_{2i,2n}, \gamma_{2i+1,2n+1} | 0 \leqslant i < n, 0 \leqslant n\}$. We introduce the following operation on the set X: $T(\gamma_{i,n}) = \gamma_{i+1,n+1}$. This operation has total degree $4s - 2$. All generators are obtained by applying T a non-negative number of times starting from one of the elements of $Y = \{\gamma_{0,2n} | n \geqslant 1\}$.

The set Y has Poincaré series $t^{4s-4}/(1 - t^{4s-4})$. So, the Poincaré series of the set X is given by $t^{4s-4}(1 - t^{4s-4})^{-1}(1 - t^{4s-2})^{-1}$. We multiply this by $(1 + t^{2s})(1 + t^{2s-1})$, make a small reduction, and obtain the Poincaré series for the third group of classes:

$$t^{4s-4}(1 + t^{2s})(1 - t^{2s-1})^{-1}(1 - t^{4s-4})^{-1}.$$

To get the Poincaré series of $\mathscr{B}$, we add the three series obtained so far.

$$P_{\mathscr{B}}(t) = \frac{t^{s-1}}{1-t^{2s-2}} + \frac{t^{2s-2}(1+t^{2s-1})}{1-t^{4s-4}} + \frac{t^{4s-4}(1+t^{2s})}{(1-t^{2s-1})(1-t^{4s-4})}.$$

The stated formula for $P_{\mathscr{B}}(t)$ follows after some reductions.

Finally, Corollary 8.9 gives that $P_\ell(t) = P_{\mathscr{B}}(t) + (1-t^2)^{-1}P_{\widetilde{\Omega}}(t)$ which leads to the stated formula after some reductions. □

9. The spectral sequences for spheres

Let X be a pointed space, $Y = \Sigma X$ it's reduced suspension. We have established a spectral sequence converging to $H^*(E\mathbb{T}\times_{\mathbb{T}}\Lambda Y; \mathbb{F}_2)$ in general. But in this special case, we are fortunate to have a direct calculation of the homology $H_*(E\mathbb{T}_+\wedge_{\mathbb{T}}\Lambda Y; \mathbb{F}_2)$. If the homology of X is of finite type, the (finite) dimensions of these homology and cohomology groups agree. So if we for the particular space X can check that the Poincaré series of the homology of $H^*(E\mathbb{T}\times_{\mathbb{T}}\Lambda Y; \mathbb{F}_2)$ as computed in [8] agrees with the Poincaré series of the E_2 term of our spectral sequence, we know that our spectral sequence collapses. (See also [1] for an easier proof of the results in [8]).

Let us now consider the special case of spheres $X = S^{s-1}$ and $Y = S^s$. The purpose of this section is to show that in this case the two Poincaré series actually agree, forcing the spectral sequence to collapse.

Theorem 9.1. *The Poincaré series of* $H^*(E\mathbb{T}\times_{\mathbb{T}}\Lambda S^s; \mathbb{F}_2)$ *is*

$$(1+t^{s-1}-t^{s+1}+t^{2s-1})(1-t^2)^{-1}(1-t^{2s-2})^{-1}.$$

Proof. We compute a sequence of related Poincaré series. First, let $A = \tilde{H}^*(S^{s-1})$ considered as a graded vector space. This has Poincaré series t^{s-1}.

For each $m \geqslant 1$, the cyclic group C_m acts on $A^{\otimes m}$. This is a 1-dimensional vector space over $\mathbb{F}_2$. We now consider the homology groups

$$H_*(C_m; A^{\otimes m}).$$

We first look at homological dimension 0. $H_0(C_m, A^{\otimes m}) \cong A^{\otimes m}$, so it has Poincaré series $t^{m(s-1)}$. In higher homological degrees, there are two cases. If m is odd, the groups all vanish, and we get a trivial Poincaré series. If m is even, and $i \geqslant 1$, $H_i(C_m, A^{\otimes m}) \cong A^{\otimes m}$. Since this single group has homological degree i, its Poincaré series is $t^{i+m(s-1)}$.

Now, recall from Carlsson and Cohen [8, Proposition 9.3], that

$$\tilde{H}_*(ES^1_+\wedge_{S^1}\Lambda S^s) \cong \oplus_{m\geqslant 1} H_*(C_m; A^{\otimes m}).$$

(Actually, we are correcting a misprint in [8] here. The homology groups on the right-hand side of the formula should not be reduced).

The Poincaré series of the right hand side contains the sum of the contribution of the homology in dimension zero. The Poincaré series of this part is $\sum_{m\geqslant 1} t^{m(s-1)} = t^{s-1}(1-t^{s-1})^{-1}$. It also contains the sums of the contributions of the reduced group homologies. Since this is trivial if m is even, we can as well put $m = 2n$, and the Poincaré series of the reduced part is

$$\sum_{i\geqslant 1}\sum_{n\geqslant 1} t^{i+2n(s-1)} = t^{2(s-1)+1}(1-t)^{-1}(1-t^{2(s-1)})^{-1}.$$

Summing, we get that the Poincaré series for $\tilde{H}_*(ES^1_+\wedge_{S^1}\Lambda(\Sigma X))$ is

$$(t^{s-1} - t^s + t^{2s-2})(1-t)^{-1}(1-t^{2s-2})^{-1}.$$

Finally, we note that there is a short exact sequence of homology groups

$$0 \to \tilde{H}_*(B\mathbb{T}) \to \tilde{H}_*(E\mathbb{T}\times_{\mathbb{T}}\Lambda S^s) \to \tilde{H}_*(E\mathbb{T}_+\wedge_{\mathbb{T}}\Lambda S^s) \to 0.$$

This shows that the Poincaré series of $H_*(E\mathbb{T}\times_{\mathbb{T}}\Lambda S^s)$ is

$$(t^{s-1} - t^s + t^{2s-2})(1-t)^{-1}(1-t^{2s-2})^{-1} + (1-t^2)^{-1}.$$

Bringing on common denominator and adding proves the theorem. □

Proposition 9.2. *The Poincaré series of* $H^*(\Lambda S^s; \mathbb{F}_2)$ *is* $(1+t^s)(1-t^{s-1})^{-1}$ *when* $s \geqslant 2$.

Proof. The mod 2 cohomology ring of ΛS^s is a special case of Theorem 2.2 of Kuribayashi and Yamaguchi [13] except for the case $s = 2$. It is, however, shown (Remark 2.6) that the Eilenberg–Moore spectral sequence also collapses when $s = 2$ so we can compute the Poincaré series from the E_2-term. It has the following form (see the proof of Theorem 2.2):

$$E_2^{*,*} \cong \Lambda(x) \otimes \Lambda(\overline{x}) \otimes \Gamma[\omega],$$

where the respective bidegrees of x, $\overline{x}$ and $\gamma_i(\omega)$ are $(0, s)$, $(-1, s)$ and $(-2i, 2is)$ such that the respective total degrees becomes s, $s-1$ and $2i(s-1)$. Thus the Poincaré series is

$$(1+t^s)(1+t^{s-1})(1-t^{2(s-1)})^{-1}$$

and the result follows by a small reduction. □

Theorem 9.3. *If we let* $X = S^s$ *with* $s \geqslant 2$ *and use* $\mathbb{F}_2$*-coefficients, then the spectral sequences of Theorem* 7.4 *collapses*. *Thus there are isomorphisms of graded* $\mathbb{F}_2$*-vector spaces*:

$$H_*(H^*(S^s); \overline{\Omega})^* \cong H^*(\Lambda S^s) \text{ and } H_*(H^*(S^s); \ell)^* \cong H^*((\Lambda S^s)_{h\mathbb{T}}).$$

Proof. By Theorems 8.10, 9.1 and Proposition 9.2 the Poincaré series of the E_2-terms agree with the Poincaré series of the targets. So the spectral sequences collapses. □

Appendix. On a filtration of the functor ℓ

In this appendix, we identify the graded object associated with the filtration

$$\ell(A) \supseteq u\ell(A) \supseteq u^2\ell(A) \supseteq \dots$$

in the case where $p = 2$ and A is a polynomial algebra.

Recall that the functors $\mathscr{L}, \widetilde{\Omega} : \mathscr{F} \to \mathscr{A}lg$ are defined by $\mathscr{L}(A) = \ell(A)/(u)$ and $\widetilde{\Omega}(A) = \mathscr{L}(A)/I_\delta(A)$ where $I_\delta(A)$ is the ideal $(\delta(x)|x \in A) \subseteq \mathscr{L}(A)$.

We want to define a map $\ell(A) \to \widetilde{\Omega}(A)[t]$ such that the elements $\phi(x)$, $q(x)$ and u in the domain are send to the elements $\phi(x)$, $q(x)$ and t^2 in the target. Unfortunately, this cannot be done by a ring map. But if we pay the penalty of changing the multiplicative structure of the target, we can almost get such a map.

Definition A.1. $\Omega_{\text{tw}}(A)$ is the free graded commutative algebra on generators $\phi(x), q(x)$ for $x \in A$ and t, of degrees $|\phi(x)| = 2|x|$, $|q(x)| = 2|x| - 1$ and $|t| = 1$, modulo the relations

$$\begin{aligned} &q(x+y) = q(x) + q(y), && \phi(x+y) = \phi(x) + \phi(y),\\ &q(xy) = \phi(x)q(y) + \phi(y)q(x), && \phi(xy) = \phi(x)\phi(y),\\ &q(x)^2 = \phi(\lambda x) + tq(\lambda x). \end{aligned}$$

Clearly, $\Omega_{\text{tw}}(A)/(t) \cong \widetilde{\Omega}(A)$. The ring $\Omega_{\text{tw}}(A)$ is just a twisted version of the polynomial ring over $\widetilde{\Omega}(A)$ in t in the following case:

Theorem A.2. *Assume that the underlying algebra of A is a polynomial algebra. Then the graded ring $Gr_*(\Omega_{\text{tw}}(A))$ corresponding to the filtration of $\Omega_{\text{tw}}(A)$ by powers of t equals $\widetilde{\Omega}(A)[t]$.*

Proof. As an intermediate step, let us consider the ring $R(A)$ which is defined exactly like $\Omega_{\text{tw}}(A)$ except that we do not include the last relation $q(x)^2 = \phi(\lambda x) + tq(\lambda x)$.

If A is a polynomial algebra on generators $\{x_i|i \in I\}$, then $R(A)$ is a polynomial ring on generators $\phi(x_i)$ and $q(x_i)$. To obtain $\Omega_{\text{tw}}(A)$ from $R(A)$, we have to add the relations $q(p)^2 = \phi(\lambda p) + tq(\lambda p)$, where p is any polynomial in the generators x_i. Actually, it is sufficient to do this for the generators themselves, as this relation for p_1p_2 follows from the relations for p_1 and p_2. Because, assume those are satisfied, then we calculate

$$\begin{aligned} q(p_1p_2)^2 &= \phi(p_1)^2q(p_2)^2 + \phi(p_2)^2q(p_1)^2\\ &= \phi(p_1)^2(\phi(\lambda p_2) + tq(\lambda p_2)) + \phi(p_2)^2(\phi(\lambda p_1) + tq(\lambda p_1))\\ &= \phi(\lambda(p_1p_2)) + tq(\lambda(p_1p_2)). \end{aligned}$$

Thus we can write $\Omega_{\text{tw}}(A)$ as an algebra

$$\mathbb{F}_2[t, \phi(x_i), q(x_i)|i \in I]/\{q(x_i)^2 = \phi(\lambda x_i) + tq(\lambda x_i)\}.$$

From this it is clear, that $\Omega_{\text{tw}}(A)$ is a free $\mathbb{F}_2[t, \phi(x_i)|i \in I]$-module, with generators

$$\{q(x_{i_1})\ldots q(x_{i_n})|i_r \neq i_s \text{ for } r \neq s,\ n \geqslant 0\}.$$

(The empty product means 1.) It follows that $Gr_*(\Omega_{\text{tw}}(A))$ is a free module over $Gr_*(\mathbb{F}_2[t, \phi(x_i)|i \in I])$ with the same generators.

So, to finish the proof, we only have to determine the multiplicative structure of $\Omega_{\text{tw}}(A)$. The multiplicative relations are given by the relations. In the graded ring they are $q(x_i)^2 = \phi(\lambda x_i)$. So, we have a presentation of the graded ring as

$$\mathbb{F}_2[t, \phi(x_i), q(x_i)|i \in I]/\{q(x_i)^2 = \phi(\lambda x_i)\}.$$

But this is exactly $\widetilde{\Omega}(A)[t]$. □

Theorem A.3. *Let A be an object in $\mathscr{F}$ and $i \geqslant 1$ an integer. Multiplication with u^i defines a natural surjective $\mathbb{F}_2$-linear map*

$$u^i : \widetilde{\Omega}(A) \to \frac{u^i \ell(A)}{u^{i+1}\ell(A)}.$$

If the underlying algebra of A is a polynomial algebra, then this map is an isomorphism and

$$Gr_*(\ell(A)) \cong \mathscr{L}(A) \oplus \bigoplus_{j \geqslant 1} u^j \otimes \widetilde{\Omega}(A).$$

Proof. Multiplication with u^i gives a surjective map $\ell(A) \to u^i\ell(A)$ and $u^i(I_\delta(A)) = 0$, $u^i(u\ell(A)) = u^{i+1}\ell(A)$ so the map factors through $\widetilde{\Omega}(A)$.

We define a natural ring map $v : \ell(A) \to \Omega_{\mathrm{tw}}(A)$ by the formulas

$$v(\phi(x)) = \phi(x) + tq(x), \quad v(q(x)) = q(x), \quad v(u) = t^2, \quad v(\delta(x)) = 0.$$

To see that v is well defined, we have to check that the relations in the definition of ℓ goes to 0. This is trivial for all relations except three which is verified as follows:

$$\begin{aligned} v(\phi(xy)) &= \phi(xy) + tq(xy) = (\phi(x) + tq(x))(\phi(y) + tq(y)) + t^2q(x)q(y) \\ &= v(\phi(x)\phi(y) + uq(x)q(y)), \\ v(q(xy)) &= q(xy) = \phi(x)q(y) + q(x)\phi(y) \\ &= (\phi(x) + tq(x))q(y) + (\phi(y) + tq(y))q(x) \\ &= v(\phi(x)q(y) + q(x)\phi(y)), \\ v(q(x)^2) &= q(x)^2 = v(\phi(\lambda x) + \delta(x^2\lambda x)). \end{aligned}$$

By the map v we get a commutative diagram as follows:

$$\begin{array}{ccc} \widetilde{\Omega}(A) & \longrightarrow & \Omega_{\mathrm{tw}}(A)/t^2\Omega_{\mathrm{tw}}(A) \\ \downarrow{\scriptstyle u^i} & & \downarrow{\scriptstyle t^{2i}} \\ u^i\ell(A)/u^{i+1}\ell(A) & \longrightarrow & t^{2i}\Omega_{\mathrm{tw}}(A)/t^{2i+2}\Omega_{\mathrm{tw}}(A) \end{array}$$

When the underlying algebra of A is a polynomial algebra, then Theorem A.2 gives that the top and the right vertical maps are injective. So in this case the left vertical map is also injective. □

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