Note

Cycle Double Covers of Graphs with Hamilton Paths

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A k-cycle double cover of a graph G is a collection \( \mathcal{Z} \) of at most k eulerian subgraphs of G such that every edge of G is an edge of exactly two subgraphs in \( \mathcal{Z} \). Presented is a short proof of the following theorem due to Tarsi (J. Combin. Theory Ser. B 41 (1986), 332–340): Every bridgeless graph containing a Hamilton path has a 6-cycle double cover.

All graphs are finite in this note. To be consistent with usage in matroid theory, a cycle in a graph is any subgraph in which every vertex has even degree. A circuit is a minimal nonempty cycle. Thus, a cycle is an edge-disjoint union of circuits. A chord of a circuit is an edge which is not in the circuit, but has both of its endpoints in the circuit. A bridge is any edge whose removal increases the number of components in the graph. A graph is cubic if it is regular of degree three. Other than this we use standard graph-theoretic terms. A k-cycle double cover (k-CDC) of a graph G is a collection \( \mathcal{Z} \) of at most k cycles in G (repetitions allowed) such that every edge of G belongs to exactly two of the cycles in \( \mathcal{Z} \).

The cycle double cover conjecture, first formulated in [5, 6], can be stated as:

Conjecture 1. Every bridgeless graph has a k – CDC for some k.

A stronger form of this conjecture was formulated by U. A. Celmins in his thesis [2].

Conjecture 2. Every bridgeless graph has a 5-CDC.

A good survey on the status of these and related conjectures appears in [3]. Here we present a short proof of the following result of Tarsi [4] in connection with Conjecture 2.

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THEOREM 3. Every bridgeless graph with a Hamilton path has a 6-CDC.

To prove this we use the following theorem proved by Bondy and Locke [1, Theorem 1].

THEOREM 4. If $P$ is a path in a 3-connected cubic graph $G$, then there exists a subgraph $H$ with $P \subseteq H \subseteq G$ such that $H$ has a 3-CDC consisting of three circuits.

Proof of Theorem 3. By standard reductions (see, for example, [2, 3] or [4]) we can assume that $G$ is a 3-connected cubic graph with a Hamilton path $P$. Apply Theorem 4 to obtain $H$, $P \subseteq H \subseteq G$, and a 3-CDC, $\mathcal{Z} = \{C_1, C_2, C_3\}$, of $H$, where each $C_i$ is a circuit. Because $H$ is a spanning subgraph of $G$, each endpoint of any edge $e$ in $E(G) - E(H)$ is contained in exactly two of the three circuits in $\mathcal{Z}$. Thus $e$ is a chord of at least one of the three circuits in $\mathcal{Z}$, and $E(G) - E(H)$ has a partition $F_1 \cup F_2 \cup F_3$ such that each edge in $F_i$ is a chord of $C_i$.

For each $i = 1, 2, 3$ such that $F_i \neq \emptyset$ do the following: Let $K_i$ be the unique cubic graph homeomorphic to $C_i \cup F_i$. As the image of $C_i$ under this homeomorphism is a Hamilton circuit in $K_i$ of even length, $K_i$ has a 3-edge coloring $E_1 \cup E_2 \cup E_3$ in which $E_1 \cup E_2$ is the image of $C_i$. Let $A_i$ and $B_i$ be the two cycles in $G$ corresponding to the cycles in $K_i$ induced by $E_1 \cup E_2$ and $E_2 \cup E_3$, respectively. Replace $C_i$ in $\mathcal{Z}$ by the pair of cycles $\{A_i, B_i\}$. Now every edge in $F_i$ is an edge of both $A_i$ and $B_i$, and every edge in $C_i$ is an edge of exactly one of $A_i$ and $B_i$.

The resulting set of at most six cycles is a 6-CDC of $G$.

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REFERENCES