Completion of Partial Matrices to Contractions

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Communicated by C. Foias

Received December 9, 1985; revised April 7, 1986

An n-by-m partially specified complex matrix is called a partial contraction if every rectangular submatrix consisting entirely of specified entries is itself a contraction. Necessary and sufficient condition are given for the pattern of specified entries such that any n-by-m partial contraction with this pattern may be completed to a full n-by-m contraction. © 1986 Academic Press, Inc.

1. INTRODUCTION

An n-by-m complex matrix A is called a contraction if the norm, \( \|A\|_2 \), of A is at most 1. Equivalently, the largest eigenvalue of \( A^*A \), or largest singular value of A, is less than or equal to 1. By a partial matrix we mean an n-by-m array in which some entries are specified (from among the complex numbers \( \mathbb{C} \)), while the remaining entries are “unspecified,” i.e., independent free variables over \( \mathbb{C} \). For example,

\[
\begin{bmatrix}
2 & ? & 1 + i \\
-1/2 & ? & \\
\end{bmatrix}
\]

* The work of this author was supported in part by National Science Foundation Grant DMS-8500372 and Office of Naval Research Contract N00014-86-K-0012.

† The work of this author was supported in part by National Science Foundation Grant DMS-8501794.

0022-1236/86 $3.00

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is a partial matrix in which the 1, 2 and 2, 3 entries are unspecified. A completion, then, of a partial matrix is simply a specification of each of the unspecified entries, resulting in a conventional matrix. Our interest here is in those partial matrices which may be completed to contractions.

Since \( \| \hat{A} \|_2 \leq \| A \|_2 \) if \( \hat{A} \) is a submatrix of the matrix \( A \), it follows that each submatrix of a contraction is again a contraction. If follows that a necessary condition for a partial matrix \( A \) to be completable to a contraction is that each submatrix consisting entirely of specified entries (a specified submatrix) must be a contraction. We call a partial matrix meeting this necessary condition a partial contraction. The question we raise here and answer below is "which patterns for the specified entries of a partial contraction guarantee completability to a contraction?" Partial results in this direction have been obtained in several previous investigations dealing with related problems [4, 8, 13, 14], including the case in which the unspecified entries occur in the form of a single rectangular submatrix, and the problem seems to date back to the Hermitian case of [13]. We were motivated in part by questions raised separately by J. W. Helton and I. Gohberg in connection with engineering control problems. Another important context in which this problem comes up is the Arveson distance formula in nest algebras (see [1, 15]).

Two \( n \)-by-\( m \) matrices (or partial matrices) are said to be permutation equivalent if one may be obtained from the other via independent reordering of the rows and columns (i.e., \( B = Q_1 A Q_2 \) for permutation matrices \( Q_1 \) and \( Q_2 \)). Clearly, permutation equivalence preserves contractions and, therefore, preserves those partial matrices which are completable to contractions and, therefore, the patterns which are completable. Thus, we often state conditions in forms achievable via permutation equivalence.

The following example is indicative of the way in which a pattern may fail to guarantee completability of partial contractions.

**Example 1.** The 2-by-3 partial contraction

\[
\begin{bmatrix}
? & 1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & ? & 1/\sqrt{2}
\end{bmatrix}
\]

has no contraction completion. Indeed, the \(?\) in the 1, 1 position is contained in both submatrices

\[
\begin{bmatrix}
? & 1/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
? & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}.
\]

In order that the former be a contraction, the \(?\) must be 0; in order that the latter be a contraction, the \(?\) must be \(-1/\sqrt{2}\). Since both conditions cannot be met simultaneously, a contraction completion is impossible.
2. The Main Result

Our principal result is the following characterization of those patterns (for the specified entries of partial contractions) which necessarily permit completion to a full contraction.

**Theorem 1.** Let P be an n-by-m pattern of specified entries. Then, there exists a contraction completion for any partial contraction with pattern P for its specified entries if and only if P is permutation equivalent to the following block "diagonal" form

\[
\begin{bmatrix}
B_1 & ? & \cdots & ? \\
? & B_2 & \cdots & \\
\vdots & \vdots & \ddots & \vdots \\
? & \cdots & ? & B_r
\end{bmatrix},
\]

possibly bordered by rows and/or columns of question marks, in which

\[
B_j = \begin{bmatrix}
B_{j11} & ? & \cdots & ? \\
B_{j22} & \cdots & \cdots & \\
\vdots & \vdots & \ddots & \vdots \\
B_{j1r} & B_{j2r} & \cdots & B_{jrr}
\end{bmatrix}, \quad j = 1, \ldots, r,
\]

and the (possibly rectangular) blocks \(B_{jsr}\), \(r \geq s \geq t \geq 1\), consist entirely of specified entries.

We observe that Theorem 1, as well as the subsequent results, are valid, with the same proofs for partial matrices and completions over the real field \(R\).

**Proof.** Assume that \(P\) is permutation equivalent to the block diagonal form as described in Theorem 1. We use the following result (proved in [8, 14]): if \(\begin{bmatrix} A \end{bmatrix}\) and \(\begin{bmatrix} C & D \end{bmatrix}\) are rectangular contractions, then there exists a matrix \(B\) such that the matrix \(\begin{bmatrix} A & B \end{bmatrix}\) is a contraction as well. Applying repeatedly this result, we prove that there exists a contraction completion for any partial contraction with a pattern as in (2). Replacement of the ?'s displayed in (1) by 0's then completes the proof of sufficiency.

Assume now that any partial contraction with pattern \(P\) admits a contraction completion. To prove that \(P\) is permutation equivalent to the block diagonal form, we shall proceed by induction on \(m + n\). It is easily seen (using the result in [8, 14] mentioned above) that Theorem 1 holds
for $m + n \leq 4$. So we can assume $m + n > 4$. Using (if necessary) permutations of rows and columns, we can assume that the first row has the form $\ast \cdots \ast ? \cdots ?$ where the $\ast$'s stand for the specified entries, and the number of $\ast$'s in the first row is maximal among all rows in $P$. Any rectangular subpattern $P'$ of $P$ also has the property that every partial contraction with pattern $P'$ admits a contraction completion (indeed, for a given partial contraction $A'$ with pattern $P'$ let $A$ be a partial contraction with pattern $P$ obtained from $A'$ by putting zeros in the specified entries in $P$ which are not in $P'$; then by our assumption $A$ admits a contraction completion.) So, by the inductive assumption, we can assume that the $(n - 1)$-by-$s$ subpattern of $P$ in the left corner is of the form (1). In fact, $r = 1$ (otherwise one can find in $P$ a $3 \times 2$ subpattern

$$\begin{bmatrix} \ast & \ast \\ \ast & ? \\ ? & \ast \end{bmatrix}$$

which is not always contraction completable as indicated in Example 1). It follows that $P$ has the following block form:

$$\begin{bmatrix} C_1 & C_2 & \cdots & C_p & ? & \cdots & ? \\ B_{11} & ? & \cdots & ? & D_{11} & \cdots & D_{1t} \\ B_{21} & B_{22} & \cdots & ? & D_{21} & \cdots & D_{2t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{p1} & B_{p2} & \cdots & B_{pp} & D_{p1} & \cdots & D_{pt} \\ ? & ? & \cdots & ? & D_{p+1,1} & \cdots & D_{p+1,r} \end{bmatrix}$$

where $[C_1, C_2, \cdots, C_p, ?, \cdots, ?]$ is the first row of $P$, and $B_{ij}$ and $C_k$ consist entirely of specified entries. If one of the entries in $D_{ij}$ $(1 \leq q \leq p - 1)$ were specified, we would find a subpattern $[\ast; \cdots; \ast]$ which contradicts our assumption on $P$ in view of Example 1. Hence all entries in $D_{ij}$ $(1 \leq q \leq p - 1)$ are unspecified. Moreover, the entries in $D_{p1}, \ldots, D_{pt}$ are unspecified as well because the first row in $P$ was chosen to have a maximal number of specified entries. Permuting rows in $P$, we obtain

$$\begin{bmatrix} B_{11} & ? & \cdots & ? & ? & \cdots & ? \\ B_{21} & B_{22} & \cdots & ? & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{p1} & B_{p2} & \cdots & B_{pp} & ? & \cdots & ? \\ C_1 & C_2 & \cdots & C_p & ? & \cdots & ? \\ ? & ? & \cdots & ? & D_{p+1,1} & D_{p+1,2} & \cdots & D_{p-1,r} \end{bmatrix}$$
It remains to use the induction hypothesis for the block

$$[D_{p+1,1} D_{p+1,2} \cdots D_{p+1,t}],$$

and Theorem 1 is proved. \[\square\]

The following Hilbert space version of the contraction completion problem deserves to be mentioned. Let $H_1, \ldots, H_m$ and $K_1, \ldots, K_n$ be complex Hilbert spaces. A partial operator matrix is a bounded linear operator $A = H_1 \oplus \cdots \oplus H_m \to K_1 \oplus \cdots \oplus K_n$, partitioned naturally $A = [A_{ij}]$ ($1 \leq j \leq m; 1 \leq i \leq n$) where some of the operators $A_{ij} = H_j \to K_i$ are specified and some are not. The definitions of partial contraction operator matrices and of their completions are given as in Section 1.

**THEOREM 2.** Let $P$ be an $n$-by-$m$ pattern of specified entries. Then there exists a contraction completion for any partial contraction operator matrix with pattern $P$ for its specified entries if and only if $P$ is permutation equivalent to the block diagonal form (1).

**Proof.** Consider first the 2-by-3 pattern $[\ast \ast \ast]$. By choosing suitable isometries $E_j$ ($1 \leq j \leq 4$), one can check (as in Example 1) that the partial operator matrix contraction $[\ast \ast \ast \ast]$, $\sqrt{2} E_j$ is not completable to a contraction. Now the proof of Theorem 2 is obtained in the same way as the proof of Theorem 1. \[\square\]

We remark that a description of all contraction completions of partial operator matrix contractions with the pattern $[\ast \ast]$ is given in [8]. A relevant problem of completing a lower triangular partial matrix to a unitary was studied in [2], where a description of all possible unitary completions (if they exist) is given as well.

**3. CONNECTION WITH CHORDAL GRAPHS AND FURTHER REMARKS**

We represent the pattern of specified entries in a partial matrix in terms of a bipartite graph. For a given $n$-by-$m$ partial matrix $A$, the vertices of the corresponding (undirected) bipartite graph $G$ consist of two disjoint subsets $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_m\}$. An (undirected) edge $(u_p, v_q)$ occurs in $G$ if and only if the $p, q$ entry of $A$ is specified; $G$ has no edges among the vertices in $U$ nor among the vertices in $V$. We shall write $G = G(\{u_1, \ldots, u_n\}, \{v_1, \ldots, v_m\})$. 
We call the \( n \)-by-\( m \) partial matrix *decomposable* if there exist permutation matrices \( Q_1 \) (\( n \)-by-\( n \)) and \( Q_2 \) (\( m \)-by-\( m \)) such that

\[
Q_1 AQ_2 = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\]

in which the blocks \( B_{12} \) and \( B_{21} \) consist entirely of unspecified entries. (Here, the blocks \( B_{11} \) and \( B_{22} \) may be rectangular, and one of the parts \([B_{12}, B_{22}]\) or \([B_{21}, B_{11}]\), but not both, may be empty). If not, \( A \) is called *indecomposable*. We shall characterize the indecomposable patterns which always admit contraction completions of partial contractions in graph-theoretic terms. We note that in general it suffices to characterize the indecomposable completable patterns, as replacement of the off-diagonal ?'s in (1) by 0's will not increase the spectral norm.

**Theorem 3.** For an \( n \)-by-\( m \) indecomposable pattern \( P \) (with corresponding bipartite graph \( G \)) for the specified entries, the following statements are equivalent:

(a) every \( n \)-by-\( m \) partial contraction with the pattern \( P \) for its specified entries can be completed to an \( n \)-by-\( m \) contraction;

(b) the undirected graph \( G \) obtained from \( G = G(\{u_1, \ldots, u_n\}, \{v_1, \ldots, v_m\}) \) via addition of the edges \((u_j, u_k), 1 \leq j < k \leq n, \) and \((v_j, v_k), 1 \leq j < k \leq m, \) is chordal, i.e., \( G \) contains no minimal simple circuit of length (measured in edges) 4 or more;

(c) the pattern \( P \) has no subpattern (lying in the intersection of 2 rows and 2 columns of \( P \)) of the form \([? \, ?]\) or \([?, ?]\) in which the blanks (resp. question marks) denote specified (resp. unspecified) entries.

(d) the pattern \( P \) has the form (2).

See [10–12] for the properties of chordal graphs and their applications to completion problems.

**Proof.** The equivalence of (a), (c), and (d) follows from Theorem 1. The equivalence (a) \( \Leftrightarrow \) (b) follows from the definition of a chordal graph taking into account that, by the indecomposability assumption, the graph \( G \) is connected.

Our next remark is the connection with the characterization of those patterns for the specified entries of partial positive definite Hermitian matrices which guarantee completability to a positive definite Hermitian matrix. For completeness we describe this result. Let \( A \) be an \( n \)-by-\( n \) partial matrix such that all diagonal entries are specified and real, and if the \( i, j \) entry is specified as \( a_{ij} \), the \( j, i \) entry must be specified as \( \bar{a}_{ji} \). Such a partial matrix \( A \) is called a *partial Hermitian matrix*. If, in addition, all (fully)
specified principal submatrices of $A$ are positive semi-definite, we say that $A$ is a \textit{partial positive semi-definite} (Hermitian) matrix. The notion of a \textit{partial positive definite matrix} is defined analogously.

With any $n$-by-$n$ partial Hermitian matrix $A$ we associate an undirected ("Hermitian") graph $G_H$ as follows: the vertices of $G_H$ are \{1, 2, ..., $n$\} and $(i, j)$ is an undirected edge in $G_H$ if and only if the $i, j$ entry (equivalently, the $j, i$ entry) of $A$ is specified. Since $G_H$ depends only upon the positions of the specified entries in $A$, and not upon their values, $G_H$ describes the \textit{pattern} of specified entries in the partial Hermitian matrix $A$.

A \textit{completion} of the partial Hermitian matrix $A$ is a Hermitian matrix obtained from $A$ by assigning values to each of the unspecified entries in $A$. The following result resolves the question of which patterns for the specified entries guarantee completability to a positive semi-definite matrix of all partial positive semi-definite matrices (whose specified entries have that pattern).

**Theorem 4** [11, 12]. For a pattern $P$ of the specified entries in a partial Hermitian matrix, the following are equivalent: (i) any partial positive semi-definite Hermitian matrix with pattern $P$ of its specified entries admits a positive semi-definite completion; (ii) any partial positive definite Hermitian matrix with pattern $P$ for its specified entries admits a positive definite completion; and (iii) the undirected graph $G_H$ corresponding to the pattern $P$ is chordal.

Completions of partial positive definite Hermitian matrices with the band pattern of specified entries were studied in [9].

The connection of Theorem 4 to the problem of completion of partial contractions to contractions is through the following straightforward observation. If $A$ is an $n$-by-$m$ matrix, then $A$ is a contraction (resp. strict contraction) if and only if the $(n + m)$-by-$(n + m)$ matrix $[I_n^* A^* A I_m]$ is positive semidefinite (resp. positive definite). Indeed,

$$
\begin{bmatrix}
I_n & A \\
A^* & I_m
\end{bmatrix} =
\begin{bmatrix}
I_n & 0 \\
0 & I_m - A^* A
\end{bmatrix}
\begin{bmatrix}
I_n & A \\
0 & I_m
\end{bmatrix}.
$$

This observation, together with Theorem 4, immediately reveals that (b) implies (a) in Theorem 3.

**Acknowledgments**

We thank F. Gilfeather and D. Larson for pointing out to us the connection between contraction completions and hyper-reflexive algebras.
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