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On Lie-Admissible Algebras Whose Commutator Lie Algebras Are Lie Subalgebras of Prime Associative Algebras

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We describe third power associative multiplications $*$ on noncentral Lie ideals of prime algebras and skew elements of prime algebras with involution provided that $x * y - y * x = [x, y]$ for all x, y and the prime algebras in question do not satisfy polynomial identities of low degree. We also obtain necessary and sufficient conditions for these multiplications to be fourth power-associative or flexible. © 2000 Academic Press

1. INTRODUCTION

In what follows $\mathscr{F}$ is an associative commutative ring with unity 1. Given elements x, y of an $\mathscr{F}$-algebra, we set

$$[x, y] = xy - yx \qquad \text{and} \qquad x \circ y = xy + yx.$$

Let $\mathscr{A}$ be an associative algebra. It is well known that $\mathscr{A}$ is a Lie (Jordan) $\mathscr{F}$-algebra with respect to *the Lie product* [,] (respectively, *the Jordan product* $\circ$). Next, a submodule $\mathscr{T}$ of $\mathscr{A}$ is called a *Lie* (*Jordan*) subalgebra of $\mathscr{A}$ if $[\mathscr{T}, \mathscr{T}] \subseteq \mathscr{T}$ (respectively, $\mathscr{T} \circ \mathscr{T} \subseteq \mathscr{T}$). Finally, a submodule $\mathscr{T}$ of $\mathscr{A}$ is said to be *a Lie ideal* of $\mathscr{A}$ if $[\mathscr{T}, \mathscr{A}] \subseteq \mathscr{T}$.

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Let $\mathscr{T}$ and $\mathscr{S}$ be Lie (Jordan) subalgebras of associative $\mathscr{F}$-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. An $\mathscr{F}$-module map $\alpha: \mathscr{T} \to \mathscr{S}$ is said to be a *Lie* (*Jordan*) map if $[x, y]^\alpha = [x^\alpha, y^\alpha]$ (respectively, $(x \circ y)^\alpha = x^\alpha \circ y^\alpha$) for all $x, y \in \mathscr{T}$.

Let $\mathscr{A}$ be an associative $\mathscr{F}$-algebra with involution $*$. We set

$$\mathscr{S}(\mathscr{A}) = \{x \in \mathscr{A} \mid x^* = x\} \qquad \text{and} \qquad \mathscr{K}(\mathscr{A}) = \{x \in \mathscr{A} \mid x^* = -x\}.$$

Clearly $\mathscr{K}(\mathscr{A})$ is a Lie subalgebra while $\mathscr{S}(\mathscr{A})$ is a Jordan subalgebra of $\mathscr{A}$. If $\mathscr{T}$ is a subset of $\mathscr{A}$, then

$$\mathscr{Z}(\mathscr{T}) = \{t \in \mathscr{T} \mid [x, t] = 0 \text{ for all } x \in \mathscr{T}\}$$

is *the center of* $\mathscr{T}$. Next, we denote by $[\mathscr{T}, \mathscr{T}]$ the submodule of $\mathscr{A}$ generated by $\{[x, y] \mid x, y \in \mathscr{T}\}$. Note that $[\mathscr{A}, \mathscr{A}]$ is a Lie ideal of $\mathscr{A}$.

Given a set $\mathscr{S}$ and a nonnegative integer n, we denote by $\mathscr{S}^n$ the nth Cartesian power of $\mathscr{S}$. It is understood that $\mathscr{S}^0 = \varnothing$.

We now set in place some further notation. Let $\mathscr{A}$ be an associative prime $\mathscr{F}$-algebra with maximal right ring of quotients $\mathscr{Q}_{\mathrm{mr}} = \mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$ and Martindale (extended) centroid $\mathscr{C} = \mathscr{C}(\mathscr{A})$ (see [11, Chap. 2]). Further, $\mathscr{A}_c = \mathscr{C}\mathscr{A} + \mathscr{C} \subseteq \mathscr{Q}_{\mathrm{mr}}$ is *the central closure of* $\mathscr{A}$. It is well known that both $\mathscr{Q}_{\mathrm{mr}}$ and $\mathscr{C}$ are $\mathscr{F}$-algebras, $\mathscr{A}$ is a subalgebra of $\mathscr{Q}_{\mathrm{mr}}$, and $\mathscr{C}$ is a field. The algebra $\mathscr{A}$ is called *centrally closed* if $\mathscr{F} = \mathscr{C}$ and $\mathscr{A} = \mathscr{A}_c$.

Let $x \in \mathscr{Q}_{\mathrm{mr}}$. By $\deg(x)$ we shall mean the degree of x over $\mathscr{C}$ (if x is algebraic over $\mathscr{C}$) or ∞ (if x is not algebraic over $\mathscr{C}$). Given a nonempty subset $\mathscr{R} \subseteq \mathscr{Q}_{\mathrm{mr}}$, we set

$$\deg(\mathscr{R}) = \sup\{\deg(x) \mid x \in \mathscr{R}\}.$$

It follows easily from the structure theory of rings with polynomial identity that $\deg(\mathscr{A}) = n < \infty$ if and only if $\mathscr{A}$ satisfies St_{2n}, the standard identity of degree $2n$, and $\mathscr{A}$ does not satisfy St_{2n-2}. Moreover, $\mathscr{A}$ satisfies St_{2n} if and only if it is a subring of $M_n(\overline{\mathscr{C}})$, the $n \times n$ matrix ring over the algebraic closure $\overline{\mathscr{C}}$ of $\mathscr{C}$ [39, 40].

Let $\mathscr{D}$ be a not necessary associative $\mathscr{F}$-algebra. Then $\mathscr{D}^{(-)}$ ($\mathscr{D}^{(+)}$) stands for $\mathscr{D}$ with multiplication $[x, y] = xy - yx$ (respectively, $x \circ y = xy + yx$), $x, y \in \mathscr{D}$. Recall that $\mathscr{D}$ is called *a noncommutative Jordan algebra* if $(xy)x^2 = x(yx^2)$ and

$$(xy)x = x(yx) \qquad \text{for all } x, y \in \mathscr{D} \tag{1}$$

[42, Sect. V.3]. An $\mathscr{F}$-algebra $\mathscr{D}$ satisfying (1) is called *flexible*. By [42, Sect. V.3], $\mathscr{D}$ is a noncommutative Jordan algebra if and only if it satisfies (1) (i.e., is *flexible*) and one of the following conditions:

$$\begin{aligned} &(x^2y)x = x^2(yx) && \text{for all } x, y \in \mathscr{D}, \\ &x^2(xy) = x(x^2y) && \text{for all } x, y \in \mathscr{D}, \\ &(yx)x^2 = (yx^2)x && \text{for all } x, y \in \mathscr{D}, \\ &\mathscr{D}^{(+)} && \text{is a (commutative) Jordan algebra.} \end{aligned} \tag{2}$$

Following [1] the algebra $\mathscr{D}$ is called *Lie-admissible* (*Jordan-admissible*) if $\mathscr{D}^{(-)}$ is a Lie algebra (respectively, $\mathscr{D}^{(+)}$ is a Jordan algebra). Further, let $\mathscr{A}$ be an associative algebra over $\mathscr{F}$ with additional multiplication $*$: $\mathscr{A}^2 \to \mathscr{A}$. Then the multiplication $*$: $\mathscr{A}^2 \to \mathscr{A}$ is called *Lie-compatible* (*Jordan-compatible*) if $(\mathscr{A}, +, *)$ is an $\mathscr{F}$-algebra and there exists an invertible element $c \in \mathscr{F}$ such that $x * y - y * x = c[x, y]$ (respectively, $x * y + y * x = cx \circ y$) for all $x, y \in \mathscr{A}$. Clearly, if $*: \mathscr{A}^2 \to \mathscr{A}$ is a Lie-compatible (Jordan-compatible) multiplication on $\mathscr{A}$, then the algebra $(\mathscr{A}, +, *)$ is Lie-admissible (respectively, Jordan-admissible). If $\mathscr{L}$ is a Lie algebra with additional multiplication $*$: $\mathscr{L}^2 \to \mathscr{L}$, then the multiplication $*$ is called Lie-compatible if $(\mathscr{L}, +, *)$ is an $\mathscr{F}$-algebra and there exists an invertible element $c \in \mathscr{F}$ such that $x * y - y * x = c[x, y]$ for all $x, y \in \mathscr{L}$. Next, the algebra $\mathscr{D}$ is called *third-power associative* if $(xx)x = x(xx)$ for all $x \in \mathscr{D}$. It is called *fourth power-associative* if

$$(xx)(xx) = ((xx)x)x = x((xx)x) = x(x(xx)) = (x(xx))x \quad \text{for all } x \in \mathscr{D}.$$

Finally, $\mathscr{D}$ is called *power-associative* if the subalgebra of $\mathscr{D}$ generated by any element is associative. If $\mathscr{F}$ is a field of characteristic 0, then $\mathscr{D}$ is power-associative if and only if it is third and fourth power-associative (see [1; 36, Lemma 1.11]). If $\mathscr{D}$ is third-power associative, then it is fourth power-associative if and only if

$$((x \circ x) \circ x) \circ x - (x \circ x) \circ (x \circ x) = 0 \qquad \text{for all } x \in \mathscr{D} \tag{3}$$

by [36, Lemma 1.10].

The study of Lie-admissible (Jordan-admissible) algebras was initiated by Albert [1] in 1949. These algebras arise naturally in various areas of mathematics and physics (for further details see [13, 14, 24, 30, 36, 41]). They have been studied by a number of authors (see for example [1, 13, 14, 15, 16, 24, 26, 27, 29, 34–38]). Albert [1] posed a problem of investigating the structure of flexible power-associative Lie-admissible algebras $\mathscr{D}$ such that $\mathscr{D}^{(-)}$ are semisimple Lie algebras. Since then, one of the main directions in studying Lie-admissible algebras has been the problem of classification of Lie-admissible structures on given class of Lie algebras.

In 1962 Laufer and Tomber [29] classified finite dimensional flexible power-associative Lie-admissible algebras $\mathscr{D}$ over algebraically closed fields of characteristic 0 such that $\mathscr{D}^{(-)}$ are semisimple, thus solving Albert's problem for algebras over such fields. Myung [34, 36] obtained a description of finite dimensional flexible power-associative Lie-admissible algebras $\mathscr{D}$ over algebraically closed fields of positive characteristic provided that $\mathscr{D}^{(-)}$ are classical Lie algebras or generalized Witt algebras. Benkart and Osborn [15] and Myung and Okubo [38] independently classified finite dimensional Lie-admissible flexible algebras $\mathscr{D}$ over algebraically closed fields of characteristic 0 such that $\mathscr{D}^{(-)}$ are semisimple. Further, Benkart and Osborn [16] described power-associative products on matrices while Benkart [14] classified third power-associative Lie-admissible algebras $\mathscr{D}$ such that $\mathscr{D}^{(-)}$ are semisimple. Recently Myung [37] described third power-associative Lie-admissible algebras associated with the Virasoro algebras while Jeong et al. [24] classified third power-associative Lie-admissible algebras $\mathscr{D}$ such that $\mathscr{D}^{(-)}$ are affine Kac–Moody algebras.

Let $\mathscr{A}$ and $\mathscr{B}$ be two associative $\mathscr{F}$-algebras and let $\alpha\colon \mathscr{A} \to \mathscr{B}$ be a Lie isomorphism. Define a new multiplication $*\colon \mathscr{A}^2 \to \mathscr{A}$ by the rule $x * y = (x^\alpha y^\alpha)^{\alpha^{-1}}$, $x, y \in \mathscr{A}$. It is easy to see that $*$ is a Lie-compatible third power-associative multiplication. Therefore the problem of classifying Lie isomorphisms of associative algebras is an important particular case of the classification of all Lie-compatible third power-associative multiplication on associative algebras. Now it is not surprising that the methods developed for the solution of the first problem (i.e., the Lie isomorphism problem) are applicable to the solution of the second one in the context of associative algebras. Setting $B(x, y) = x * y$, one immediately gets $[B(x, x), x] = 0$ for all $x \in \mathscr{A}$, an important functional identity investigated by Brešar in [18] in prime rings with connection with the Lie isomorphism problem. Note that, earlier, studying third power-associative Lie-compatible multiplications on matrix algebras over fields, Benkart and Osborn [16] actually solved the same functional identity in this class of algebras, though their results were not stated in such a form.

Our approach to the study of Lie-compatible multiplications is based on recent results in the theory of functional identities obtained in [7, 10] and especially in [8, 9]. The reader is referred to [7, 8, 10] for the historical account of this newly developed theory.

The following four theorems are motivated by the aforesaid results on Lie-admissible algebras.

THEOREM 1.1. *Let $\mathscr{F}$ be a commutative ring with* 1 *and* $\frac{1}{2}$, *let $\mathscr{L}$ be a prime Lie $\mathscr{F}$-algebra*, *and let Ω be a class of Lie $\mathscr{F}$-algebras $\mathscr{R}$ satisfying one* *of the following two conditions*:

(i) *There exists a prime* $\mathscr{F}$*-algebra* $\mathscr{A} = \mathscr{A}(\mathscr{R})$ *with extended centroid* $\mathscr{C}$ *and with central closure* $\mathscr{A}_c$ *such that* $\deg(\mathscr{A}) \geq 5$, $\mathscr{R}$ *is a noncentral Lie ideal of the algebra* $\mathscr{A}$, $\mathscr{C} \cap (\mathscr{C}\mathscr{R}) = 0$, *and* $\mathscr{C}\mathscr{R} + \mathscr{C}$ *is not a subring of* $\mathscr{A}_c$.

(ii) *There exists a prime* $\mathscr{F}$*-algebra* $\mathscr{A} = \mathscr{A}(\mathscr{R})$ *with involution* # *of the first kind, with extended centroid* $\mathscr{C}$ *and with* $\deg(\mathscr{A}) \geq 10$ *such that* $\mathscr{R} = \mathscr{K}(\mathscr{A})$.

Let $\cdot\colon \mathscr{L}^2 \to \mathscr{L}$ *be a Lie-compatible multiplication on* $\mathscr{L}$ *and let* $\mathscr{T}$ *be the extended centroid of the Lie algebra* $\mathscr{L}$. *Suppose that the Lie algebra* $\mathscr{L}$ *is a subdirect product of algebras from the class* Ω. *Then* $\mathscr{T}$ *is a field and we have*:

(a) *The multiplication* $\cdot\colon \mathscr{L}^2 \to \mathscr{L}$ *is third power-associative if and only if there exist an invertible element* $t \in \mathscr{F}$ *and an* $\mathscr{F}$*-linear map* $\mu\colon \mathscr{L} \to \mathscr{T}$ *such that*

$$xy = \tfrac{1}{2}\{t[x, y] + \mu(x)y + \mu(y)x\} \qquad \text{for all } x, y \in \mathscr{L}. \tag{4}$$

(b) *The algebra* $(\mathscr{L}, +, \cdot)$ *is flexible if and only if* (4) *is fulfilled and* $\mu([\mathscr{L}, \mathscr{L}]) = 0$.

(c) *If* $[\mathscr{L}, \mathscr{L}] = \mathscr{L}$, *then the algebra* $(\mathscr{L}, +, \cdot)$ *is flexible if and only if there exists an invertible element* $t \in \mathscr{F}$ *such that* $xy = t[x, y]$ *for all* $x, y \in \mathscr{L}$.

It follows easily from [2] that a free Lie algebra $\mathscr{L}$ over a field generated by an infinite set is a subdirect product of Lie algebras belonging to families B_l, C_l, and D_l, $l = 1, 2, \ldots$. Therefore the above theorem is applicable to $\mathscr{L}$ and in particular a third power-associative multiplication on $\mathscr{L}$ must be given as in (4).

THEOREM 1.2. *Let* $\mathscr{F}$ *be a field with* $\operatorname{char}(\mathscr{F}) \neq 2$, *let* $\mathscr{R}$ *be a Lie subalgebra of a centrally closed prime* $\mathscr{F}$*-algebra* $\mathscr{A}$, *and let* $*\colon \mathscr{R}^2 \to \mathscr{R}$ *be a Lie-compatible multiplication on* $\mathscr{R}$. *Suppose that one of the following two conditions is satisfied*:

(i) $\mathscr{R}$ *is a noncentral Lie ideal of* $\mathscr{A}$, $\deg(\mathscr{A}) \geq 5$, $\mathscr{F} \cap \mathscr{R} = 0$, *and* $\mathscr{R} + \mathscr{F}$ *is not a subalgebra of* $\mathscr{A}$.

(ii) $\deg(\mathscr{A}) \geq 10$, *the algebra* $\mathscr{A}$ *has an involution* # *of the first kind, and* $\mathscr{R} = \mathscr{K}(\mathscr{A})$.

Then we have:

(a) *The multiplication* $*$ *is third power-associative if and only if there exist an invertible element* $t \in \mathscr{F}$ *and an* $\mathscr{F}$*-linear map* $\mu\colon \mathscr{R} \to \mathscr{F}$ *such that*

$$x * y = \tfrac{1}{2}\{t[x, y] + \mu(x)y + \mu(y)x\} \qquad \text{for all } x, y \in \mathscr{R}. \tag{5}$$

If (5) *is fulfilled, then* $(\mathscr{R}, +, \bullet)$ *is a Jordan algebra, where* $x \bullet y = x * y + y * x$ *for all* $x, y \in \mathscr{R}$. *In particular,* (5) *implies the power-associativity of* $*$.

(b) *The algebra* $(\mathscr{R}, +, *)$ *is flexible if and only if* (5) *is fulfilled and* $\mu([\mathscr{R}, \mathscr{R}]) = 0$. *If the algebra* $(\mathscr{R}, +, *)$ *is flexible, then it is a noncommutative Jordan algebra.*

(c) *If* $[\mathscr{R}, \mathscr{R}] = \mathscr{R}$, *then the algebra* $(\mathscr{R}, +, *)$ *is flexible if and only if there exists an invertible element* $t \in \mathscr{F}$ *such that* $x * y = t[x, y]$ *for all* $x, y \in \mathscr{R}$.

THEOREM 1.3. *Let* $\mathscr{F}$ *be a field with* $\operatorname{char}(\mathscr{F}) \neq 2$, *let* $\mathscr{A}$ *be a centrally closed prime* $\mathscr{F}$*-algebra with unity* 1 *and with* $\deg(\mathscr{A}) \geq 3$, *and let* $*: \mathscr{A}^2 \to \mathscr{A}$ *be a Lie-compatible multiplication on* $\mathscr{A}$. *Then the multiplication* $*$ *is third power-associative if and only if there exist a nonzero element* $t \in \mathscr{F}$, *an element* $\lambda \in \mathscr{F}$, *an* $\mathscr{F}$*-linear map* $\mu: \mathscr{A} \to \mathscr{F}$, *and a symmetric* $\mathscr{F}$*-bilinear map* $\tau: \mathscr{A}^2 \to \mathscr{F}$ *such that*

$$x * y = \tfrac{1}{2}\{t[x, y] + \lambda x \circ y + \mu(x)y + \mu(y)x + \tau(x, y)\} \quad \text{for all } x, y \in \mathscr{A}. \tag{6}$$

Further, we have:

(a) *Suppose that* $\deg(\mathscr{A}) \geq 4$. *Then the algebra* $(\mathscr{A}, +, *)$ *is flexible if and only if* (6) *is fulfilled and*

$$\mu([x, y]) = 0 = \tau(x, [x, y]) \qquad \text{for all } x, y \in \mathscr{A}. \tag{7}$$

(b) *Suppose that* $\deg(\mathscr{A}) \geq 5$, (6) *is fulfilled, and* $\lambda \neq 0$. *Then we have*:

(i) *Assume that the multiplication* $*$ *is fourth power-associative. Then*

$$\mu(x \circ y) + \mu(x)\mu(y) + \tau(x, y)\{2\lambda + \mu(1)\} = 0 \qquad \text{for all } x, y \in \mathscr{A}. \tag{8}$$

Define a multiplication $\bullet: \mathscr{A}^2 \to \mathscr{A}$ *as*

$$x \bullet y = x * y + y * x = \lambda x \circ y + \mu(x)y + \mu(y)x + \tau(x, y), \qquad x, y \in \mathscr{A},$$

and let the map $\beta: \mathscr{A} \to \mathscr{A}$ *be given by the rule* $x^\beta = \lambda x + \frac{1}{2}\mu(x)$, $x \in \mathscr{A}$. *Then* β *is a homomorphism of algebras* $(\mathscr{A}, +, \bullet)$ *and* $(\mathscr{A}, +, \circ)$. *If in addition* $(\mathscr{A}, +, *)$ *is flexible and the multiplication* $\diamondsuit: \mathscr{A}^2 \to \mathscr{A}$ *is given by the rule* $x \diamondsuit y = \frac{1}{2}\{\frac{t}{\lambda}[x, y] + x \circ y\}$, $x, y \in \mathscr{A}$, *then* β *is a homomorphism of algebras* $(\mathscr{A}, +, *)$ *and* $(\mathscr{A}, +, \diamondsuit)$.

(ii) *Assume that* $2\lambda + \mu(1) \neq 0$. *Then the multiplication* $*$ *is fourth power-associative if and only if* (8) *is fulfilled. If* (8) *is fulfilled, then* β *is an isomorphism of algebras* $(\mathscr{A}, +, \bullet)$ *and* $(\mathscr{A}, +, \circ)$. *If in addition* $(\mathscr{A}, +, *)$ *is flexible, then* β *is an isomorphism of algebras* $(\mathscr{A}, +, *)$ *and* $(\mathscr{A}, +, \diamond)$.

(iii) *If* $\mathscr{A}$ *is a simple algebra and the multiplication* $*$ *is fourth power-associative, then* $2\lambda + \mu(1) \neq 0$.

(iv) *Suppose that the multiplication* $*$ *is fourth power-associative and* $2\lambda + \mu(1) = 0$. *Then* $\mathscr{I} = \ker(\mu)$ *is an ideal of* $\mathscr{A}$ *such that* $\mathscr{I} \cap \mathscr{F} = 0$ *and* $\mathscr{A} = \mathscr{I} + \mathscr{F}$. *Further,* $\mathscr{F} = \ker(\beta)$ *and is an ideal of the algebra* $(\mathscr{A}, +, \bullet)$ *and* β *induces an isomorphism of the corresponding factor algebra and* $(\mathscr{I}, +, \circ)$. *Finally, if in addition* $(\mathscr{A}, +, *)$ *is flexible, then* $\mathscr{F} = \ker(\beta)$ *is an ideal of the algebra* $(\mathscr{A}, +, *)$ *and* β *induces an isomorphism of the corresponding factor algebra and* $(\mathscr{I}, +, \diamond)$.

We remark that the theorem generalizes results of Benkart and Osborn [16] obtained for matrix algebras over fields. Further, the fourth power-associativity of the multiplication $\bullet$ given by the rule

$$x \bullet y = x * y + y * x = \mu(x)y + \mu(y)x + \tau(x, y)$$

was characterized in [16] (see also [36, Theorem 2.14]). In view of (3), a characterization of the fourth power-associativity of the multiplication $*$ in the case $\lambda = 0$ follows from their result (see Theorem 1.3(b)).

THEOREM 1.4. *Let* $\mathscr{F}$ *be a field with* $\operatorname{char}(\mathscr{F}) \neq 2$, *let* $\mathscr{A}$ *be a centrally closed prime* $\mathscr{F}$*-algebra with unity* 1 *and with* $\deg(\mathscr{A}) \geq 4$, *let* $\mathscr{R}$ *be a Lie ideal of the algebra* $\mathscr{A}$ *such that* $\mathscr{R} \cap \mathscr{F} = 0$ *and* $\mathscr{R} + \mathscr{F} = \mathscr{A}$, *let* $\pi\colon \mathscr{A} \to \mathscr{F}$ *be the canonical projection of the direct sum* $\mathscr{R} + \mathscr{F}$ *of vector spaces* $\mathscr{R}$ *and* $\mathscr{F}$ *onto* $\mathscr{F}$, *and let* $*\colon \mathscr{R}^2 \to \mathscr{R}$ *be a Lie-compatible multiplication on* $\mathscr{R}$. *Then the multiplication* $*$ *is third power-associative if and only if there exist a nonzero element* $t \in \mathscr{F}$, *an element* $\lambda \in \mathscr{F}$, *and an* $\mathscr{F}$*-linear map* $\mu\colon \mathscr{R} \to \mathscr{F}$ *such that*

$$x * y = \tfrac{1}{2}\{t[x, y] + \lambda x \circ y + \mu(x)y + \mu(y)x - \lambda(x \circ y)^{\pi}\} \quad \text{for all } x, y \in \mathscr{R}, \qquad (9)$$

where $x \circ y = xy + yx$. *Further, assume that* $\deg(\mathscr{A}) \geq 5$. *Then*:

(a) *The algebra* $(\mathscr{R}, +, *)$ *is flexible if and only if* (9) *is fulfilled and* $\mu([\mathscr{R}, \mathscr{R}]) = 0$.

(b) *Suppose that* (9) *is fulfilled and* $\lambda = 0$. *Then the multiplication* $*$ *is power-associative.*

(c) *Suppose that* $(\mathcal{R}, +, *)$ *is flexible and* $[\mathcal{R}, \mathcal{R}] = \mathcal{R}$. *Then* $\mu = 0$. *Further, if* $\deg(\mathcal{A}) \geq 6$, *the multiplication* $*$ *is fourth power-associative if and only if either* $\lambda = 0$ *or* $\mathcal{R}$ *is an ideal of the algebra* $\mathcal{A}$. *If the multiplication* $*$ *is fourth power-associative, then it is power-associative.*

(d) *Suppose that* $\deg(\mathcal{A}) \geq 6$, (9) *is fulfilled,* $\lambda \neq 0$, *and the map* β: $\mathcal{R} \to \mathcal{A}$ *is given by the rule* $x^{\beta} = \lambda x + \frac{1}{2}\mu(x)$, $x \in \mathcal{R}$. *We have*:

(i) *Assume that the multiplication* $*$ *is fourth power-associative. Then* β *is an isomorphism of algebras* $(\mathcal{R}, +, \bullet)$ *and* $(\mathcal{R}^{\beta}, +, \circ)$, *where*

$$x \bullet y = x * y + y * x = \lambda x \circ y + \mu(x) y + \mu(y) x - \lambda(x \circ y)^{\pi}$$

and $\mathcal{R}^{\beta}$ *is an ideal of the algebra* $\mathcal{A}$ *such that* $\mathcal{R}^{\beta} \cap \mathcal{F} = 0$ *and* $\mathcal{R}^{\beta} + \mathcal{F} = \mathcal{A}$. *If in addition* $(\mathcal{R}, +, *)$ *is flexible and the multiplication* $\diamond$: $(\mathcal{R}^{\beta})^2 \to \mathcal{R}^{\beta}$ *is given by the rule*

$$x \diamond y = \frac{1}{2}\left\{\frac{t}{\lambda}[x, y] + x \circ y\right\} \qquad \text{for all } x, y \in \mathcal{R}^{\beta},$$

then β *is an isomorphism of algebras* $(\mathcal{R}, +, *)$ *and* $(\mathcal{R}^{\beta}, +, \diamond)$.

(ii) *If* β *is an isomorphism of algebras* $(\mathcal{R}, +, \bullet)$ *and* $(\mathcal{R}^{\beta}, +, \circ)$, *then the multiplication* $*$ *is fourth power-associative.*

(iii) *If* β *is an isomorphism of algebras* $(\mathcal{R}, +, *)$ *and* $(\mathcal{R}^{\beta}, +, \diamond)$, *then the multiplication* $*$ *is power-associative and* $(\mathcal{R}, +, *)$ *is a noncommutative Jordan algebra.*

(e) *If* $\mathcal{A}$ *is a simple algebra with* $\deg(\mathcal{A}) \geq 6$ *and* (9) *is fulfilled, then the multiplication* $*$ *is fourth power-associative if and only if* $\lambda = 0$.

The study of Jordan maps of associative rings goes back to Ancochea [3, 4], Kaplansky [25], Hua [21], and Jacobson and Rickart [23]. In 1956 Herstein [20] described surjective Jordan maps onto prime rings of characteristic not 2 and 3. Smiley [45] removed the restriction $\mathrm{char}(\mathcal{A}) \neq 3$. Further results were obtained by Baxter and Martindale [6], Brešar [17], Lagutina [28], Martindale [31, 32], and McCrimmon [33]. The following theorem is motivated by the aforesaid results.

THEOREM 1.5. *Let* $\mathcal{A}$ *be a prime ring with maximal right ring of quotients* $\mathcal{Q}_{\mathrm{mr}}$ *and with Martindale symmetric ring of quotients* $\mathcal{Q}_s = \mathcal{Q}_s(\mathcal{A})$, *let* $\mathcal{R}$ *be a Jordan subring of* $\mathcal{Q}_{\mathrm{mr}}$, *let* $\mathcal{D}$ *be a flexible ring, and let* α: $\mathcal{D} \to \mathcal{R}$ *be an epimorphism of additive groups such that* $(x \circ y)^{\alpha} = x^{\alpha} \circ y^{\alpha}$ *for all* $x, y \in \mathcal{D}$. *Suppose* $\mathrm{char}(\mathcal{A}) \neq 2$. *We have*:

(a) *If* $\mathcal{R} \supseteq \mathcal{A}$ *and* $\deg(\mathcal{A}) \geq 4$, *then there exists an element* $t \in \mathcal{C}$ *such that*

$$(xy)^{\alpha} = t x^{\alpha} y^{\alpha} + (1 - t) y^{\alpha} x^{\alpha} \qquad \text{for all } x, y \in \mathcal{D}.$$

If in addition $\mathscr{D}$ *is associative*, *then* $t = 0, 1$.

(b) *If* $\mathscr{A}$ *is a prime ring with involution* $*$ *and with* $\deg(\mathscr{A}) \geq 10$, *and* $\mathscr{S}(\mathscr{Q}_s) \supseteq \mathscr{R} \supseteq \mathscr{S}(\mathscr{A})$, *then there exists an element* $t \in \mathscr{C}$ *such that* $t + t^* = 1$

$$(xy)^{\alpha} = tx^{\alpha}y^{\alpha} + t^* y^{\alpha}x^{\alpha} \qquad \textit{for all } x, y \in \mathscr{D}.$$

The paper is organized as follows. All the necessary results and definitions from [8, 9] are listed in the second section. In the third section we prove Theorem 1.5 and discuss its applications to noncommutative Jordan algebras. The last section is devoted to the proof of Theorems 1.1–1.4.

2. PRELIMINARY RESULTS

For the sake of completeness we now state a few results from [8, 9] upon which our paper is based on. We first set in place some notation. In what follows $\mathscr{N}$ is the set of all nonnegative integers, $\mathscr{N}^* = \mathscr{N} \setminus \{0\}$, $\mathscr{F}$ is a commutative ring with unity, $\mathscr{Q}$ is an $\mathscr{F}$-algebra with 1 and with center $\mathscr{C}$, and $\mathscr{C}^*$ is the group of invertible elements of the ring $\mathscr{C}$. Next, $\mathscr{R}$ is a nonempty subset of $\mathscr{Q}$, $\mathscr{S}$ is a nonempty set, and $\alpha: \mathscr{S} \to \mathscr{R}$ is a surjective map. Given $n \in \mathscr{N}^*$ and $s_1, s_2, \ldots, s_n \in \mathscr{S}$, we set $\bar{s}_n = (s_1, s_2, \ldots, s_n) \in \mathscr{S}^n$ where $\mathscr{S}^n$ is the nth Cartesian power of $\mathscr{S}$. Further, let $m \in \mathscr{N}^*$, let $1 \leq i \leq m$, and let $E: \mathscr{S}^{m-1} \to \mathscr{Q}$ be a map (it is understood that E is a constant belonging to $\mathscr{Q}$ whenever $m = 1$). We define a map $E^i: \mathscr{S}^m \to \mathscr{Q}$ by the rule

$$E^i(\bar{x}_m) = E^i(x_1, x_2, \ldots, x_m) = E(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

for all $\bar{x}_m \in \mathscr{S}^m$. Now let $m \geq 2$, $1 \leq i < j \leq m$, and $F: \mathscr{S}^{m-2} \to \mathscr{Q}$. We define a map $F^{ij}: \mathscr{S}^m \to \mathscr{Q}$ by the rule

$$F^{ij}(\bar{x}_m) = F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$$

for all $\bar{x}_m \in \mathscr{S}_m$ and set $F^{ji} = F^{ij}$.

Let $\mathscr{I}, \mathscr{J} \subseteq \{1, 2, \ldots, m\}$ and $E_i, F_j: \mathscr{R}^{m-1} \to \mathscr{Q}$, $i \in \mathscr{I}$, $j \in \mathscr{J}$. Consider *functional identities on* $\mathscr{R}$ of the following form:

$$\sum_{i \in \mathscr{I}} E_i^i(\bar{x}_m)x_i + \sum_{j \in \mathscr{J}} x_j F_j^j(\bar{x}_m) = 0 \qquad \text{for all } \bar{x}_m \in \mathscr{R}^m, \tag{10}$$

$$\sum_{i \in \mathscr{I}} E_i^i(\bar{x}_m)x_i + \sum_{j \in \mathscr{J}} x_j F_j^j(\bar{x}_m) \in \mathscr{C} \qquad \text{for all } \bar{x}_m \in \mathscr{R}^m. \tag{11}$$

Next, consider the following condition: there exist maps

$$p_{ij}\colon \mathscr{R}^{m-2} \to \mathscr{Q}, \qquad i \in \mathscr{I}, \quad j \in \mathscr{J}, \quad i \neq j,$$
$$\lambda_l\colon \mathscr{R}^{m-1} \to \mathscr{C}, \qquad l \in \mathscr{I} \cup \mathscr{J},$$

such that

$$\begin{aligned} E_i^i(\bar{x}_m) &= \sum_{\substack{j \in \mathscr{J},\\ j \neq i}} x_j p_{ij}^{ij}(\bar{x}_m) + \lambda_i^i(\bar{x}_m), \\ F_j^j(\bar{x}_m) &= -\sum_{\substack{i \in \mathscr{I},\\ i \neq j}} p_{ij}^{ij}(\bar{x}_m) x_i - \lambda_j^j(\bar{x}_m) \\ & i \in \mathscr{I}, \qquad j \in \mathscr{J}, \\ & \lambda_l = 0 \qquad \text{if } l \notin \mathscr{I} \cap \mathscr{J}. \end{aligned} \tag{12}$$

It is understood that all the p_{ij}'s are equal to 0 if $m = 1$.

Given $d \in \mathscr{N}^*$, following [8, Definition 1] the subset $\mathscr{R}$ of $\mathscr{Q}$ is called *d-free* if the following conditions are satisfied:

(i) For all $m \in \mathscr{N}^*$ and $\mathscr{I}, \mathscr{J} \subseteq \{1, 2, \dots, m\}$ with $\max\{|\mathscr{I}|, |\mathscr{J}|\} \leq d$, we have that (10) implies (12).

(ii) For all $m \in \mathscr{N}^*$ and $\mathscr{I}, \mathscr{J} \subseteq \{1, 2, \dots, m\}$ with $\max\{|\mathscr{I}|, |\mathscr{J}|\} \leq d - 1$, we have that (11) implies (12).

Assume for a while that $\mathscr{A}$ is a prime ring with maximal right ring of quotients $\mathscr{Q} = \mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$ and with Martindale centroid $\mathscr{C}$ (see [11, Chap. 2]). According to [8, Theorems 2.4 and 2.20] a subset $\mathscr{R}$ of $\mathscr{Q}$ is d-free provided that one of the following conditions is fulfilled:

$$\mathscr{R} = \mathscr{A} \text{ and } \deg(\mathscr{A}) \geq d; \tag{13}$$

$$\mathscr{R} \text{ is a noncentral Lie ideal of } \mathscr{A} \text{ and } \deg(\mathscr{A}) \geq d + 1; \tag{14}$$

$$\mathscr{A} \text{ has an involution}, \deg(\mathscr{A}) \geq 2d + 2, \text{ and } \mathscr{R} \in \{\mathscr{K}(\mathscr{A}), \mathscr{S}(\mathscr{A})\}. \tag{15}$$

THEOREM 2.1 [8, Theorem 2.8]. *Let $d \in \mathscr{N}^*$ and let $\mathscr{B} \subseteq \mathscr{R} \subseteq \mathscr{Q}$ be subsets. Suppose that $\mathscr{B}$ is d-free. Then $\mathscr{R}$ is a d-free subset of $\mathscr{Q}$ as well.*

Let $\{x_1, x_2, \dots, x_m\}$ be noncommuting variables, let $M = x_{i_1} x_{i_2} \cdots x_{i_k}$, $k \leq m$, be a multilinear monomial of degree k in the $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\} \subseteq \{x_1, x_2, \dots, x_m\}$ and let $\bar{s}_m \in \mathscr{S}^m$. We set

$$M(\bar{s}_m) = s_{i_1}^{\alpha}, s_{i_2}^{\alpha} \cdots s_{i_k}^{\alpha} \in \mathscr{Q}.$$

Next, let λ: $\mathscr{S}^{m-k} \to \mathscr{C}$ be a map. Then we denote by λM the map $\mathscr{S}^m \to \mathscr{Q}$ given by the rule

$$(\lambda M)(\bar{s}_m) = \lambda(s_{j_1}, s_{j_2}, \ldots, s_{j_{m-k}}) M(\bar{s}_m) \in \mathscr{Q},$$

where $\{j_1, j_2, \ldots, j_{m-k}\} = \{1, 2, \ldots, m\} \setminus \{i_1, i_2, \ldots, i_k\}$ and $j_1 < j_2 < \cdots < j_{m-k}$. The map λM is called *a multilinear quasi-monomial* of degree $\leq m$ with the coefficient λ of the monomial M. If $\lambda \neq 0$, then we shall say that λM is a multilinear quasi-monomial of degree m. A sum $q(\bar{x}_m) = \sum \lambda_i M_i$ of multilinear quasi-monomials $\lambda_i M_i$ of degree $\leq m$ (where $M_i \neq M_j$ for $i \neq j$) is called *a multilinear quasi-polynomial* of degree $\leq m$ and each λ_i is called the coefficients of the monomial M_i in q. Finally, the coefficient of the monomial 1 in q is called *the constant term of q*.

The following result is particular case of [9, Theorems 1.1] while the next two ones are particular cases of [9, Theorems 1.2].

THEOREM 2.2. *Let $\mathscr{R}$ be a nonempty subset of $\mathscr{Q}$, let $\mathscr{S}$ be a set, let α: $\mathscr{S} \to \mathscr{R}$ be a surjective map of sets, and let q be a multilinear quasi-polynomial of degree $\leq m$ such that $q(\bar{s}_m) = 0$ for all $\bar{s}_m \in \mathscr{S}^m$. Suppose that either the constant term of q is equal to 0 and $\mathscr{R}$ is an m-free subset of $\mathscr{Q}$, or $\mathscr{R}$ is an $(m+1)$-free subset of $\mathscr{Q}$. Then the coefficients of all the monomials in q are equal to 0.*

THEOREM 2.3. *Let $\mathscr{R}$ be a submodule of the $\mathscr{F}$-module $\mathscr{Q}$, let $\mathscr{S}$ be an $\mathscr{F}$-module, let α: $\mathscr{S} \to \mathscr{R}$ be an epimorphism of $\mathscr{F}$-modules, and let B: $\mathscr{S}^2 \to \mathscr{Q}$ be a bilinear map. Suppose that either $\mathscr{R}$ is a 4-free subset of $\mathscr{Q}$ and*

$$[B(s_1, s_2), s_3^\alpha] + [B(s_2, s_3), s_1^\alpha] + [B(s_3, s_1), s_2^\alpha] \in \mathscr{C} \quad \text{for all } \bar{s}_3 \in \mathscr{S}^3,$$

or $\mathscr{R}$ is a 3-free subset of $\mathscr{Q}$ and

$$[B(s_1, s_2), s_3^\alpha] + [B(s_2, s_3), s_1^\alpha] + [B(s_3, s_1), s_2^\alpha] = 0 \quad \text{for all } \bar{s}_3 \in \mathscr{S}^3.$$

Then $B(\bar{s}_2)$ is a multilinear quasi-polynomial of degree ≤ 2 and the coefficients of all the monomials in B are $\mathscr{F}$-multilinear maps.

THEOREM 2.4. *Let $\mathscr{R}$ be a submodule of the $\mathscr{F}$-module $\mathscr{Q}$, let $\mathscr{S}$ be an $\mathscr{F}$-module, let α: $\mathscr{S} \to \mathscr{R}$ be an epimorphism of $\mathscr{F}$-modules, and let B: $\mathscr{S}^2 \to \mathscr{Q}$ be a bilinear map such that*

$$[B(x,u),[v^\alpha, y^\alpha]] + [B(x,v),[u^\alpha, y^\alpha]] + [B(y,u),[v^\alpha, x^\alpha]] + [B(y,v),[u^\alpha, x^\alpha]] = 0$$

for all $x, y, u, v \in \mathscr{S}$. Suppose that $\mathscr{R}$ is a 4-free subset of $\mathscr{Q}$. Then $B(x, y)$ is a multilinear quasi-polynomial of degree ≤ 2 and the coefficients of all the monomials in B are $\mathscr{F}$-multilinear maps.

3. JORDAN-ADMISSIBLE ALGEBRAS

We begin our discussion of Jordan-compatible multiplications with the following result which is inspired by [19, Theorem 3.3].

THEOREM 3.1. *Let $\mathscr{D}$ be a not necessary associative ring, let $\mathscr{R}$ be a 4-free subset of $\mathscr{Q}$, let $\alpha: \mathscr{D} \to \mathscr{R}$ be a surjective map of sets, and let $c \in \mathscr{F}$ be an invertible element such that*

$$(x \circ y)^{\alpha} = cx^{\alpha} \circ y^{\alpha} \qquad \text{for all } x, y \in \mathscr{D}. \tag{16}$$

Further, let $B: \mathscr{D}^2 \to \mathscr{Q}$ be a map such that $B(x, y) = -B(y, x)$ and

$$B(x, y \circ z) = cB(x, y) \circ z^{\alpha} + cB(x, z) \circ y^{\alpha} \qquad \text{for all } x, y, z \in \mathscr{D}.$$

Suppose that $\frac{1}{2} \in \mathscr{C}$. Then there exists an element $\lambda \in \mathscr{C}$ such that $B(x, y) = \lambda[x^{\alpha}, y^{\alpha}]$ for all $x, y \in \mathscr{D}$.

Proof. Since $B(x, y) = -B(y, x)$, it is easy to see that

$$B(x \circ y, z) = cx^{\alpha} \circ B(y, z) + cy^{\alpha} \circ B(x, z) \qquad \text{for all } x, y, z \in \mathscr{D}.$$

On one hand, we have that

$$\begin{aligned} B(x \circ y, u \circ v) &= cB(x, u \circ v) \circ y^{\alpha} + cB(y, u \circ v) \circ x^{\alpha} \\ &= c^2\{B(x, u) \circ v^{\alpha} + B(x, v) \circ u^{\alpha}\} \circ y^{\alpha} \\ &\quad + c^2\{B(y, u) \circ v^{\alpha} + B(y, v) \circ u^{\alpha}\} \circ x^{\alpha} \end{aligned} \tag{17}$$

for all $x, y, u, v \in \mathscr{D}$. On the other hand,

$$\begin{aligned} B(x \circ y, u \circ v) &= cB(x \circ y, u) \circ v^{\alpha} + cB(x \circ y, v) \circ u^{\alpha} \\ &= c^2\{B(x, u) \circ y^{\alpha} + B(y, u) \circ x^{\alpha}\} \circ v^{\alpha} \\ &\quad + c^2\{B(x, v) \circ y^{\alpha} + B(y, v) \circ x^{\alpha}\} \circ u^{\alpha} \end{aligned} \tag{18}$$

for all $x, y, u, v \in \mathscr{D}$. Since $(a \circ b) \circ d - (a \circ d) \circ b = [a, [b, d]]$ for all $a, b, d \in \mathscr{Q}$, subtracting (18) from (17) we get

$$\begin{aligned} &[B(x, u), [v^{\alpha}, y^{\alpha}]] + [B(x, v), [u^{\alpha}, y^{\alpha}]] + [B(y, u), [v^{\alpha}, x^{\alpha}]] \\ &\qquad + [B(y, v), [u^{\alpha}, x^{\alpha}]] = 0 \end{aligned}$$

for all $x, y, u, v \in \mathscr{D}$. As $\mathscr{R}$ is 4-free, Theorem 2.4 yields that there exist elements $\lambda_1, \lambda_2 \in \mathscr{C}$ and maps $\mu_1, \mu_2: \mathscr{D} \to \mathscr{C}$ and $\nu: \mathscr{D}^2 \to \mathscr{C}$ such that

$$B(x, y) = \lambda_1 x^\alpha y^\alpha + \lambda_2 y^\alpha x^\alpha + \mu_1(x) y^\alpha + \mu_2(y) x^\alpha + \nu(x, y) \quad \text{for all } x, y \in \mathscr{D}.$$

Recalling that $B(x, y) = -B(y, x)$ for all $x, y \in \mathscr{D}$, we infer from Theorem 2.2 that $\lambda_1 = -\lambda_2$, $\mu_1 = -\mu_2$, and $\nu(x, y) = -\nu(y, x)$ for all $x, y \in \mathscr{D}$. Setting $\lambda = \lambda_1$ and $\mu = \mu_1$, we see that

$$B(x, y) = \lambda[x^\alpha, y^\alpha] + \mu(x) y^\alpha - \mu(y) x^\alpha + \nu(x, y) \quad \text{for all } x, y \in \mathscr{D}. \tag{19}$$

Substituting $y \circ z$ for y in (19) and making use of (16), we get

$$\begin{aligned} &c\{\lambda[x^\alpha, y^\alpha] + \mu(x) y^\alpha - \mu(y) x^\alpha + \nu(x, y)\} \circ z^\alpha \\ &\quad + c\{\lambda[x^\alpha, z^\alpha] + \mu(x) z^\alpha - \mu(z) x^\alpha + \nu(x, z)\} \circ y^\alpha \\ &\quad = cB(x, y) \circ z^\alpha + cB(x, z) \circ y^\alpha = B(x, y \circ z) \\ &\quad = \lambda c[x^\alpha, y^\alpha \circ z^\alpha] + c\mu(x) y^\alpha \circ z^\alpha - \mu(y \circ z) x^\alpha + \nu(x, y \circ z) \end{aligned}$$

and so Theorem 2.2 implies in particular that $\mu = 0$ and $\nu = 0$. The proof is thereby complete.

COROLLARY 3.2. *Let $\mathscr{A}$ be a prime ring with maximal right ring of quotients $\mathscr{Q}_{mr}$, with Martindale centroid $\mathscr{C}$, and with a Jordan subring $\mathscr{R}$. Let $B: \mathscr{R}^2 \to \mathscr{Q}$ be a biadditive map such that $B(x, y) = -B(y, x)$ and*

$$B(x, y \circ z) = B(x, y) \circ z + B(x, z) \circ y \qquad \text{for all } x, y, z \in \mathscr{R}.$$

Suppose that $\operatorname{char}(\mathscr{A}) \neq 2$ *and either* $\mathscr{R} = \mathscr{A}$ *and* $\deg(\mathscr{A}) \geq 4$, *or* $\mathscr{A}$ *is a prime ring with involution*, $\mathscr{R} = \mathscr{S}(\mathscr{A})$, *and* $\deg(\mathscr{A}) \geq 10$. *Then there exists an element $\lambda \in \mathscr{C}$ such that $B(x, y) = \lambda[x, y]$ for all $x, y \in \mathscr{A}$.*

Proof. By (13) and (15), $\mathscr{R}$ is a 4-free subset of $\mathscr{Q}_{mr}$. The result now follows from Lemma 3.1 (with $\alpha = \mathrm{id}_{\mathscr{R}}$ and $c = 1$).

PROPOSITION 3.3. *Let $\mathscr{D}$ be a flexible $\mathscr{F}$-algebra, let $\mathscr{R}$ be a 4-free Jordan subalgebra of $\mathscr{Q}$, let $\alpha: \mathscr{D} \to \mathscr{R}$ be an epimorphism of $\mathscr{F}$-modules, and let $c \in \mathscr{F}$ be an invertible element such that $(x \circ y)^\alpha = cx^\alpha \circ y^\alpha$ for all $x, y \in \mathscr{D}$. Suppose $\frac{1}{2} \in \mathscr{C}$. Then there exists an element $t \in \mathscr{C}$ such that*

$$(xy)^\alpha = ctx^\alpha y^\alpha + c(1 - t) y^\alpha x^\alpha \qquad \text{for all } x, y \in \mathscr{D}.$$

Proof. By [36, Lemma 1.5],

$$[x \circ y, z] = [y, z] \circ x + [x, z] \circ y \qquad \text{for all } x, y, z \in \mathscr{D}. \tag{20}$$

Define a function B: $\mathscr{D}^2 \to \mathscr{Q}$ by the rule $B(x, y) = [x, y]^{\alpha}$ for all $x, y \in \mathscr{D}$. Clearly $B(x, y) = -B(y, x)$ for all $x, y \in \mathscr{D}$. Applying α to both hands of (20), we get

$$B(x \circ y, z) = cB(y, z) \circ x^{\alpha} + cB(x, z) \circ y^{\alpha} \qquad \text{for all } x, y, z \in \mathscr{D}.$$

Therefore all the conditions of Lemma 3.1 are fulfilled and so there exists an element $a \in \mathscr{C}$ such that $B(x, y) = a[x^{\alpha}, y^{\alpha}]$ for all $x, y \in \mathscr{D}$. Since $2xy = x \circ y + [x, y]$, we see that

$$(xy)^{\alpha} = ctx^{\alpha}y^{\alpha} + c(1 - t)y^{\alpha}x^{\alpha} \qquad \text{for all } x, y \in \mathscr{D},$$

where $t = (ac^{-1} + 1)/2$. The proof is now complete.

PROPOSITION 3.4. *Let $\mathscr{R}$ be a 4-free Jordan $\mathscr{F}$-subalgebra of $\mathscr{Q}$ and let $*$: $\mathscr{R} \times \mathscr{R} \to \mathscr{R}$ be a Jordan-compatible multiplication on $\mathscr{R}$ such that $(\mathscr{R}, +, *)$ is a flexible $\mathscr{F}$-algebra. Suppose that $\frac{1}{2} \in \mathscr{C}$. Then there exist an invertible element $c \in \mathscr{F}$ and an element $t \in \mathscr{C}$ such that*

$$x * y = ctxy + c(1 - t)yx \qquad \textit{for all } x, y \in \mathscr{R}.$$

Proof. Since $*$ is Jordan-compatible, there exists an invertible element $c \in \mathscr{F}$ such that $x * y + y * x = cx \circ y$ for all $x, y \in \mathscr{R}$. The result now follows from Proposition 3.3 with $\alpha = \mathrm{id}_{\mathscr{R}}$.

THEOREM 3.5. *Let $\mathscr{A}$ be a prime $\mathscr{F}$-algebra with maximal right ring of quotients $\mathscr{Q}_{\mathrm{mr}}$ and with extended centroid $\mathscr{C}$. Let $\mathscr{R}$ be a Jordan $\mathscr{F}$-subalgebra of $\mathscr{Q}_{\mathrm{mr}}$ and let $*$: $\mathscr{R} \times \mathscr{R} \to \mathscr{R}$ be a Jordan-compatible multiplication on $\mathscr{R}$ such that $(\mathscr{R}, +, *)$ is flexible. Suppose that* $\mathrm{char}(\mathscr{A}) \neq 2$ *and either $\mathscr{R} \supseteq \mathscr{A}$ and* $\deg(\mathscr{A}) \geq 4$, *or $\mathscr{A}$ is a prime algebra with involution, $\mathscr{R} \supseteq \mathscr{S}(\mathscr{A})$, and* $\deg(\mathscr{A}) \geq 10$. *Then there exist an invertible element $c \in \mathscr{F}$ and an element $t \in \mathscr{C}$ such that*

$$x * y = ctxy + c(1 - t)yx \qquad \textit{for all } x, y \in \mathscr{R}.$$

Moreover, $(\mathscr{R}, +, *)$ is a noncommutative Jordan $\mathscr{F}$-algebra.

Proof. It follows from both (13) and (15) together with Theorem 2.1 that $\mathscr{R}$ is a 4-free subset of $\mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$. The first part of the theorem follows now from Proposition 3.4. The last statement is verified easily.

Probably the above theorem can also be obtained from results of Skosyrskii [43, 44].

PROPOSITION 3.6. *Let $\mathscr{D}$ be an associative ring, let $\mathscr{R}$ be a 4-free Jordan subring of $\mathscr{Q}$, and let $\alpha: \mathscr{D} \to \mathscr{R}$ be a surjective Jordan homomorphism. Suppose that $\frac{1}{2} \in \mathscr{C}$. Then $\mathscr{R}$ is a subring of $\mathscr{Q}$ and α is either a homomorphism of rings, or an antihomomorphism of rings.*

Proof. Since every associative ring is flexible, by Proposition 3.3 there exists an element $t \in \mathscr{C}$ such that

$$(xy)^{\alpha} = tx^{\alpha}y^{\alpha} + (1-t)y^{\alpha}x^{\alpha} \qquad \text{for all } x, y \in \mathscr{D}.$$

Finally, the identity $\{(xy)z\}^{\alpha} = \{x(yz)\}^{\alpha}$ together with Theorem 2.2 implies that $t(1-t) = 0$ and whence $t = 0, 1$. Thus α is a homomorphism (if $t = 1$) or antihomomorphism of rings (if $t = 0$). The proof is now complete.

Proof of Theorem 1.5. If $\mathscr{R} \supseteq \mathscr{A}$ and $\deg(\mathscr{A}) \geq 4$, then $\mathscr{A}$ is a 4-free subset of $\mathscr{Q}_{\mathrm{mr}}$ by (13) and so $\mathscr{R}$ is 4-free by Theorem 2.1. If $\mathscr{A}$ is a prime ring with involution and with $\deg(\mathscr{A}) \geq 10$, and $\mathscr{R} \supseteq \mathscr{S}(\mathscr{A})$, then $\mathscr{S}(\mathscr{A})$ is a 4-free subset of $\mathscr{Q}_{\mathrm{mr}}$ by (15) and whence $\mathscr{R}$ is also 4-free by Theorem 2.1. Therefore in both cases $\mathscr{R}$ is a 4-free subset of $\mathscr{Q}_{\mathrm{mr}}$ and so by Proposition 3.3 there exists an element $t \in \mathscr{C}$ such that

$$(xy)^{\alpha} = tx^{\alpha}y^{\alpha} + (1-t)y^{\alpha}x^{\alpha} \qquad \text{for all } x, y \in \mathscr{D}.$$

Now assume that $\mathscr{A}$ is a prime ring with involution $*$ and $\mathscr{R} = \mathscr{S}(\mathscr{A})$. Since

$$tx^{\alpha}y^{\alpha} + (1-t)y^{\alpha}x^{\alpha} = (xy)^{\alpha} = \left\{(xy)^{\alpha}\right\}^{*} = t^{*}y^{\alpha}x^{\alpha} + (1-t^{*})x^{\alpha}y^{\alpha}$$

for all $x, y \in \mathscr{D}$, Theorem 2.2 yields that in particular $t^{*} = 1 - t$ and so $t + t^{*} = 1$. Finally, assume that $\mathscr{D}$ is an associative ring. Then the identity $\{(xy)z\}^{\alpha} = \{x(yz)\}^{\alpha}$ together with Theorem 2.2 implies that $t(1-t) = 0$ and whence $t = 0, 1$. The proof is now complete.

We conclude our discussion of Jordan-compatible multiplications with the following useful technical result.

LEMMA 3.7. *Let $\mathscr{D}$ be a ring such that $(x \circ x) \circ (x \circ x) = \{(x \circ x) \circ x\} \circ x$ for all $x \in \mathscr{D}$, let $\mathscr{R}$ be a 5-free subset of $\mathscr{Q}$, let $\alpha: \mathscr{D} \to \mathscr{R}$ be an additive surjective map of sets, let $\tau: \mathscr{D}^2 \to \mathscr{C}$ be a symmetric additive map, let $\mu: \mathscr{D} \to \mathscr{C}$ be an additive map, let $\lambda \in \mathscr{C}$ be an invertible element, and let the map $\beta: \mathscr{D} \to \mathscr{Q}$ be given by the rule $x^{\beta} = \lambda x^{\alpha} + \mu(x)$. Suppose that $\frac{1}{2} \in \mathscr{C}$ and*

$$(x \circ y)^{\beta} = x^{\beta} \circ y^{\beta} + \tau(x, y) \qquad \textit{for all } x, y \in \mathscr{D}.$$

Then $\tau = 0$.

Proof. Since $(x \circ x)\circ(x \circ x) = \{(x \circ x)\circ x\}\circ x$ for all $x \in \mathscr{D}$, we have

$$\begin{aligned} 0 &= \left[(x \circ x)\circ(x \circ x) - \{(x \circ x)\circ x\}\circ x\right]^{\beta} \\ &= \{x^{\beta}\circ x^{\beta} + \tau(x,x)\}\circ\{x^{\beta}\circ x^{\beta} + \tau(x,x)\} + \tau(x\circ x, x\circ x) \\ &\quad -\{\{x^{\beta}\circ x^{\beta} + \tau(x,x)\}\circ x^{\beta} + \tau(x\circ x, x)\}\circ x^{\beta} - \tau(\{x\circ x\}\circ x, x) \\ &= 4\tau(x,x)\{x^{\beta}\}^2 - 4\tau(x^2,x)x^{\beta} + 2\tau(x,x)^2 + 4\tau(x^2,x^2) \\ &\quad - \tau(\{x\circ x\}\circ x, x) \end{aligned} \tag{21}$$

for all $x \in \mathscr{D}$. Substituting $\lambda x^{\alpha} + \mu(x)$ for x^{β}, we see that the coefficient of $(x^{\alpha})^2$ in (21) is equal to $4\lambda^2\tau(x,x)$. Both (21) and Theorem 2.2 now imply that $2\tau(x,y) = \tau(x,y) + \tau(y,x) = 0$. The proof is thereby complete.

4. LIE-ADMISSIBLE ALGEBRAS

We start our discussion with the following general result.

PROPOSITION 4.1. *Let* $\mathscr{D}$ *be a (not necessary associative)* $\mathscr{F}$*-algebra, let* β *be an* $\mathscr{F}$*-submodule of* $\mathscr{Q}$, *let* $\alpha\colon \mathscr{D} \to \mathscr{B}$ *be an epimorphism of* $\mathscr{F}$*-modules, let* $\epsilon\colon \mathscr{D}\times\mathscr{D} \to \mathscr{C}$ *be an* $\mathscr{F}$*-bilinear map, and let* $t \in \mathscr{C}$ *be an invertible element such that*

$$[x,y]^{\alpha} = t[x^{\alpha}, y^{\alpha}] + \epsilon(x,y) \qquad \textit{for all } x, y \in \mathscr{D}.$$

Suppose that $\frac{1}{2} \in \mathscr{C}$, $\mathscr{D}$ *is a third power-associative algebra, and either* $\mathscr{B}$ *is a* 4*-free subset of* $\mathscr{Q}$ *or* $\mathscr{B}$ *is a* 3*-free subset of* $\mathscr{Q}$ *and* $\epsilon = 0$. *We have*:

(a) *There exist an element* $\lambda \in \mathscr{C}$, *an* $\mathscr{F}$*-linear map* $\mu\colon \mathscr{D} \to \mathscr{C}$, *and a symmetric* $\mathscr{F}$*-bilinear map* $\tau\colon \mathscr{D}^2 \to \mathscr{C}$ *such that*

$$(x\circ y)^{\alpha} = \lambda x^{\alpha}\circ y^{\alpha} + \mu(x)y^{\alpha} + \mu(y)x^{\alpha} + \tau(x,y) \qquad \textit{for all } x, y \in \mathscr{D}. \tag{22}$$

(b) *If* $\mathscr{B}$ *is* 4*-free and* $\mathscr{D}$ *is a flexible algebra, then*

$$2\lambda\epsilon(x,y) = -\mu([x,y]) \qquad \textit{and}$$
$$\epsilon(x^2,y) = \mu(x)\epsilon(x,y) + \tau(x,[x,y])$$

for all $x, y \in \mathscr{D}$.

(c) *Suppose that* $\mathscr{B}$ *is* 5-*free*, $\mathscr{D}$ *is a fourth power-associative algebra, and* λ *is invertible*. *Then*

$$\tfrac{1}{2}\mu(x \circ y) - \tfrac{1}{2}\mu(x)\mu(y) + \lambda\tau(x, y) = 0 \qquad \textit{for all } x, y \in \mathscr{D} \quad (23)$$

Moreover, if the map $\beta\colon \mathscr{D} \to \mathscr{B}_c = \mathscr{B}\mathscr{C} + \mathscr{C}$ *is given by the rule* $x^\beta = \lambda x^\alpha + \frac{1}{2}\mu(x)$, $x \in \mathscr{D}$, *then*

$$(x \circ y)^\beta = x^\beta \circ y^\beta \qquad \textit{for all } x, y \in \mathscr{D}.$$

Finally, if $\mathscr{D}$ *is flexible, then*

$$[x, y]^\beta = \frac{t}{\lambda}[x^\beta, y^\beta] \qquad \textit{for all } x, y \in \mathscr{D}.$$

Proof. (a) Define a map $B\colon \mathscr{D} \times \mathscr{D} \to \mathscr{Q}$ by the rule $B(x, y) = (x \circ y)^\alpha$, $x, y \in \mathscr{D}$. Clearly B is an $\mathscr{F}$-bilinear map. Since $\mathscr{D}$ is third power-associative, $[x^2, x] = 0$. Linearizing, we see that

$$[x \circ y, z] + [y \circ z, x] + [z \circ x, y] = 0 \qquad \text{for all } x, y, z \in \mathscr{D}.$$

Applying α to both hands of the equation, we get

$$[B(x, y), z^\alpha] + [B(y, z), x^\alpha] + [B(z, x), y^\alpha] \in \mathscr{C} \qquad \text{for all } x, y, z \in \mathscr{D}.$$

Moreover, the left hand of the above equation is equal to 0 provided that $\epsilon = 0$. Now Theorem 2.3 implies that there exists elements $a, b \in \mathscr{C}$, $\mathscr{F}$-linear maps $\mu, \nu\colon \mathscr{D} \to \mathscr{C}$, and an $\mathscr{F}$-bilinear map $\tau\colon \mathscr{D} \times \mathscr{D} \to \mathscr{C}$ such that

$$(x \circ y)^\alpha = B(x, y) = ax^\alpha y^\alpha + by^\alpha x^\alpha + \mu(x)y^\alpha + \nu(y)x^\alpha + \tau(x, y) \qquad \text{for all } x, y \in \mathscr{D}.$$

Recalling that $B(x, y) = B(y, x)$ for all $x, y \in \mathscr{D}$, we infer from Theorem 2.2 that $a = b$, $\mu = \nu$, and $\tau(x, y) = \tau(y, x)$ for all $x, y \in \mathscr{D}$. Setting $\lambda = a$, we get

$$(x \circ y)^\alpha = B(x, y) = \lambda x^\alpha \circ y^\alpha + \mu(x)y^\alpha + \mu(y)x^\alpha + \tau(x, y) \qquad \text{for all } x, y \in \mathscr{D}$$

and so (a) is proved.

(b) By [36, Lemma 1.5], $[x \circ x, y] = 2x \circ [x, y]$ for all $x, y \in \mathscr{D}$. Therefore (22) implies that

$$\frac{t}{2}[\lambda x^{\alpha} \circ x^{\alpha} + 2\mu(x)x^{\alpha}, y^{\alpha}] + \epsilon(x^2, y)$$
$$= \frac{1}{2}[x \circ x, y]^{\alpha} = (x \circ [x, y])^{\alpha}$$
$$= t\lambda x^{\alpha} \circ [x^{\alpha}, y^{\alpha}] + 2\lambda\epsilon(x, y)x^{\alpha} + t\mu(x)[x^{\alpha}, y^{\alpha}]$$
$$+ \mu(x)\epsilon(x, y) + \mu([x, y])x^{\alpha} + \tau(x, [x, y])$$

and whence

$$\{2\lambda\epsilon(x, y) + \mu([x, y])\}x^{\alpha} - \epsilon(x^2, y) + \mu(x)\epsilon(x, y)$$
$$+ \tau(x, [x, y]) = 0$$

for all $x, y \in \mathscr{D}$. It now follows from Theorem 2.2 that in particular

$$2\lambda\epsilon(x, y) = -\mu([x, y]) \quad \text{and}$$
$$\epsilon(x^2, y) = \mu(x)\epsilon(x, y) + \tau(x, [x, y]) \quad \text{for all } x, y \in \mathscr{D}$$

(see also [8, Corollary 2.11]).

(c) Let $\beta\colon \mathscr{D} \to \mathscr{B}_c$ be as in the proposition. Set

$$\sigma(x, y) = \tfrac{1}{2}\mu(x \circ y) - \tfrac{1}{2}\mu(x)\mu(y) + \lambda\tau(x, y)$$

for all $x, y \in \mathscr{D}$. It is easy to see that $(x \circ y)^{\beta} = x^{\beta} \circ y^{\beta} + \sigma(x, y)$ for all $x, y \in \mathscr{D}$. Since $\mathscr{D}$ is third and fourth power-associative, (3) implies that $(x \circ x) \circ (x \circ x) = \{(x \circ x) \circ x\} \circ x$ for all $x \in \mathscr{D}$. It now follows from Lemma 3.7 that $\sigma = 0$. Next, assume that $\mathscr{D}$ is flexible. Then by (b), $2\lambda\epsilon(x, y) = -\mu([x, y])$ for all $x, y \in \mathscr{D}$ and so

$$[x, y]^{\beta} = \lambda[x, y]^{\alpha} + \frac{1}{2}\mu([x, y])$$
$$= t\lambda[x^{\alpha}, y^{\alpha}] + \lambda\epsilon(x, y) + \frac{1}{2}\mu([x, y])$$
$$= t\lambda[x^{\alpha}, y^{\alpha}] = \frac{t}{\lambda}[x^{\beta}, y^{\beta}].$$

The proof is now complete.

THEOREM 4.2. *Let $\mathscr{R}$ be a Lie subalgebra of the $\mathscr{F}$-algebra $\mathscr{Q}$, let $\mathscr{L}$ be a Lie $\mathscr{F}$-algebra, let $\alpha: \mathscr{L} \to \mathscr{R}$ be a surjective homomorphism of Lie algebras, and let $\cdot: \mathscr{L}^2 \to \mathscr{L}$ be a Lie-compatible multiplication on $\mathscr{L}$. Suppose that $\frac{1}{2} \in \mathscr{C}$, $\mathscr{R}$ is a 3-free subset of $\mathscr{Q}$, and the multiplication $\cdot: \mathscr{L}^2 \to \mathscr{L}$ is third power-associative. We have*:

(a) *There exist an invertible element $t \in \mathscr{F}$, an element $\lambda \in \mathscr{C}$, an $\mathscr{F}$-linear map $\mu: \mathscr{L} \to \mathscr{C}$, and a symmetric $\mathscr{F}$-bilinear map $\tau: \mathscr{L}^2 \to \mathscr{C}$ such that*

$$(xy)^{\alpha} = \tfrac{1}{2}\{t[x^{\alpha}, y^{\alpha}] + \lambda x^{\alpha} \circ y^{\alpha} + \mu(x) y^{\alpha} + \mu(y) x^{\alpha} + \tau(x, y)\}$$

for all $x, y \in \mathscr{L}$.

(b) *If in addition $\mathscr{R}$ is 4-free and $(\mathscr{L}, +, \cdot)$ is flexible, then*

$$\mu([x, y]) = 0 = \tau(x, [x, y]) \qquad \text{for all } x, y \in \mathscr{L}.$$

(c) *If in addition $\mathscr{R}$ is 5-free, the multiplication $\cdot: \mathscr{L}^2 \to \mathscr{L}$ is fourth power-associative, λ is invertible, and the map $\beta: \mathscr{L} \to \mathscr{R}_c = \mathscr{R}\mathscr{C} + \mathscr{C}$ is given by the rule $x^{\beta} = \lambda x^{\alpha} + \frac{1}{2}\mu(x)$, $x \in \mathscr{L}$, then*

$$\tfrac{1}{2}\mu(x \circ y) - \tfrac{1}{2}\mu(x)\mu(y) + \lambda\tau(x, y) = 0 \qquad \text{for all } x, y \in \mathscr{L}$$

and $(x \circ y)^{\beta} = x^{\beta} \circ y^{\beta}$ for all $x, y \in \mathscr{L}$. Finally, if in addition $(\mathscr{L}, +, \cdot)$ is flexible, then

$$[x, y]^{\beta} = \frac{t}{\lambda}[x^{\beta}, y^{\beta}] \qquad \text{for all } x, y \in \mathscr{L}.$$

Proof. By assumption there exists an invertible element $t \in \mathscr{F}$ such that $xy - yx = t[x, y]$ and so $(xy - yx)^{\alpha} = t[x^{\alpha}, y^{\alpha}]$ for all $x, y \in \mathscr{L}$. Clearly $xy = \frac{1}{2}(x \circ y + t[x, y])$ for all $x, y \in \mathscr{L}$. The result now follows from Proposition 4.1 with $\epsilon = 0$, $\mathscr{D} = (\mathscr{L}, +, \cdot)$ and $\mathscr{B} = \mathscr{R}$.

Given a (not necessary associative) $\mathscr{F}$-algebra $\mathscr{D}$, we denote by $M^*(\mathscr{D})$ ($M(\mathscr{D})$) the subalgebra of the $\mathscr{F}$-algebra $\mathrm{End}({}_{\mathscr{F}}\mathscr{D})$ generated by all left and right multiplications (respectively, by $M^*(\mathscr{D})$ and $\mathrm{id}_{\mathscr{D}}$). The algebra $M^*(\mathscr{D})$ ($M(\mathscr{D})$) is called *the multiplication ideal* (respectively, *the multiplication algebra*) of $\mathscr{D}$. It is well known that $\mathscr{D}$ is a left unital module over $M(\mathscr{D})$ and $M^*(\mathscr{D})$ is an ideal of $M(\mathscr{D})$ (see [47]). Given $p \in M(\mathscr{D})$ and $x \in \mathscr{D}$, we denote by $p \cdot x$ their product.

The concept of the extended centroid of a nonassociative semiprime ring was introduced by Baxter and Martindale [5]. A different approach to the definition of the extended centroid and the central closure of such rings was found by Wisbauer [46]. The reader is referred to [5, 46, 47] for the definition and basic properties of the extended centroid and central closure of semiprime rings. Let $\mathscr{D}$ be a (not necessary associative) semiprime $\mathscr{F}$-algebra with extended centroid $\mathscr{T}$. By [47, 32.1], $\mathscr{T}$ is a commutative von Neumann regular self-injective $\mathscr{F}$-algebra. Moreover $\mathscr{T}$ is a field provided that $\mathscr{D}$ is a prime algebra. Given a nonempty subset $\mathscr{S} \subseteq \mathscr{D}$, there exists a uniquely determined idempotent $E(\mathscr{S}) \in \mathscr{T}$ such that

$$r(\mathscr{T}; \mathscr{S}) = \{c \in \mathscr{T} \mid c\mathscr{S} = 0\} = (1 - E(\mathscr{S}))\mathscr{T}$$

(see [12, 33.3; 47, 32.3(3)]). Further, let $\mathscr{I}_{\mathscr{S}}$ be the ideal of the algebra $\mathscr{D}$ generated by the subset $\mathscr{S}$ and let $\mathscr{J}$ be an ideal of $\mathscr{D}$ such that $\mathscr{J} \cap \mathscr{I}_{\mathscr{S}} = 0$ and $\mathscr{I}_{\mathscr{S}} + \mathscr{J}$ is an essential ideal of $\mathscr{D}$. Then $E(\mathscr{S})$ is a projection of the left $M(\mathscr{D})$-module $\mathscr{I}_{\mathscr{S}} \oplus \mathscr{J}$ onto $\mathscr{I}_{\mathscr{S}}$ (see [47, 32.3(3); 12, 33.3]). That is to say,

$$E(\mathscr{S})(\mathscr{I}_{\mathscr{S}} + \mathscr{J}) = \mathscr{I}_{\mathscr{S}}. \tag{24}$$

Given $c \in \mathscr{T}$, we set $(c : \mathscr{D}) = \{d \in \mathscr{D} \mid cd \in \mathscr{D}\}$. Clearly $(c : \mathscr{D})$ is an ideal of $\mathscr{D}$. Moreover, $(c : \mathscr{D})$ is an essential ideal of $\mathscr{D}$ by [47, 32.1(3)].

We continue with the following general observation.

PROPOSITION 4.3. *Let $\mathscr{D}$ be a nonassociative $\mathscr{F}$-algebra, let $\{\mathscr{D}_i \mid i \in \mathscr{I}\}$ be a family of nonassociative prime $\mathscr{F}$-algebras, let $\pi_i\colon \mathscr{D} \to \mathscr{D}_i$, $i \in \mathscr{I}$, be surjective homomorphisms of $\mathscr{F}$-algebras, let $\mathscr{T} = \mathscr{C}(\mathscr{D})$ be the extended centroid of $\mathscr{D}$, let $\mathscr{T}_i = \mathscr{C}(\mathscr{D}_i)$, $i \in \mathscr{I}$, let $n \in \mathscr{N}^*$, let $a_1, a_2, \dots, a_n \in \mathscr{D}$, and let*

$$\mathscr{T}^{(i)} = \{\lambda \in \mathscr{T} \mid (\lambda : \mathscr{D}) \not\subseteq \ker(\pi_i)\}, \qquad i \in \mathscr{I}.$$

Suppose that $\bigcap_{i \in \mathscr{I}} \ker(\pi_i) = 0$. Then $\mathscr{T}$ is a commutative von Neumann regular self-injective ring and we have:

(a) *Suppose that $a_n^{\pi_i} \in \sum_{k=1}^{n-1} \mathscr{T}_i a_k^{\pi_i}$ for all $i \in \mathscr{I}$. Then $a_n \in \sum_{k=1}^{n-1} \mathscr{T} a_k$.*

(b) *For every $i \in \mathscr{I}$, $\mathscr{T}^{(i)}$ is a subalgebra of the $\mathscr{F}$-algebra $\mathscr{T}$ and there exists a homomorphism of $\mathscr{F}$-algebras $\beta_i\colon \mathscr{T}^{(i)} \to \mathscr{T}_i$ such that $(\lambda x)^{\pi_i} = \lambda^{\beta_i} x^{\pi_i}$ for all $\lambda \in \mathscr{T}^{(i)}$ and $x \in (\lambda : \mathscr{D})$. Further, if $x \in \mathscr{D}$ with $x^{\pi_i} \neq 0$, then $E(x) \in \mathscr{T}^{(i)}$ and $E(x)^{\beta_i} = 1$.*

(c) *Let $\mathscr{J}$ be an essential ideal of $\mathscr{D}$, let $a, b \in \mathscr{T}$ with $a\mathscr{J} \cup b\mathscr{J} \subseteq \mathscr{D}$, and let $\mathscr{K}(\mathscr{J}) = \{i \in \mathscr{I} \mid \mathscr{J}^{\pi_i} \neq 0\}$. Then $\bigcap_{i \in \mathscr{K}(\mathscr{J})} \ker(\pi_i) = 0$. If $a^{\beta_i} = b^{\beta_i}$ for all $i \in \mathscr{K}(\mathscr{J})$, then $a = b$.*

Proof. Clearly $\mathscr{D}$ is a semiprime ring and so $\mathscr{T}$ is a commutative von Neumann regular self-injective ring.

(a) Suppose that $a_n \notin \sum_{k=1}^{n-1} \mathscr{T} a_k$. By [47, 32.2(7)] there exists $p \in M(\mathscr{D})$ such that $a = p \cdot a_n \neq 0$ and $p \cdot a_k = 0$ for all $k = 1, 2, \dots, n-1$.

If $M^*(\mathscr{D}) \cdot a = 0$, then $\mathscr{F}a$ is an ideal of $\mathscr{D}$ with $(\mathscr{F}a)^2 = 0$ which is impossible. Therefore $p' \cdot a \neq 0$ for some $p' \in M^*(\mathscr{D})$. Set $q = p'p$. Then $q \in M^*(\mathscr{D})$, $q \cdot a_n \neq 0$, and $q \cdot a_k = 0$ for all $k = 1, 2, \dots, n-1$. By assumption there exists $i \in \mathscr{I}$ such that $(q \cdot a_n)^{\pi_i} \neq 0$. Clearly π_i induces a surjective homomorphism of $\mathscr{F}$-algebras $\gamma\colon M^*(\mathscr{D}) \to M^*(\mathscr{D}_i)$ such that $(f \cdot x)^{\pi_i} = f^\gamma \cdot x^{\pi_i}$ for all $f \in M^*(\mathscr{D})$ and $x \in \mathscr{D}$. Set $g = q^\gamma$ and $b_k = a_k^{\pi_i}$, $k = 1, 2, \dots, n$. Then $b_n = \sum_{k=1}^{n-1} \lambda_k b_k$ for some $\lambda_k \in \mathscr{T}_i$ by our assumption. Therefore

$$0 \neq (q \cdot a_n)^{\pi_i} = g \cdot b_n = \sum_{k=1}^{n-1} \lambda_k (g \cdot b_k) = \sum_{k=1}^{n-1} \lambda_k (q \cdot a_k)^{\pi_i} = 0,$$

a contradiction. Therefore (a) is proved.

(b) Let $\lambda_1, \lambda_2 \in \mathscr{T}^{(i)}$ and $u, v \in \mathscr{F}$. Set $\mathscr{U} = (\lambda_1 : \mathscr{D}) \cap (\lambda_2 : \mathscr{D})$. Since $\ker(\pi_i)$ is a prime ideal of the algebra $\mathscr{D}$ and $(\lambda_1 : \mathscr{D}), (\lambda_2 : \mathscr{D}) \not\subseteq \ker(\pi_i)$, we conclude that $\mathscr{U} \not\subseteq \ker(\pi_i)$. Clearly $\mathscr{U}$ is an essential ideal of $\mathscr{D}$ and $(u\lambda_1 + v\lambda_2)\mathscr{U} \subseteq \mathscr{D}$. Therefore $u\lambda_1 + v\lambda_2 \in \mathscr{T}^{(i)}$ and so $\mathscr{T}^{(i)}$ is an $\mathscr{F}$-submodule of $\mathscr{T}$.

Let $\lambda \in \mathscr{T}^{(i)}$. Set $\mathscr{J} = (\lambda : \mathscr{D})$, $\pi = \pi_i$, $\mathscr{B} = \mathscr{D}_i$, and $\mathscr{K} = \mathscr{J}^\pi$. We define a map $f_\lambda\colon \mathscr{K} \to \mathscr{B}$ by the rule $f_\lambda(x) = \{\lambda x^{\pi^{-1}}\}^\pi$, $x \in \mathscr{K}$. We claim that f_λ is a well-defined map. Indeed, let $a \in \mathscr{J} \cap \ker(\pi)$ and let $\mathscr{L} = M(\mathscr{D}) \cdot (\lambda a)$. Clearly $\mathscr{L}$ is a nonzero ideal of the algebra $\mathscr{D}$ and $\mathscr{J}\mathscr{L} = (\lambda\mathscr{J})(M(\mathscr{D}) \cdot a) \subseteq \ker(\pi)$. Since $\mathscr{J}^\pi \neq 0$ and $\mathscr{B}$ is a prime ring, we conclude that $\mathscr{L}^\pi = 0$ and so $\lambda a \in \ker(\pi)$. Therefore f_λ is well defined. Obviously f_λ is homomorphism of $M(\mathscr{B})$-modules and so there exists a uniquely defined element $c_\lambda \in \mathscr{T}_i$ such that $f_\lambda(x) = c_\lambda x$ for all $x \in \mathscr{K}$. Define a map $\beta = \beta_i\colon \mathscr{T}^{(i)} \to \mathscr{T}_i$ by the rule $\lambda^\beta = c_\lambda$. Clearly $(\lambda x)^\pi = \lambda^\beta x^\pi$ for all $\lambda \in \mathscr{T}^{(i)}$ and $x \in (\lambda : \mathscr{D})$.

Let $\zeta, \eta \in \mathscr{T}^{(i)}$. Set $\mathscr{M} = (\zeta : \mathscr{D})$ and $\mathscr{N} = (\eta : \mathscr{D})$. Since $\mathscr{M}^\pi \neq 0 \neq \mathscr{N}^\pi$ and $\mathscr{B}$ is a prime algebra, we conclude that $(\mathscr{M} \cap \mathscr{N})^\pi \neq 0$. Let $\mathscr{U} = \mathscr{M} \cap \mathscr{N}$. We have

$$\zeta^\beta x^\pi + \eta^\beta x^\pi = (\zeta x)^\pi + (\eta x)^\pi = \{(\zeta + \eta)x\}^\pi = (\zeta + \eta)^\beta x^\pi$$

for all $x \in \mathscr{U}$ and so $\{\zeta^\beta + \eta^\beta - (\zeta + \eta)^\beta\}\mathscr{U}^\pi = 0$. Since $\mathscr{U}^\pi$ is a nonzero ideal of the prime algebra $\mathscr{B}$, we conclude that $(\zeta + \eta)^\beta = \zeta^\beta + \eta^\beta$. Analogously one can show that $(f\zeta)^\beta = f\zeta^\beta$ for all $f \in \mathscr{F}$ and $\zeta \in \mathscr{T}^{(i)}$. Finally, $(\mathscr{M}\mathscr{N})^\pi = \mathscr{M}^\pi\mathscr{N}^\pi \neq 0$ because $\mathscr{B}$ is prime. Let $\mathscr{W}$ be the ideal of $\mathscr{D}$ generated by $\mathscr{M}\mathscr{N}$. Then $\mathscr{W} \not\subseteq \ker(\pi)$ and $\zeta\mathscr{W} \cup \eta\mathscr{W} \cup \zeta\eta\mathscr{W} \subseteq \mathscr{D}$. We show that $\mathscr{W}$ is an essential ideal of $\mathscr{D}$. Indeed, let $\mathscr{K}$ be a nonzero ideal of $\mathscr{D}$. As $\mathscr{D}$ is semiprime, $0 \neq (\mathscr{K} \cap \mathscr{M} \cap \mathscr{N})^2 \subseteq \mathscr{W} \cap \mathscr{K}$ and so $\mathscr{W}$ is an essential ideal of $\mathscr{D}$. Therefore $\zeta\eta \in \mathscr{T}^{(i)}$ and whence $\mathscr{T}^{(i)}$ is a subalgebra of $\mathscr{T}$. Further,

$$(\zeta\eta)^{\beta} x^{\pi} = (\zeta\eta x)^{\pi} = [\zeta(\eta x)]^{\pi} = \zeta^{\beta}(\eta x)^{\pi} = \zeta^{\beta}\eta^{\beta}x^{\pi}$$

for all $x \in \mathscr{W}$ and whence $[(\zeta\eta)^{\beta} - \zeta^{\beta}\eta^{\beta}]\mathscr{W}^{\pi} = 0$. Thus $(\zeta\eta)^{\beta} = \zeta^{\beta}\eta^{\beta}$.

Finally, let $x \in \mathscr{D}$. Let $\mathscr{I}_x$ be the ideal of $\mathscr{D}$ generated by x and let $\mathscr{J}$ be an ideal of $\mathscr{D}$ maximal with respect to the property $\mathscr{I}_x \cap \mathscr{J} = 0$. Clearly $\mathscr{I}_x + \mathscr{J}$ is an essential ideal of $\mathscr{D}$ contained in $(E(x) : \mathscr{D})$ by (24). Since $x^{\pi} \neq 0$, $\mathscr{I}_x^{\pi} \neq 0$ as well and so $E(x) \in \mathscr{T}^{(i)}$. Further,

$$x^{\pi_i} = (E(x)x)^{\pi_i} = E(x)^{\beta_i} x^{\pi_i}$$

and whence $E(x)^{\beta_i} = 1$ because $\mathscr{T}_i$ is a field.

(c) Let $\mathscr{U} = \bigcap_{i \in \mathscr{I}(\mathscr{J})} \ker(\pi_i)$ and let $\mathscr{V} = \bigcap_{i \in \mathscr{I} \setminus \mathscr{I}(\mathscr{J})} \ker(\pi_i)$. Then $\mathscr{J} \subseteq \mathscr{V}$ and so $\mathscr{V}$ is an essential ideal of $\mathscr{D}$. Since $\mathscr{U} \cap \mathscr{V} = \bigcap_{i \in \mathscr{I}} \ker(\pi_i) = 0$, we conclude that $\mathscr{U} = 0$.

Suppose that $a^{\beta_i} = b^{\beta_i}$ for all $i \in \mathscr{I}(\mathscr{J})$. Set $c = a - b$. Then $c\mathscr{J} \subseteq \mathscr{D}$ and $c^{\beta_i} = 0$ for all $i \in \mathscr{I}(\mathscr{J})$. Assume that $c \neq 0$. Then $cy \neq 0$ for some $y \in \mathscr{J}$ and so there exists $i \in \mathscr{I}(\mathscr{J})$ with $(cy)^{\pi_i} \neq 0$. But $(cy)^{\pi_i} = c^{\beta_i} y^{\pi_i} = 0$, a contradiction. The proof is thereby complete.

Proof of Theorem 1.1. Let $\{(\mathscr{R}_p; \pi_p) \mid p \in \mathscr{P}\}$ be a family of Lie algebras $\mathscr{R}_p \in \Omega$ together with surjective Lie homomorphisms $\pi_p : \mathscr{L} \to \mathscr{R}_p$ such that $\bigcap_{p \in \mathscr{P}} \ker(\pi_p) = 0$. Further, let $\mathscr{A}_p = \mathscr{A}(\mathscr{R}_p)$ and $\mathscr{C}_p$ be as in Theorem 1.1(i) or (ii), $p \in \mathscr{P}$. By (14), (15), each $\mathscr{R}_p$ is a 4-free subset of $\mathscr{Q}_{\text{mr}}(\mathscr{A}_p)$.

Given any nonzero ideal $\mathscr{I}$ of the Lie algebra $\mathscr{L}$, we set

$$P(\mathscr{I}) = \{p \in \mathscr{P} \mid \mathscr{I}^{\pi_p} \neq 0\}.$$

Next, we note that $\mathscr{T}$ is a field because $\mathscr{L}$ is prime, and set $\hat{\mathscr{L}} = \mathscr{L}\mathscr{T}$ (see [47, 32.2(1)]). Finally, let $\mathscr{T}^{(p)}$ and $\beta_p : \mathscr{T}^{(p)} \to \mathscr{C}_p$ be as in Proposition 4.3.

(a) First suppose that $\cdot : \mathscr{L}^2 \to \mathscr{L}$ is third power-associative. We claim that

$$x \circ y = xy + yx \in \mathscr{T}x + \mathscr{T}y \qquad \text{for all } x, y \in \mathscr{L}. \tag{25}$$

Indeed, by Proposition 4.3(a) it is enough to show that $(x \circ y)^{\pi_p} \in \mathscr{C}_p x^{\pi_p} + \mathscr{C}_p y^{\pi_p}$ for all $p \in \mathscr{P}$. Since $\mathscr{R}_p$ is a 4-free subset of $\mathscr{Q}_{\text{mr}}(\mathscr{A}_p)$, Theorem 4.2(a) yields that there exist an element $\lambda_p \in \mathscr{C}_p$, an $\mathscr{F}$-linear map ν_p: $\mathscr{L} \to \mathscr{C}_p$, and a symmetric $\mathscr{F}$-bilinear map τ_p: $\mathscr{L}^2 \to \mathscr{C}_p$ such that

$$(u \circ v)^{\pi_p} = \lambda_p u^{\pi_p} \circ v^{\pi_p} + \nu_p(u)v^{\pi_p} + \nu_p(v)u^{\pi_p} + \tau_p(u, v)$$

for all $u, v \in \mathscr{L}$.

Clearly $\nu_p(u)v^{\pi_p} + \nu_p(v)u^{\pi_p} \in \mathscr{C}_p\mathscr{R}_p$. Now assume that $\mathscr{R}_p$ and $\mathscr{A}_p$ are as in Theorem 1.1(i). If $\lambda_p \neq 0$, then $u^{\pi_p} \circ v^{\pi_p} \in \mathscr{C}_p\mathscr{R}_p + \mathscr{C}_p$ for all $u, v \in \mathscr{L}$. Since π_p is surjective, we conclude that $\mathscr{R}_p \circ \mathscr{R}_p \subseteq \mathscr{C}_p\mathscr{R}_p + \mathscr{C}_p$ and whence $\mathscr{C}_p\mathscr{R}_p + \mathscr{C}_p$ is a subring of $\mathscr{C}_p\mathscr{A}_p + \mathscr{C}_p$, a contradiction. Therefore $\lambda_p = 0$ and so $\tau_p(u,v) \in \mathscr{C}_p \cap (\mathscr{C}_p\mathscr{R}_p) = 0$ for all $u, v \in \mathscr{L}$. Hence $(x \circ y)^{\pi_p} \in \mathscr{C}_p x^{\pi_p} + \mathscr{C}_p y^{\pi_p}$. Finally, assume that $\mathscr{R}_p$ and $\mathscr{A}_p$ are as in Theorem 1.1(ii). It is well known that the involution # can be uniquely extended up to the involution of the same kind of $\mathscr{C}_p\mathscr{A}_p + \mathscr{C}_p$ (see [11, Proposition 2.5.4]). Then $\lambda_p u^{\pi_p} \circ u^{\pi_p} + \tau_p(u,v)$ is a symmetric element of $\mathscr{C}_p\mathscr{A}_p + \mathscr{C}_p$ belonging to $\mathscr{C}_p\mathscr{R}_p \subseteq \mathscr{K}(\mathscr{C}_p\mathscr{A}_p + \mathscr{C}_p)$ and so it is equal to 0. Therefore again $(x \circ y)^{\pi_p} \in \mathscr{C}_p x^{\pi_p} + \mathscr{C}_p y^{\pi_p}$ which proves our claim. It also follows from our discussion that

$$(u \circ v)^{\pi_p} = \nu_p(u)v^{\pi_p} + \nu_p(v)u^{\pi_p} \qquad \text{for all } u, v \in \mathscr{L}. \tag{26}$$

It follows from (25) that $x \circ x \in \mathscr{T}x$ for all $x \in \mathscr{L}$. Define a map $\mu: \mathscr{L} \to \mathscr{T}$ by the rule $x \circ x = 2\mu(x)x$, $x \in \mathscr{L}$. Since $\mathscr{T}$ is a field, the map μ is well defined. Clearly $\mu(fx) = f\mu(x)$ for all $f \in \mathscr{F}$ and $x \in \mathscr{L}$.

Let $x \in \mathscr{L}$ and $\mathscr{I} = (\mu(x) : \mathscr{L})$. Since $x \circ x = 2\mu(x)x$, $x \in \mathscr{I}$. We now claim that

$$\mu(x)^{\beta_p} = \nu_p(x) \qquad \text{for all } p \in P(\mathscr{I}). \tag{27}$$

Indeed, given $p \in P(\mathscr{I})$, $\mu(x) \in \mathscr{T}^{(p)}$ and so it follows from Proposition 4.3(b) that $(\mu(x)x)^{\pi_p} = \mu(x)^{\beta_p}x^{\pi_p}$. By (26) we have

$$2\nu_p(x)x^{\pi_p} = (x \circ x)^{\pi_p} = (2\mu(x)x)^{\pi_p} = 2\mu(x)^{\beta_p}x^{\pi_p}$$

and so $\mu(x)^{\beta_p} = \nu_p(x)$.

Now let $x, y \in \mathscr{L}$ and $\mathscr{J} = (\mu(x) : \mathscr{L}) \cap (\mu(y) : \mathscr{L}) \cap (\mu(x+y) : \mathscr{L})$. Clearly $P(\mathscr{J}) \subseteq P((\mu(x) : \mathscr{L})) \cap P((\mu(y) : \mathscr{L})) \cap P((\mu(x+y) : \mathscr{L}))$. Given $p \in P(\mathscr{J})$, it follows from (27) that

$$\begin{aligned}\mu(x+y)^{\beta_p} &= \nu_p(x+y) = \nu_p(x) + \nu_p(y) = \mu(x)^{\beta_p} + \mu(y)^{\beta_p} \\ &= \{\mu(x) + \mu(y)\}^{\beta_p}\end{aligned}$$

and so $\mu(x+y) = \mu(x) + \mu(y)$ by Proposition 4.3(c). We conclude that $\mu: \mathscr{L} \to \mathscr{T}$ is an $\mathscr{F}$-linear map. We now have

$$\begin{aligned}2x \circ y &= (x+y)\circ(x+y) - x \circ x - y \circ y \\ &= 2\mu(x+y)(x+y) - 2\mu(x)x - 2\mu(y)y = 2\mu(x)y + 2\mu(y)x\end{aligned}$$

and so $x \circ y = \mu(x)y + \mu(y)x$ for all $x, y \in \mathscr{L}$. Since $\cdot\colon \mathscr{L}^2 \to \mathscr{L}$ is Lie-compatible, there exists an invertible element $t \in \mathscr{F}$ such that $xy - yx = t[x, y]$ for all $x, y \in \mathscr{L}$. Finally, $xy = \frac{1}{2}\{xy - yx + xy + yx\} = \frac{1}{2}\{t[x, y] + \mu(x)y + \mu(y)x\}$ and so (4) is satisfied.

Conversely, suppose that (4) is fulfilled. Then $x^2 = \mu(x)x$ and so $[x^2, x] = 0$ for all $x \in \mathscr{L}$. Thus $\mathscr{L}$ is third power-associative.

(b) Assume that $\mathscr{L}$ is flexible. Then it is third power-associative and so (4) is fulfilled by (a). Next, let $x, y \in \mathscr{L}$. It follows from Theorem 4.2(b) that $\nu_p([x, y]) = 0$ for all $p \in \mathscr{P}$. Now both (27) and Proposition 4.3(c) imply that $\mu([x, y]) = 0$.

Conversely, assume that $\mu([\mathscr{L}, \mathscr{L}]) = 0$ and (4) is fulfilled. Then

$$\begin{aligned}
4x(yx) &= 2x\{t[y, x] + \mu(x)y + \mu(y)x\} \\
&= t[x, t[y, x] + \mu(x)y] + t\mu(x)[y, x] + \mu(x)^2 y \\
&\quad + \mu(x)\mu(y)x + \mu(\mu(x)y + \mu(y)x)x \\
&= t^2[x, [y, x]] + \mu(x)^2 y + \mu(x)\mu(y)x \\
&\quad + \mu(\mu(x)y + \mu(y)x)x \qquad \text{and} \\
4(xy)x &= 2\{t[x, y] + \mu(x)y + \mu(y)x\}x \\
&= t[t[x, y] + \mu(x)y, x] + \mu(\mu(x)y + \mu(y)x)x \\
&\quad + t\mu(x)[x, y] + \mu(x)^2 y + \mu(x)\mu(y)x \\
&= t^2[[x, y], x] + \mu(\mu(x)y + \mu(y)x)x + \mu(x)^2 y \\
&\quad + \mu(x)\mu(y)x
\end{aligned}$$

because $\mu([x, y]) = 0$. Therefore $x(yx) = (xy)x$ for all $x, y \in \mathscr{L}$ and whence the algebra $(\mathscr{L}, +, \cdot)$ is flexible. The last statement follows from both (a) and (b).

Proof of Theorem 1.2. (a) In view of Theorem 1.1 it is enough to prove the last statement. To this end, suppose that (5) is satisfied. Then $x \bullet y = \mu(x)y + \mu(y)x$ for all $x, y \in \mathscr{R}$. Since $(\mathscr{R}, +, *)$ is an $\mathscr{F}$-algebra, $(\mathscr{R}, +, \bullet)$ is also an $\mathscr{F}$-algebra. Recalling that $\mu(\mathscr{R}) \subseteq \mathscr{F}$, we see that

$$\begin{aligned}
[(x \bullet x) \bullet y] \bullet x &= 2\mu(x)(x \bullet y) \bullet x = 2\mu(x) x \bullet (y \bullet x) \\
&= (x \bullet x) \bullet (y \bullet x)
\end{aligned}$$

for all $x, y \in \mathscr{R}$. Therefore, $(\mathscr{R}, +, \bullet)$ is a Jordan algebra. Let $x \in \mathscr{R}$ and let $\mathscr{B}$ be the subalgebra of $(\mathscr{R}, +, *)$ generated by x. It is easy to see that

$\mathscr{B} = \mathscr{F}x$ and $(\mathscr{B}, +, *)$ is an associative algebra. Therefore $*$ is power-associative.

(b) The result follows from (a), Theorem 1.1(b), and (2). Finally, (c) follows from Theorem 1.1(c).

LEMMA 4.4. *Let $\mathscr{F}$ be a field with* $\operatorname{char}(\mathscr{F}) \neq 2$, *let $\mathscr{A}$ be an $\mathscr{F}$-algebra with unity* 1, *and let $\mathscr{I}$ be a subspace of $\mathscr{A}$ such that $\mathscr{I} \cap \mathscr{F} = 0$ and $\mathscr{I} + \mathscr{F} = \mathscr{A}$. Suppose that $x \circ y \in \mathscr{I}$ for all $x, y \in \mathscr{I}$, where $x \circ y = xy + yx$. Then $\mathscr{I}$ is an ideal of the algebra $\mathscr{A}$.*

Proof. Let $x, y \in \mathscr{I}$ and $a \in \mathscr{F}$. Then $x \circ (y + a) = x \circ y + 2ax \in \mathscr{I}$ and so $\mathscr{I} \circ \mathscr{A} \subseteq \mathscr{I}$. Next, let $x, z \in \mathscr{A}$ and $y \in \mathscr{I}$. Then on the one hand

$$[y, [x, z]] = (x \circ y) \circ z - x \circ (y \circ z) \in \mathscr{I}$$

and so $[\mathscr{I}, [\mathscr{A}, \mathscr{A}]] \subseteq \mathscr{I}$. On the other hand $\mathscr{I} \circ [\mathscr{A}, \mathscr{A}] \subseteq \mathscr{I}$. Therefore $\mathscr{I}[\mathscr{A}, \mathscr{A}] \cup [\mathscr{A}, \mathscr{A}]\mathscr{I} \subseteq \mathscr{I}$ and whence $[\mathscr{I} \circ \mathscr{I}, \mathscr{A}] \subseteq \mathscr{I}$. Since $(\mathscr{I} \circ \mathscr{I}) \circ \mathscr{A} \subset \mathscr{I}$, we conclude that the ideal $\mathscr{J}$ of $\mathscr{A}$ generated by $\mathscr{I} \circ \mathscr{I}$ is contained in $\mathscr{I}$. We now see that $\mathscr{I}/\mathscr{J}$ is the prime radical of the factor algebra $\mathscr{A}/\mathscr{J}$ and so $\mathscr{I}$ is an ideal of $\mathscr{A}$.

Proof of Theorem 1.3. Since $\deg(\mathscr{A}) \geq 3$, $\mathscr{A}$ is a 3-free subset of $\mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$ by (13). Suppose that $*$ is third power-associative. Take $\mathscr{L} = \mathscr{A}$ and $\alpha = \mathrm{id}_{\mathscr{A}}$. Then (6) follows from Theorem 4.2(a). Conversely, assume that (6) is fulfilled. Then $x * x = \lambda x^2 + \mu(x)x + \frac{1}{2}\tau(x, x)$. Therefore $(x * x) * x - x * (x * x) = t[x * x, x] = 0$ and whence $*$ is third power-associative.

Assume that $\deg(\mathscr{A}) \geq 4$. Then $\mathscr{A}$ is a 4-free subset of $\mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$ by (13).

(a) Suppose that the algebra $(\mathscr{A}, +, *)$ is flexible. Then it is third power-associative and so (6) is fulfilled. Next, (7) follows from Theorem 4.2(b). If both (6) and (7) are satisfied, then as in the proof of Theorem 1.2 one may easily check that $(\mathscr{A}, +, *)$ is flexible. Thus (a) is proved.

Now suppose that $\deg(\mathscr{A}) \geq 5$, (6) is satisfied, and $\lambda \neq 0$. Then $\mathscr{A}$ is a 5-free subset of $\mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$.

(i) Set $\mathscr{D} = (\mathscr{A}, +, *)$, $\mathscr{B} = \mathscr{A}$, $\alpha = \mathrm{id}_{\mathscr{A}}$, and $\epsilon = 0$. Now the result follows from Proposition 4.1(c) because by (23) we have

$$\begin{aligned} 0 &= \mu(x \bullet y) - \mu(x)\mu(y) + 2\lambda\tau(x, y) \\ &= \lambda\mu(x \circ y) + \mu(x)\mu(y) + \tau(x, y)\{2\lambda + \mu(1)\} \end{aligned}$$

for all $x, y \in \mathscr{A}$. Moreover, $(x \bullet y)^{\beta} = x^{\beta} \circ y^{\beta}$ for all $x, y \in \mathscr{A}$ by Proposition 4.1(c).

If in addition $(\mathscr{A}, +, *)$ is flexible, then $(x * y - y * x)^{\beta} = \frac{t}{\lambda}[x^{\beta}, y^{\beta}]$, by Proposition 4.1(c), and so

$$(x * y)^{\beta} = \frac{1}{2}\{x * y - y * x + x \bullet y\}^{\beta} = \frac{1}{2}\left\{\frac{t}{\lambda}\left[x^{\beta}, y^{\beta}\right] + x^{\beta} \circ y^{\beta}\right\}$$

$$= x^{\beta} \diamond y^{\beta} \qquad \text{for all } x, y \in \mathscr{A}.$$

(ii) Let $2\lambda - \mu(1) \neq 0$. Assume that (8) is fulfilled. Since $*$ is third power-associative, it is fourth power-associative if and only if the multiplication $\bullet$ is fourth power-associative (see (3)). As $(\mathscr{A}, +, \circ)$ is power-associative, it is enough to show that β is an isomorphism of algebras $(\mathscr{A}, +, \bullet)$ and $(\mathscr{A}, +, \circ)$. In view of (i), we have only to show that β is bijective. If $x^{\beta} = 0$, then $\lambda x = -\frac{1}{2}\mu(x) \in \mathscr{F}$ and so $x \in \mathscr{F}$. Therefore, $\mu(x) = \mu(1)x$,

$$0 = x^{\beta} = \lambda x + \tfrac{1}{2}\mu(x) = \tfrac{1}{2}\{2\lambda + \mu(1)\}x,$$

and whence $x = 0$. We see that β is injective. Next, let $y \in \mathscr{A}$. Set

$$x = \frac{1}{\lambda}y - \frac{\mu(y)}{\lambda(2\lambda + \mu(1))}.$$

Clearly $x^{\beta} = y$ and so β is bijective.

If in addition $(\mathscr{A}, +, *)$ is flexible, then β is an isomorphism of algebras $(\mathscr{A}, +, *)$ and $(\mathscr{A}, +, \diamond)$ by (i) and the above result.

Conversely, suppose that $*$ is fourth-power associative. Then (8) is fulfilled by (i).

(iv) It follows from (8) that

$$\mu(x \circ y) + \mu(x)\mu(y) = 0 \qquad \text{for all } x, y \in \mathscr{A}. \tag{28}$$

Therefore if $x \in \mathscr{I} = \ker(\mu)$ and $y \in \mathscr{A}$, then $x \circ y \in \mathscr{I}$ by (28). Since $\mu(1) = -2\lambda \neq 0$, $\ker(\mu) \cap \mathscr{F} = 0$ and so $\mathscr{I} \cap \mathscr{F} = 0$. Clearly $\dim_{\mathscr{F}}(\mathscr{A}/\mathscr{I}) = 1$ and whence $\mathscr{A} = \mathscr{I} + \mathscr{F}$. By Lemma 4.4, $\mathscr{I}$ is an ideal of the algebra $\mathscr{A}$. Obviously $\ker(\beta) = \mathscr{F}$. By (i), β is a homomorphism of algebras $(\mathscr{A}, +, \bullet)$ and $(\mathscr{A}, +, \circ)$. Therefore $\mathscr{F} = \ker(\beta)$ is an ideal of the algebra $(\mathscr{A}, +, \bullet)$. We now have $\mathscr{A}^{\beta} = (\mathscr{I} + \mathscr{F})^{\beta} = \mathscr{I}$ and the result follows from the first homomorphism theorem. If in addition $(\mathscr{A}, +, *)$ is flexible, then β is a homomorphism of $(\mathscr{A}, +, *)$ into $(\mathscr{A}, +, \diamond)$ and so it induces an isomorphism of $\mathscr{A}/\mathscr{F}$ and $(\mathscr{I}, +, \diamond)$. Finally, (iii) follows from (iv).

Proof of Theorem 1.4. First we remark that if $\deg(\mathscr{A}) \geq 4$ ($\deg(\mathscr{A}) \geq 5$, $\deg(\mathscr{A}) \geq 6$), then $\mathscr{R}$ is a 3-free (respectively, 4-free, 5-free) subset of $\mathscr{Q}_{\mathrm{mr}}(\mathscr{A})$ by (14).

Suppose that the multiplication $*$ is third power-associative. Take $\mathscr{L} = \mathscr{R}$ and $\alpha = \mathrm{id}_{\mathscr{R}}$. By Theorem 4.2(a),

$$x * y = \tfrac{1}{2}\{t[x, y] + \lambda x \circ y + \mu(x) y + \mu(y) x + \tau(x, y)\} \qquad \text{for all } x, y \in \mathscr{R}.$$

Since $t[x, y] + \mu(x)y + \mu(y)x \in \mathscr{R}$, we conclude that $\lambda x \circ y + \tau(x, y) \in \mathscr{R}$. Therefore

$$0 = \{\lambda x \circ y + \tau(x, y)\}^{\pi} = \lambda(x \circ y)^{\pi} + \tau(x, y)$$

and so $\tau(x, y) = -\lambda(x \circ y)^{\pi}$ which proves (9).

Both the converse implication and the statement (a) are proved analogously to that of Theorem 1.3 because $\tau(x, [x, y]) = -\lambda(x \circ [x, y])^{\pi} = -\lambda[x^2, y]^{\pi} = 0$ for all $x, y \in \mathscr{R}$.

(b) Let $x \in \mathscr{R}$ and $\zeta, \eta \in \mathscr{F}$. Then $(\zeta x) * (\eta x) = \zeta\eta\mu(x)x$ and so the subalgebra of $(\mathscr{R}, +, *)$ generated by x is equal to $\mathscr{F}x$ and is associative. Thus $*$ is power-associative.

(c) It follows from (a) that $\mu = 0$. Assume that the multiplication $*$ is fourth power-associative. Put $\mathscr{L} = \mathscr{R}$ and $\alpha = \mathrm{id}_{\mathscr{R}}$. It follows from Theorem 4.2(c) that $\lambda^2(x \circ y)^{\pi} = -\lambda\tau(x, y) = 0$ for all $x, y \in \mathscr{R}$. If $\lambda = 0$, then there is nothing to prove. Assume that $\lambda \neq 0$. Then $(x \circ y)^{\pi} = 0$ for all $x, y \in \mathscr{R}$ and so $\mathscr{R} \circ \mathscr{R} \subseteq \mathscr{R}$. It now follows from Lemma 4.4 that $\mathscr{R}$ is an ideal of $\mathscr{A}$.

Conversely, if $\lambda = 0$, then $*$ is power-associative by (b). Suppose that $\mathscr{R}$ is an ideal of $\mathscr{A}$ and $\lambda \neq 0$. Then $(x \circ y)^{\pi} = 0$ and so

$$x * y = \tfrac{1}{2}\{t[x, y] + \lambda x \circ y\} \qquad \text{for all } x, y \in \mathscr{R}$$

which is easily verified to be power-associative.

Now suppose that the condition (d) of Theorem 1.4 is satisfied.

(i) It follows from Theorem 4.2(c) that β is a homomorphism of algebras $(\mathscr{R}, +, \bullet)$ and $(\mathscr{R}^{\beta}, +, \circ)$. Since $\ker(\beta) \subseteq \mathscr{F} \cap \mathscr{R} = 0$, we conclude that β is an isomorphism. Since $\mathscr{R} + \mathscr{F} = \mathscr{A}$, $\mathscr{R}^{\beta} + \mathscr{F} = \mathscr{A}$ by the definition of β. Clearly $\mathscr{R}^{\beta} \cap \mathscr{F} = 0$. By Lemma 4.4, $\mathscr{R}^{\beta}$ is an ideal of $\mathscr{A}$. Finally, if in addition $(\mathscr{R}, +, *)$ is flexible, then β is an isomorphism of algebras $(\mathscr{R}, +, *)$ and $(\mathscr{R}^{\beta}, +, \diamond)$ by Theorem 4.2(d).

(ii) Since $(\mathscr{R}^{\beta}, +, \circ)$ is power-associative, the result follows from (3).

(iii) The result follows from the obvious fact that $(\mathscr{R}^{\beta}, +, \diamond)$ is flexible and power-associative. Finally, (e) follows from (i) and (b).

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