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Fusion algebras for imprimitive complex reflection groups

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Abstract

We prove that the Fourier matrices for the imprimitive complex reflection groups introduced by Malle in [Gunter Malle, Unipotente Grade imprimitiver komplexer Spiegelungsgruppen, *J. Algebra* 177 (3) (1995) 768–826] define fusion algebras with not necessarily positive but integer structure constants. Hence they define \mathbb{Z} -algebras. As a result, we obtain that all known Fourier matrices belonging to spetses define algebras with integer structure constants.

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1. Introduction

In his classification of irreducible characters of a finite group of Lie type, Lusztig develops a theory in which a so-called non-abelian Fourier transform emerges. This is a matrix which only depends on the Weyl group of the group of Lie type. Geck and Malle [6] set up a system of axioms based on the properties such a Fourier matrix has. Using this system Broué, Malle and Michel construct analogous transformations for the spetses, which until now remain mysterious objects.

In [9] Malle defines unipotent degrees for the imprimitive complex reflection groups. The transformation matrix from the fake degrees to these unipotent degrees defines an algebra via the formula of Verlinde. In the present article, we show that these matrices yield algebras with

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integer structure constants. We prove this by looking at exterior powers of group rings of cyclic groups.

We start by giving a definition of the specific type of \mathbb{Z} -algebra we will study. It has roughly the properties of a table algebra, though it is not a \mathbb{C} -algebra and its structure constants may be negative. The based rings Lusztig introduces in [8] are also a variation of the algebras we look at. A much more extensive investigation of such algebras can be found in the authors dissertation [4].

We then recall the definition of the matrices from [9] and explain their relation to exterior powers. In the following section we examine the algebras belonging to exterior powers so that we can prove the result in the next section. Finally, we show a connection to the Kac–Peterson matrices attached to affine Kac–Moody algebras: exterior powers of matrices of type $A_1^{(1)}$ are matrices of type $C_l^{(1)}$.

2. Basic definitions

Definition 2.1. Let R be a finitely generated commutative \mathbb{Z} -algebra which is a free \mathbb{Z} -module with basis $B = \{b_0 = 1, \dots, b_{n-1}\}$ and structure constants

$$b_i b_j = \sum_k N_{ij}^k b_k, \quad N_{ij}^k \in \mathbb{Z},$$

for $0 \leq i, j < n$. Assume that there is an involution $\sim : R \rightarrow R$ which is a \mathbb{Z} -module homomorphism such that

$$\tilde{B} = B, \quad N_{\tilde{i}\tilde{j}}^k = N_{ij}^{\tilde{k}}, \quad N_{ij}^0 = \delta_{i,j},$$

for all $0 \leq i, j, k < n$, where \tilde{i} is the index with $\tilde{b}_i = b_{\tilde{i}}$. Then we call (R, B) a \mathbb{Z} -based ring.

Remark that if the involution \sim exists, then it is unique by the third equation above. The second equation expresses that \sim is an algebra homomorphism. Remark also, that if we replace an element $b \in B$ by $-b$, then the new basis spans the same algebra, but \sim does not necessarily exist anymore (with respect to the new basis).

Example 1. Let G be a finite group. Then the character ring of G is a \mathbb{Z} -based ring with basis $\text{Irr}(G)$ (the irreducible characters) and non-negative structure constants, where multiplication is just tensor product. The involution \sim is complex conjugation on the characters.

Example 2. The representation ring of the quantum double of a finite group (see [3]) is a \mathbb{Z} -based ring where the basis is again the set of irreducible representations.

The \mathbb{Z} -based rings are a generalization of algebras with the properties of representation rings. There are many other such generalizations. One of them are the table algebras, to which the \mathbb{Z} -based rings with non-negative structure constants belong (viewed as \mathbb{C} -algebras). All \mathbb{Z} -based rings are generalized table algebras (GT-algebras, [1]). But GT-algebras do not have the properties which we will need. Another structure is the so-called C -algebra, which has an elaborate structure theory [2]. Unfortunately, the \mathbb{Z} -based rings of the present article are not always C -algebras.

If (R, B) is a \mathbb{Z} -based ring, then we have a linear map $\tau : R \rightarrow \mathbb{C}$ defined by $\tau(b_i) = \delta_{0,i}$. The map

$$\langle , \rangle : R \times R \rightarrow \mathbb{Z}, \quad \langle r, r' \rangle := \tau(\tilde{r}r'),$$

for $r, r' \in R$ behaves like an inner product with orthonormal basis B because $r = \sum_{b \in B} \langle b, r \rangle b$ for all $r \in R$. The set \tilde{B} is the basis dual to B with respect to this inner product. Extending \langle , \rangle to the \mathbb{C} -algebra $R_{\mathbb{C}} := R \otimes_{\mathbb{Z}} \mathbb{C}$, one can prove that $R_{\mathbb{C}}$ is semisimple (compare [4, 1.2]).

Proposition 2.2. *Let R be a \mathbb{Z} -based ring. Then the algebra $R_{\mathbb{C}} := R \otimes_{\mathbb{Z}} \mathbb{C}$ is semisimple.*

Proof. Extend \sim and τ to $R_{\mathbb{C}}$:

$$\widetilde{r \otimes z} := \tilde{r} \otimes \bar{z}, \quad \tau' : r \otimes z \mapsto z\tau(r),$$

where $r \in R, z \in \mathbb{C}$. This yields a hermitian positive definite sesquilinear form $\langle r, r' \rangle := \tau'(\tilde{r}r')$. If \mathfrak{J} is a left ideal in $R_{\mathbb{C}}$, then the orthogonal complement

$$\mathfrak{J}^{\perp} := \{r \in R_{\mathbb{C}} \mid \langle r, r' \rangle = 0 \forall r' \in \mathfrak{J}\}$$

is a left ideal too:

$$\langle tr, r' \rangle = \tau'(\tilde{t}rr') = \tau'(\tilde{r}\tilde{t}r') = \langle r, \tilde{t}r' \rangle = 0,$$

for all $r \in \mathfrak{J}^{\perp}, t \in R_{\mathbb{C}}$ and $r' \in \mathfrak{J}$. The claim follows. \square

Now $R_{\mathbb{C}}$ is a commutative semisimple algebra over an algebraically closed field, so by the theorem of Wedderburn–Artin it is isomorphic as a \mathbb{C} -algebra to \mathbb{C}^n with componentwise multiplication. By choosing B as a basis for $R_{\mathbb{C}}$ and the canonical basis $\{e_i\}_i$ with $e_i e_j = \delta_{i,j} e_i$ for all i, j for \mathbb{C}^n , an isomorphism φ is described by a matrix s which we will call an s -matrix of (R, B) :

$$\varphi(b_i) = \sum_k s_{ki} e_k.$$

Remark that this matrix depends on the choice of the isomorphism φ . Another isomorphism would differ from φ by a \mathbb{C} -algebra automorphism of \mathbb{C}^n , so an s -matrix is unique up to a permutation of rows.

The rows of s are the one-dimensional representations of R because $s_{ki} s_{kj} = \sum_l N_{ij}^l s_{kl}$ for all k, i, j . They are orthogonal (see [4, 1.2.3], the proof is the same as for the orthogonality relation for irreducible characters of finite groups). By normalizing them,

$$d := s\bar{s}^t, \quad S_{ij} := \frac{s_{ij}}{\sqrt{d_{ii}}},$$

where $\sqrt{d_{ii}}$ is the positive root, we get an orthonormal matrix S which we call S -matrix or *Fourier matrix* of (R, B) . We can recover the structure constants of R (and the involution \sim) from the S -matrix via the formula of Verlinde:

$$N^l_{ij} = \sum_k \frac{S_{ki} S_{kj} \overline{S_{kl}}}{S_{k0}} \tag{1}$$

(this follows immediately by transporting the multiplication via φ from \mathbb{C}^n to R). The columns of the s -matrix are the image of B in \mathbb{C}^n under φ . Conversely, given a matrix S and a column (here column 0), we may define quantities N^l_{ij} via (1), which will be structure constants of a \mathbb{Z} -based ring if S satisfies certain properties.

This is equivalent to the following construction: start with a matrix $S \in \mathbb{C}^{n \times n}$ with $S\overline{S}^t = 1$ and choose a column i_0 in which all entries are non-zero. Divide each row by the entry of column i_0 to get a matrix s . The columns of s span a \mathbb{Z} -lattice in \mathbb{C}^n which is free since S is invertible. If this lattice is closed under componentwise multiplication, then it is a \mathbb{Z} -algebra R with the columns of s as a basis. In this case, we say that the matrix S (or s) with unit i_0 define the \mathbb{Z} -algebra R . The involution \sim corresponds to complex conjugation on the columns of S .

Example 3. Let G be a finite group. An s -matrix of the character ring of G is the transposed character table of G .

Example 4. Untwisted affine Kac–Moody algebras have for each level k a Kac–Peterson matrix which is the S -matrix of a \mathbb{Z} -based ring with non-negative structure constants (see [7, 13.8]).

We will need the following lemma later on.

Lemma 2.3. *Let $S \in \mathbb{C}^{n \times n}$ with $S\overline{S}^t = 1$ such that $N^l_{ij} = \sum_k \frac{S_{ki} S_{kj} \overline{S_{kl}}}{S_{k0}} \in \mathbb{Z}$ and $S_{i0} \in \mathbb{R}^\times$ for all i, j, l . If the set of columns of S is invariant under complex conjugation, then S defines a \mathbb{Z} -based ring.*

Proof. Let \sim be the permutation of the columns given by complex conjugation. Then

$$N^0_{ij} = \sum_k \frac{S_{ki} S_{kj} \overline{S_{k0}}}{S_{k0}} = \sum_k S_{ki} \overline{S_{k\tilde{j}}} = \delta_{i,\tilde{j}}$$

because $S_{k0} \in \mathbb{R}$ and $S\overline{S}^t = 1$. We have to check that \sim is multiplicative:

$$N^m_{\tilde{i}\tilde{j}} = \sum_k \frac{S_{k\tilde{i}} S_{k\tilde{j}} \overline{S_{km}}}{S_{k0}} = \sum_k \frac{\overline{S_{k\tilde{i}}} \overline{S_{k\tilde{j}}} S_{km}}{\overline{S_{k0}}} = \sum_k \frac{S_{ki} S_{kj} \overline{S_{k\tilde{m}}}}{S_{k0}} = N^{\tilde{m}}_{ij}$$

because $N^m_{\tilde{i}\tilde{j}}, S_{k0} \in \mathbb{R}$. \square

3. Fourier matrices for imprimitive complex reflection groups

Let us define the Fourier matrices for the imprimitive complex reflection groups $G(e, 1, n)$ (compare with [9]). The original definition is slightly technical, but it takes a simple form if we express it by means of exterior powers of the S -matrix of a group ring of a cyclic group.

3.1. Definition of the Fourier matrices

We use the notation of [9, 4A]. Let $e \geq 1$ and Y be a totally ordered set with d elements. Consider the set

$$\Psi := \{ \psi : Y \rightarrow \{0, \dots, e - 1\} \}$$

and a map $\pi : Y \rightarrow \mathbb{N}$. In [9], ‘ Ψ ’ is a subset of our Ψ ; we will restrict to that subset later. We define an equivalence relation \sim_π on Ψ :

$$\phi \sim_\pi \psi \iff \pi(\phi^{-1}(i)) = \pi(\psi^{-1}(i)) \quad \text{for all } 0 \leq i < e,$$

for $\phi, \psi \in \Psi$, $\psi^{-1}(i) := \{y \in Y \mid \psi(y) = i\}$, and denote the class of ψ by $[\psi]$. Now call an element $\psi \in \Psi$ π -admissible if for all $y, y' \in Y$ with $\pi(y) = \pi(y')$ and $\psi(y) = \psi(y')$ we have $y = y'$.

A π -admissible ψ can be interpreted in the sense of [9] as an e -symbol with entries in $\pi(Y)$: an e -symbol is an ordered sequence $\mathcal{S} = (L_0, \dots, L_{e-1})$ of e strictly increasing finite sequences of natural numbers $L_i = (\lambda_{i,1}, \dots, \lambda_{i,m_i})$, written

$$\mathcal{S} = \begin{pmatrix} \lambda_{0,1} & \dots & \lambda_{0,m_0} \\ \lambda_{1,1} & \dots & \lambda_{1,m_1} \\ \vdots & \ddots & \vdots \\ \lambda_{e-1,1} & \dots & \lambda_{e-1,m_{e-1}} \end{pmatrix}.$$

For $0 \leq i < e$, the set of entries of L_i is $\pi(\psi^{-1}(i))$.

We define a matrix \mathbb{S} indexed by the classes of π -admissible elements of Ψ (compare with [9, 4.10]):

$$\mathbb{S}_{[\phi],[\psi]} := \frac{(-1)^{m(e-1)}}{\tau(e)^m} \sum_{\vartheta \in [\phi]} \epsilon(\vartheta) \epsilon(\psi) \prod_{y \in Y} \zeta^{-\vartheta(y)\psi(y)},$$

where $\zeta = \exp(2\pi i/e)$, $m := \lfloor \frac{d}{e} \rfloor \in \mathbb{Z}$ and

$$\epsilon(\psi) := (-1)^{|\{(y,y') \in Y \times Y \mid y < y', \psi(y) < \psi(y')\}|}, \quad \tau(e) := \prod_{i=0}^{e-1} \prod_{j=i+1}^{e-1} (\zeta^i - \zeta^j).$$

The Fourier matrices of [9] are submatrices of \mathbb{S} . Let $r := |\pi(Y)|$ and $w_1, \dots, w_r \in \mathbb{N}$ be such that $\pi(Y) = \{w_1, \dots, w_r\}$. Then $n_i := |\pi^{-1}(w_i)| = |\psi(\pi^{-1}(w_i))|$ if ψ is π -admissible. Remark that if $\vartheta, \phi \in \Psi$ are equivalent (and π -admissible) then there is a permutation $\sigma \in \text{Sym}(Y)$ such that

$\phi = \vartheta \circ \sigma$ and $\pi \circ \sigma = \pi$. Then $\epsilon(\vartheta)\epsilon(\phi) = \epsilon_\sigma$ holds, where ϵ_σ is the sign of the permutation σ . Using this we get

$$\mathbb{S}_{[\phi],[\psi]} = \frac{(-1)^{m(e-1)}}{\tau(e)^m} \epsilon(\phi)\epsilon(\psi) \sum_{\substack{\sigma \in \text{Sym}(Y) \\ \pi \circ \sigma = \pi}} \epsilon_\sigma \prod_{y \in Y} \zeta^{-\phi(\sigma(y))\psi(y)},$$

and by defining $c := (-1)^{m(e-1)} i^{-\binom{e-1}{2}m} \sqrt{e}^{d-em}$

$$\mathbb{S}_{[\phi],[\psi]} = c\epsilon(\phi)\epsilon(\psi) \sum_{\substack{\sigma \in \text{Sym}(Y) \\ \pi \circ \sigma = \pi}} \epsilon_\sigma \prod_{y \in Y} \frac{1}{\sqrt{e}} \zeta^{-\phi(\sigma(y))\psi(y)}$$

because $\tau(e) = i^{\binom{e-1}{2}} \sqrt{e}^e$. Call a π -admissible $\psi \in \Psi$ ordered, if for all $y, y' \in Y, y < y'$ with $\pi(y) = \pi(y')$ we have $\psi(y) < \psi(y')$. Then each class $[\psi]$ has exactly one ordered representative. So the set indexing \mathbb{S} is in bijection with

$$\mathcal{E} := \{ \psi \mid \psi \in \Psi, \psi \text{ } \pi\text{-admissible and ordered} \}$$

and we will only consider elements from \mathcal{E} from now on. Take $\psi_1, \psi_2 \in \mathcal{E}$ and let $\psi_1(\pi^{-1}(w_\mu)) = \{i_1^\mu, \dots, i_{n_\mu}^\mu\}, \psi_2(\pi^{-1}(w_\mu)) = \{j_1^\mu, \dots, j_{n_\mu}^\mu\}$ such that $i_1^\mu < \dots < i_{n_\mu}^\mu, j_1^\mu < \dots < j_{n_\mu}^\mu$. We finally get

$$\overline{\mathbb{S}_{\psi_1, \psi_2}} = \bar{c}\epsilon(\psi_1)\epsilon(\psi_2) \prod_{\mu=1}^r \sum_{\sigma \in S_{n_\mu}} \epsilon_\sigma \prod_{v=1}^{n_\mu} \frac{1}{\sqrt{e}} \zeta^{i_v^\mu j_{\sigma(v)}^\mu}. \tag{2}$$

3.2. Connection to exterior powers

Let $S \in \mathbb{C}^{e \times e}$ be the S -matrix of the group ring of the cyclic group $\mathbb{Z}/e\mathbb{Z}$, so $S = (\frac{\zeta^{ij}}{\sqrt{e}})_{i,j}$. Denote by $\Lambda^n \mathbb{C}^e, n \leq e$, the subspace of $\otimes_{i=1}^n \mathbb{C}^e$ spanned by

$$C_n := \left\{ \sum_{\sigma \in S_n} \epsilon_\sigma e_{i_{\sigma(1)}} \otimes \dots \otimes e_{i_{\sigma(n)}} \mid 0 \leq i_1 < \dots < i_n \leq e-1 \right\},$$

where e_0, \dots, e_{e-1} is the canonical basis of \mathbb{C}^e . The basis C_n is indexed by the set of n -tuples (i_1, \dots, i_n) with $0 \leq i_1 < \dots < i_n < e$; we will therefore write $\vec{i} := (i_1, \dots, i_n)$ for the corresponding element of the basis. The restriction of $\otimes_{i=1}^n S$ to $\Lambda^n \mathbb{C}^e$ defines an automorphism corresponding to the matrix

$$(\Lambda^n S)_{\vec{i}, \vec{j}} = \sum_{\sigma \in S_n} \epsilon_\sigma \prod_{v=1}^n S_{i_v, j_{\sigma(v)}} = \det((S_{i_v, j_{v'}})_{1 \leq v, v' \leq n}) \tag{3}$$

with respect to C_n . Now consider the matrix $\Lambda^{n_1} S \otimes \dots \otimes \Lambda^{n_r} S$ on the space $\Lambda^{n_1} \mathbb{C}^e \otimes \dots \otimes \Lambda^{n_r} \mathbb{C}^e$ with the basis

$$E := \{ \vec{i}^1 \otimes \dots \otimes \vec{i}^r \mid \vec{i}^\mu \in C_{n_\mu}, 1 \leq \mu \leq r \}.$$

We identify E with the set \mathcal{E} by

$$E \rightarrow \mathcal{E}, \quad \bar{i}^1 \otimes \cdots \otimes \bar{i}^r \mapsto \psi,$$

where ψ is the element of \mathcal{E} with $\psi(\pi^{-1}(w_\mu)) = \{i_1^\mu, \dots, i_{n_\mu}^\mu\}$ for $1 \leq \mu \leq r$ (remember that $\pi(Y) = \{w_1, \dots, w_r\}$). Formula (2) becomes

$$\overline{\mathbb{S}}_{\psi_1, \psi_2} = \bar{c} \epsilon(\psi_1) \epsilon(\psi_2) (A^{n_1} S \otimes \cdots \otimes A^{n_r} S)_{\psi_1, \psi_2},$$

which explains why we will first concentrate on exterior powers of S to find out what the structure constants of submatrices of \mathbb{S} look like. Note that since we are only interested in the integrality of structure constants, we can multiply the column of ψ_2 by $\epsilon(\psi_2)$ for each ψ_2 and do not have to care about these signs anymore. The $\epsilon(\psi_1)$ in the row of ψ_1 has no effect on the structure constants because it is canceled in the formula of Verlinde.

4. Exterior powers

Consider again the matrix $A^n S$ where $S = (\frac{\zeta^{ij}}{\sqrt{e}})_{i,j}$ and $e, n \in \mathbb{N}, e \geq n$. This matrix represents the restriction of $\bigotimes^n S$ to $A^n \mathbb{C}^e$ with respect to the basis C_n defined above. Our goal is to prove that $A^n S$ defines a \mathbb{Z} -algebra which is a \mathbb{Z} -based ring for an adequate basis (see Theorem 4.3), so first we need to see that for suitable \bar{i}_0 the structure constants

$$N_{\bar{j}, \bar{m}}^{\bar{k}} = \sum_{\bar{i} \in C_n} \frac{(A^n S)_{\bar{i}, \bar{j}} (A^n S)_{\bar{i}, \bar{m}} \overline{(A^n S)_{\bar{i}, \bar{k}}}}{(A^n S)_{\bar{i}, \bar{i}_0}},$$

$\bar{j}, \bar{m}, \bar{k} \in C_n$, given by the formula of Verlinde are integers. We take $\bar{i}_0 := (0, \dots, n - 1)$. Notice that the above formula is then well defined because $(A^n S)_{\bar{i}, \bar{i}_0}$ in the denominator is a Vandermonde determinant and thus unequal to 0.

4.1. Connection to Schur functions

We begin by analyzing the quotient

$$D_{\bar{i}, \bar{j}} := \frac{(A^n S)_{\bar{i}, \bar{j}}}{(A^n S)_{\bar{i}, \bar{i}_0}}.$$

The theorem about Jacobi–Trudi determinants (see [10, Theorem 4.5.1]) says [10, Lemma 4.6.1 and Corollary 4.6.2], that $D_{\bar{i}, \bar{j}}$ is the Schur function $s_{\bar{j}}(\bar{x}) \in \mathbb{C}[[\bar{x}]]$, $\bar{x} = \{x_1, x_2, \dots\}$ evaluated at $x_1 = \zeta^{i_1}, \dots, x_n = \zeta^{i_n}, x_{n+1} = 0, \dots$, where \bar{j}' is the partition $\bar{j}' := (j_n - (n - 1), \dots, j_2 - 1, j_1)$. There is an elementary proof of this statement in [4, 5.1.2].

The definition of the Schur function $s_{\bar{j}'}$ (see [10, 4.4.1]) is $s_{\bar{j}'} = \sum_{T \in \mathcal{T}_{\bar{j}'}} \bar{x}^T$, where $\mathcal{T}_{\bar{j}'}$ is the set of semistandard \bar{j}' -tableaux, that means tableaux of shape (Ferrer diagram) \bar{j}' with weakly

increasing rows, strictly increasing columns and entries in \mathbb{N} . If T_{h_1, h_2} are the entries of some $T \in \mathcal{T}_{\bar{j}}$, then $\bar{x}^T := \prod_{h_1, h_2} x_{T_{h_1, h_2}}$. This can also be written as

$$\bar{x}^T = \prod_{v \in \mathbb{N}} x_v^{w_{v+1}(T)}$$

with suitable $w_{v+1}(T) \in \mathbb{N}$. In our setting, x_{n+1}, x_{n+2}, \dots are all equal to 0, so from now on we consider

$$s_{\bar{j}} = \sum_{T \in \mathcal{T}_{\bar{j}}} \prod_{v=1}^n x_v^{w_{v+1}(T)}$$

(by abuse of notation). In the proof of the next theorem we will need the following lemma.

Lemma 4.1. *Let $a \in \mathbb{N}$ and $z_v(a) := \{T \in \mathcal{T}_{\bar{j}} \mid w_{v+1}(T) = a\}$ for $1 \leq v \leq n$. Then*

$$|z_v(a)| = |z_{v'}(a)|$$

for all $1 \leq v' \leq n$.

Proof. Proposition 4.4.2 in [10] says that $s_{\bar{j}}(\bar{x})$ is a symmetric function. This means, that $s_{\bar{j}} = s_{\bar{j}}(x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})$ for all permutations $\pi \in S_n$ and hence

$$\sum_{T \in \mathcal{T}_{\bar{j}}} \prod_{v=1}^n x_v^{w_{\pi(v)+1}(T)} = \sum_{T \in \mathcal{T}_{\bar{j}}} \prod_{v=1}^n x_v^{w_{v+1}(T)}.$$

Therefore, for all $1 \leq v \leq n$, $a \in \mathbb{N}$ and $T \in \mathcal{T}_{\bar{j}}$ with $w_{v+1}(T) = a$ there is a $T' \in \mathcal{T}_{\bar{j}}$ with $w_{\pi(v)+1}(T') = w_{v+1}(T)$. But then we have $|z_v(a)| = |z_{v'}(a)|$ for all $1 \leq v' \leq n$. \square

4.2. The structure constants are integers

For $\bar{j} \in C_n$, we will write $\mathcal{T}_{\bar{j}}$ instead of $\mathcal{T}_{\bar{j}'}$. Here is the main theorem:

Theorem 4.2. *The structure constants $N_{\bar{j}, \bar{m}}^{\bar{k}}$ defined above are integers.*

Proof. First notice that by Eq. (3) we have

$$(\Lambda^n S)_{\bar{i}, \bar{j}} = \frac{1}{\sqrt{e^n}} \sum_{\sigma \in S_n} \varepsilon_\sigma \prod_{v=1}^n \zeta^{i_v j_{\sigma(v)}}$$

so we get

$$N_{\bar{j}, \bar{m}}^{\bar{k}} = \frac{1}{e^n} \sum_{0 \leq i_1 < \dots < i_n \leq e-1} \frac{P_{\bar{i}, \bar{j}} \overline{P_{\bar{i}, \bar{k}}} P_{\bar{i}, \bar{m}}}{P_{\bar{i}, \bar{i}_0}}$$

if $P_{\bar{i}, \bar{j}} := \det(\zeta^{i_\mu j_\nu})_{\mu, \nu}$, $\bar{i}_0 := (0, \dots, n - 1)$. We have seen that $D_{\bar{i}, \bar{m}} := \frac{P_{\bar{i}, \bar{m}}}{P_{\bar{i}, \bar{i}_0}}$ can be expressed as

$$D_{\bar{i}, \bar{m}} = \sum_{T \in \mathcal{T}_{\bar{m}}} \prod_{\nu=1}^n \zeta^{i_\nu w_{\nu+1}(T)}.$$

This remains well defined if we take any tuple (i_1, \dots, i_n) , $0 \leq i_1, \dots, i_n \leq e - 1$ instead of restricting to those with $i_1 < \dots < i_n$. In general, $D_{\bar{i}, \bar{m}}$ can be non-zero for some \bar{i} with two equal entries. But the term $P_{\bar{i}, \bar{j}} \overline{P_{\bar{i}, \bar{k}}} D_{\bar{i}, \bar{m}}$ is still 0 in this case because the determinants $P_{\bar{i}, \bar{j}}$, $P_{\bar{i}, \bar{k}}$ vanish then. Furthermore, we know that $D_{\bar{i}, \bar{m}}$ is invariant under permutation of the i_1, \dots, i_n , if $0 \leq i_1 < \dots < i_n \leq e - 1$. Under permutation, the determinants $P_{\bar{i}, \bar{j}}$ and $P_{\bar{i}, \bar{k}}$ are modified by signs which cancel each other. So we are allowed to write

$$N_{\bar{j}, \bar{m}}^{\bar{k}} = \frac{1}{e^n} \sum_{0 \leq i_1 < \dots < i_n \leq e-1} P_{\bar{i}, \bar{j}} \overline{P_{\bar{i}, \bar{k}}} D_{\bar{i}, \bar{m}} = \frac{1}{e^n n!} \sum_{0 \leq i_1, \dots, i_n \leq e-1} P_{\bar{i}, \bar{j}} \overline{P_{\bar{i}, \bar{k}}} D_{\bar{i}, \bar{m}}.$$

We want to prove that $a := \sum_{0 \leq i_1, \dots, i_n \leq e-1} P_{\bar{i}, \bar{j}} \overline{P_{\bar{i}, \bar{k}}} D_{\bar{i}, \bar{m}}$ is an integer and congruent 0 modulo $e^n n!$. Substitute

$$P_{\bar{i}, \bar{j}} = \sum_{\sigma \in S_n} \varepsilon_\sigma \prod_{\nu=1}^n \zeta^{i_\nu j_{\sigma(\nu)}}$$

to get

$$a = \sum_{\sigma_1, \sigma_2 \in S_n} \varepsilon_{\sigma_1} \varepsilon_{\sigma_2} \sum_{T \in \mathcal{T}_{\bar{m}}} \sum_{0 \leq i_1, \dots, i_n \leq e-1} \prod_{\nu=1}^n \zeta^{i_\nu (j_{\sigma_1(\nu)} - k_{\sigma_2(\nu)} + w_{\nu+1}(T))},$$

where the inner sum can be rewritten as

$$\sum_{i_1=0}^{e-1} \zeta^{i_1 (j_{\sigma_1(1)} - k_{\sigma_2(1)} + w_{1+1}(T))} \sum_{i_2=0}^{e-1} \zeta^{i_2 (j_{\sigma_1(2)} - k_{\sigma_2(2)} + w_{2+1}(T))} \dots \sum_{i_n=0}^{e-1} \zeta^{i_n (j_{\sigma_1(n)} - k_{\sigma_2(n)} + w_{n+1}(T))}.$$

For a pair (σ_1, σ_2) , this is not zero if and only if all brackets $(j_{\sigma_1(\nu)} - k_{\sigma_2(\nu)} + w_{\nu+1}(T))$ are congruent 0 modulo e . By Lemma 4.1, $w_{\nu+1}(T)$, $T \in \mathcal{T}_{\bar{m}}$, take the same values for all ν with the same multiplicities. Hence if (σ_1, σ_2) is an adequate pair (for which the sum is not zero) then $(\sigma_1 \tau, \sigma_2 \tau)$, $\tau \in S_n$, is also adequate ($\varepsilon_{\sigma_1 \tau} \varepsilon_{\sigma_2 \tau} = \varepsilon_{\sigma_1} \varepsilon_{\sigma_2}$). Every adequate pair gives a contribution of e^n at the end. So a is congruent 0 modulo $e^n n!$. Hence $N_{\bar{j}, \bar{m}}^{\bar{k}} \in \mathbb{Z}$. \square

4.3. Negative structure constants

Here is an example in which the ring defined by $\Lambda^n S$ has negative structure constants:

Example 5. Take $e = 4$, $n = 2$. We have 6 elements in the basis. We write the multiplication table as the list of matrices $(N_{0,i}^j)_{i,j}, \dots, (N_{5,i}^j)_{i,j}$. If we write ‘.’ for ‘0,’ then it is

$$\begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{bmatrix}, \quad \begin{bmatrix} . & 1 & . & . & . & . \\ . & . & 1 & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \\ -1 & . & . & . & . & . \\ . & -1 & . & . & . & . \end{bmatrix}, \quad \begin{bmatrix} . & . & 1 & . & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \\ -1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & . & -1 & . & . \end{bmatrix}, \\
 \begin{bmatrix} . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ -1 & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & -1 & . & . & . & . \\ . & . & -1 & . & . & . \end{bmatrix}, \quad \begin{bmatrix} . & . & . & . & 1 & . \\ -1 & . & . & . & . & 1 \\ . & -1 & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & -1 & -1 & . & . \\ . & . & . & . & -1 & . \end{bmatrix}, \quad \begin{bmatrix} . & . & . & . & . & 1 \\ . & -1 & . & . & . & . \\ . & . & . & -1 & . & . \\ . & . & -1 & . & . & . \\ . & . & . & . & -1 & . \\ 1 & . & . & . & . & . \end{bmatrix}.$$

This is not a \mathbb{Z} -based ring, because there exists no involution \sim as required. Applying substitutions $b \mapsto -b$ rectifies this. But it is not possible to obtain a \mathbb{Z} -based ring with non-negative structure constants just by applying such substitutions (the computer easily checks all 2^6 sign changes).

It is unknown for which e, n it is possible to get non-negative structure constants by applying sign changes. Computations show that rings corresponding to e, n with at most 50 base elements have negative structure constants if and only if both e and n are even and $1 < n < e$.

As it is impossible to check all 2^{50} sign changes, we apply another method: If there is an appropriate sign change, then the new structure constants will be the absolute values of the old ones and define a \mathbb{Z} -based ring. So to decide if a given ring has such a sign change, we try to compute an s -matrix for these new structure constants (which fails if they do not define an algebra) and then compare the s -matrices.

A sign change as in the example above corresponds to multiplying a column in the Fourier matrix by -1 , which does not change the \mathbb{Z} -algebra. As we see in the example, the matrix $\Lambda^n S$ does not define a \mathbb{Z} -based ring in general. However, we can prove that there are sign changes such that we obtain a \mathbb{Z} -based ring. We want to modify the matrix in such a way that we can apply Lemma 2.3.

Theorem 4.3. *Let $\Lambda^n S$ be as above where $S = (\frac{\zeta^{ij}}{\sqrt{e}})_{i,j}$ and $e, n \in \mathbb{N}, e \geq n$. Then $\Lambda^n S$ with the column \bar{i}_0 as unit defines a \mathbb{Z} -algebra R and a basis B . Applying suitable sign changes to B , we get a basis B' such that (R, B') is a \mathbb{Z} -based ring.*

Proof. First we define the involution \sim . Let $\bar{i} = (i_1, \dots, i_n) \in C_n$ be an element of the basis. Define $\bar{i}' := (n - 1 - i_1, \dots, n - 1 - i_n)$ with entries taken modulo e . Permuting \bar{i}' we get an element $\tilde{\bar{i}} \in C_n$ and we will denote the sign of this permutation by $\gamma_{\bar{i}}$. If as above $P_{\bar{k},\bar{i}} := \det(\zeta^{k_\mu i_\nu})_{\mu,\nu}$ then

$$\overline{P_{\bar{k},\bar{i}}} = P_{\bar{k},\bar{i}'} = P_{\bar{k},\tilde{\bar{i}}} \prod_{\nu=1}^n \zeta^{-k_\nu(n-1)} = \gamma_{\bar{i}} P_{\bar{k},\tilde{\bar{i}}} \prod_{\nu=1}^n \zeta^{-k_\nu(n-1)}.$$

With $\theta_{\bar{k}} := \sqrt{\gamma_{i_0}} \prod_{v=1}^n \zeta^{-\frac{k_v(n-1)}{2}}$ (for some choice of square root of γ_{i_0}) it follows that $\overline{P_{\bar{k}, i_0} \theta_{\bar{k}}} = P_{\bar{k}, \bar{i}_0} \theta_{\bar{k}}$ and

$$\overline{P_{\bar{k}, i_0} \theta_{\bar{k}}} = \gamma_i P_{\bar{k}, \bar{i}} \sqrt{\gamma_{i_0}} \prod_{v=1}^n \zeta^{-\frac{k_v(n-1)}{2}} = \gamma_i P_{\bar{k}, \bar{i}} \sqrt{\gamma_{i_0}} \sqrt{\gamma_{i_0}}^{-1} \theta_{\bar{k}} = \gamma_i \gamma_{i_0} P_{\bar{k}, \bar{i}} \theta_{\bar{k}}.$$

Remember that $P_{\bar{k}, \bar{i}}$ are the entries of $\Lambda^n S$ up to a factor that is a real number. So multiplying each row \bar{k} by $\theta_{\bar{k}}$, which is a root of unity, we get a matrix M whose columns we will denote by $v_{\bar{i}}$. This matrix satisfies $MM^t = 1$, all entries in $v_{\bar{i}_0}$ are real and for every $v_{\bar{i}}$ either $\bar{v}_{\bar{i}}$ or $-\bar{v}_{\bar{i}}$ is a column of M . The matrix M defines the same algebra with the same basis as $\Lambda^n S$ because we only have multiplied rows with roots of unity.

Now for each set $\{\bar{i}, \bar{i}\}$ such that $\bar{v}_{\bar{i}} = -\bar{v}_{\bar{i}}$ choose \bar{i} or \bar{i} . Multiply each column $v_{\bar{i}}$ of M by -1 if \bar{i} is a chosen element. The set of columns in the resulting matrix is now closed under complex conjugation. By Theorem 4.2, this matrix defines a ring with integer structure constants. So all assumptions of Lemma 2.3 are satisfied and we obtain a \mathbb{Z} -based ring. \square

5. The fusion algebras for the complex reflection groups $G(e, 1, n)$

The Fourier matrices for the complex reflection groups $G(e, 1, n)$ decompose into blocks which are submatrices of the matrix \mathbb{S} considered above. Here, Y has $d = em + 1$ elements and we restrict to the subset

$$E' := \left\{ \bar{i}^1 \otimes \dots \otimes \bar{i}^r \in E \mid \sum_{y=1}^r \sum_{v=1}^{n_y} i_v^y \equiv m \binom{e}{2} \pmod{e} \right\}$$

of E , so $\mathbb{S}' := (\mathbb{S}_{\xi_1, \xi_2})_{\xi_1, \xi_2 \in E'}$ is the matrix we will look at now. Let $a := m \binom{e}{2}$.

Proposition 5.1. *Choose $a_y \in \mathbb{Z}$, $1 \leq y \leq r$, such that*

$$\xi_0 = (a_1, \dots, a_1 + n_1 - 1) \otimes \dots \otimes (a_r, \dots, a_r + n_r - 1) \in E'.$$

Then the structure constants

$$N_{\xi_1, \xi_2}^{\xi_3} := \sum_{\xi \in E'} \frac{\mathbb{S}'_{\xi, \xi_1} \mathbb{S}'_{\xi, \xi_2} \overline{\mathbb{S}'_{\xi, \xi_3}}}{\mathbb{S}'_{\xi, \xi_0}}$$

are integers for all $\xi_1, \xi_2, \xi_3 \in E'$.

Proof. Using the notation and the arguments of Theorem 4.2, we see that for $\xi_1 = \bar{j}^1 \otimes \dots \otimes \bar{j}^r$, $\xi_2 = \bar{k}^1 \otimes \dots \otimes \bar{k}^r$, $\xi_3 = \bar{l}^1 \otimes \dots \otimes \bar{l}^r \in E'$

$$N_{\xi_1, \xi_2}^{\xi_3} = e \sum_{\bar{i}^1 \otimes \dots \otimes \bar{i}^r \in E'} \prod_{\mu=1}^r \frac{1}{e^{n_\mu}} P_{i^\mu, j^\mu} \overline{P_{i^\mu, k^\mu}} D_{i^\mu, l^\mu}$$

because $c\bar{c} = e^{d-em} = e$ and therefore

$$N_{\xi_1, \xi_2}^{\xi_3} = \frac{e}{(n_r - 1)!} \left(\prod_{\mu=1}^{r-1} \frac{1}{n_\mu!} \right) \times \sum_{0 \leq i_1^1, \dots, i_{n_1}^1 \leq e-1} \cdots \sum_{0 \leq i_1^{r-1}, \dots, i_{n_{r-1}}^{r-1} \leq e-1} \sum_{\substack{0 \leq i_2^r, \dots, i_{n_r}^r \leq e-1 \\ (i_1^r = a - i_2^r - \dots - i_{n_r}^r \\ - \sum_{\mu=1}^{r-1} \sum_{v=1}^{n_\mu} i_v^\mu)}} \prod_{\mu=1}^r \frac{1}{e^{n_\mu}} P_{i^\mu, \bar{j}^\mu} \overline{P_{i^\mu, \bar{k}^\mu}} D_{\bar{i}^\mu, \bar{j}^\mu}.$$

At the heart we find a power of ζ (see the proof of Theorem 4.2) with exponent of the form $\sum_{\mu=1}^r \sum_{v=1}^{n_\mu} i_v^\mu \cdot w_{\mu,v}$, where the coefficient in front of i_1^r equals

$$w := w_{r,1} = (j_{\sigma_1(1)}^r - k_{\sigma_2(1)}^r + w_{n+1}^r(T^r)).$$

Using the relation

$$i_1^r w = \left(a - i_2^r - \dots - i_n^r - \sum_{\mu=1}^{r-1} \sum_{v=1}^{n_\mu} i_v^\mu \right) w$$

we can eliminate i_1^r by subtracting w from each coefficient $w_{\mu,v}$ belonging to $i_1^1, \dots, i_{n_{r-1}}^{r-1}, i_2^r, \dots, i_n^r$; the factor $\zeta^{aw} = (\zeta^{\binom{e}{2}})^{mw} = (\pm 1)^{mw}$ remains, which lies in \mathbb{Z} because $2 \cdot \binom{e}{2}$ is divisible by e .

Then the proof goes on as in Theorem 4.2. A pair (σ_1, σ_2) may only be modified by elements of the stabilizer of 1 in S_{n_r} ; we obtain the desired factor $(n_r - 1)!$. The last sum yields only e^{n_r-1} because there is no sum indexed by i_1^r . Together with the e in front, this cancels against the factor $\frac{1}{e^{n_r}}$. \square

As we did for the exterior powers, we want to see that this \mathbb{Z} -algebra is a \mathbb{Z} -based ring for a suitable basis. In order to be able to use Lemma 2.3, we need to prove that the rows of S' are orthogonal (this is implicit in [9, 4A]):

Proposition 5.2. *We have $S' \overline{S'}^t = I$.*

Proof. We want to prove $\sum_{\xi \in E'} S'_{\xi_1, \xi} S'_{\xi_2, \xi} = \delta_{\xi_1, \xi_2}$, where $\xi = \bar{i}^1 \otimes \dots \otimes \bar{i}^r$ runs through E' , so

$$\sum_{\mu=1}^r \sum_{v=1}^{n_\mu} i_v^\mu \equiv a \pmod{e}. \tag{*}$$

As in Proposition 5.1 we are allowed to sum over all $0 \leq i_1^\mu, \dots, i_{n_\mu}^\mu < e$ instead of $0 \leq i_1^\mu < \dots < i_{n_\mu}^\mu < e$ for all $1 \leq \mu < r$. This yields a factor $\frac{1}{n_\mu!}$. The i_v^μ are only related by Eq. (*). We can therefore restrict to the case $r = 1$ without loss of generality, which simplifies the subsequent equations considerably. Now, for $\xi_1 := \bar{k} := (k_1, \dots, k_n)$, $\xi_2 := \bar{j} := (j_1, \dots, j_n)$ and $\xi := \bar{i} := (i_1, \dots, i_n)$ we have

$$\begin{aligned} & \sum_{\xi \in E'} \mathbb{S}'_{\xi_1, \xi} \overline{\mathbb{S}'_{\xi_2, \xi}} \\ &= \frac{e}{n!} \sum_{\substack{0 \leq i_1, \dots, i_n \leq e-1 \\ \sum_{v=1}^n i_v \equiv a}} \sum_{\sigma_1 \in \mathcal{S}_n} \sum_{\sigma_2 \in \mathcal{S}_n} \varepsilon_{\sigma_1 \sigma_2} \prod_{v=1}^n \frac{1}{e} \zeta^{k_v i_{\sigma_1(v)} - j_v i_{\sigma_2(v)}} \\ &= \frac{e}{n!} \sum_{\sigma_1, \sigma_2 \in \mathcal{S}_n} \varepsilon_{\sigma_1 \sigma_2} \sum_{0 \leq i_2, \dots, i_n \leq e-1} \zeta^{a(k_{\sigma_1(1)} - j_{\sigma_2(1)})} \frac{1}{e^n} \prod_{v=2}^n \zeta^{i_v(k_{\sigma_1(v)} - j_{\sigma_2(v)} - k_{\sigma_1(1)} + j_{\sigma_2(1)})}, \end{aligned}$$

which is not zero if and only if

$$k_{\sigma_1(v)} - j_{\sigma_2(v)} \equiv k_{\sigma_1(1)} - j_{\sigma_2(1)} \pmod{e} \tag{**}$$

for all $1 \leq v \leq n$. But $0 \leq k_1 < \dots < k_n < e$ and $0 \leq j_1 < \dots < j_n < e$, so this holds only if $\sigma := \sigma_1 = \sigma_2$. If (**) is satisfied, then because of (*)

$$nf \equiv \sum_{v=1}^n k_v - j_v \equiv a - a \equiv 0 \pmod{e},$$

where $f := k_{\sigma(1)} - j_{\sigma(1)}$. On the other hand, $n = em + 1 \equiv 1 \pmod{e}$ by assumption. Consequently, the inner sum in the above formula is zero only if $\sigma_1 = \sigma_2$ and (**) are true, and in this case $f = k_{\sigma_1(1)} - j_{\sigma_2(1)} \equiv 0$. Hence

$$\sum_{\xi \in E'} \mathbb{S}'_{\xi_1, \xi} \overline{\mathbb{S}'_{\xi_2, \xi}} = \frac{e}{n!} \sum_{\sigma \in \mathcal{S}_n} \zeta^{af} \frac{1}{e} = 1.$$

Conversely, from (**), $d = 0$ and $\sigma_1 = \sigma_2$ follow $\bar{j} = \bar{k}$. \square

Theorem 5.3. *The Fourier matrices \mathbb{S}' for the imprimitive complex reflection group $G(e, 1, n)$ define \mathbb{Z} -algebras and bases which are \mathbb{Z} -based rings by applying sign changes to the bases.*

Proof. As in Proposition 5.1, we choose

$$\xi_0 = (a_1, \dots, a_1 + n_1 - 1) \otimes \dots \otimes (a_r, \dots, a_r + n_r - 1).$$

It remains to prove that a suitable involution \sim exists. We proceed exactly as in Theorem 4.3. For $\xi = (i_1^1, \dots, i_{n_1}^1) \otimes \dots \otimes (i_1^r, \dots, i_{n_r}^r) \in E'$, let ξ' be

$$\xi' := (w_1 - i_1^1, \dots, w_1 - i_{n_1}^1) \otimes \dots \otimes (w_r - i_1^r, \dots, w_r - i_{n_r}^r),$$

with $w_\mu := n_\mu - 1 + 2a_\mu$, $\mu = 1, \dots, r$. Define $\tilde{\xi}$ to be the element of E which we get by sorting each bracket increasingly. Then it lies in E' , because

$$\begin{aligned} \sum_{\mu=1}^r \sum_{\nu=1}^{n_\mu} (n_\mu - 1 + 2a_\mu) - i_\nu^\mu &= \left(\sum_{\mu} n_\mu (n_\mu - 1) + 2n_\mu a_\mu \right) - a \\ &= 2 \left(\sum_{\mu} \left(\sum_{\nu=0}^{n_\mu-1} \nu + a_\mu \right) \right) - a \equiv 2a - a \pmod{e}, \end{aligned}$$

where the last congruence comes from $\xi_0 \in E'$.

It is easy to check that for $\xi := \bar{k}^1 \otimes \dots \otimes \bar{k}^r$, $\xi_1 := \bar{i}^1 \otimes \dots \otimes \bar{i}^r$

$$\overline{\mathbb{S}_{\xi, \xi_1}} = \mathbb{S}_{\xi, \tilde{\xi}_1} \prod_{\mu=1}^r \gamma_{\tilde{i}^\mu} \prod_{\nu=1}^{n_\mu} \zeta^{-k_\nu^\mu (n_\mu - 1 + 2a_\mu)},$$

where $\gamma_{\tilde{i}^\mu}$ is the sign of the permutation which sorts the tuple of ξ'_1 belonging to \tilde{i}^μ (as in Theorem 4.3). From now on, the proof continues exactly as in Theorem 4.3. Remark that we need Proposition 5.2 at the end. \square

5.1. Eigenvalues and representation of $SL_2(\mathbb{Z})$

As Lusztig did it for his non-abelian Fourier matrix, Malle also defines a matrix \mathbb{T} of eigenvalues of Frobenius associated to a Fourier matrix S' for the complex reflection groups $G(e, 1, n)$ (see [9, 4B]). The matrices S' and \mathbb{T} define an $SL_2(\mathbb{Z})$ -representation.

For a given S -matrix S , we will call a diagonal matrix T such that S and T define a representation of $SL_2(\mathbb{Z})$, i.e.

$$S^4 = 1, \quad (ST)^3 = 1, \quad [S^2, T] = 1$$

a T -matrix associated to S . The pair (S, T) is then called *modular datum* (this is not exactly the usual definition: we do not require that the structure constants defined by Verlinde’s formula are non-negative, and S is not necessarily symmetric). Remark that a T -matrix is in general not uniquely determined by S .

There exists also a T -matrix for the exterior power $\Lambda^n S$ as above. It is obtained by taking the exterior power of a T -matrix corresponding to S (compare [9, 4B]):

Proposition 5.4. *Let $S := (\frac{\zeta^{ij}}{\sqrt{e}})_{i,j}$ be the S -matrix of the group ring of $\mathbb{Z}/e\mathbb{Z}$, where $e \in \mathbb{N}$ and $\zeta := \exp(2\pi i/e)$, $\zeta_{24} := \exp(2\pi i/24)$. Then the diagonal matrix T with*

$$T_{i,i} = \zeta_{24}^{e-1} \zeta^{\frac{i^2+ei}{2}}$$

for $0 \leq i < e$ is a T -matrix for S .

Proof. The equation $[S^2, T] = 1$ is satisfied since T is diagonal and S^2 is the permutation corresponding to complex conjugation on the columns of S . We have to verify $((ST)^3)_{i,j} = \delta_{i,j}$. Define $t_i := T_{i,i}$. Because

$$\zeta^{\frac{i^2+ei}{2}} = \zeta^{\frac{(i+e)^2+e(i+e)}{2}},$$

t_i only depends on the class of $i \pmod e$. This allows us to substitute k by $k - l - i$ in the following equation:

$$\begin{aligned} ((ST)^3)_{i,j} &= \frac{1}{e\sqrt{e}} \sum_{k,l=0}^{e-1} \zeta^{ik+kl+l_j} t_k t_l t_j = \sum_{k,l} \zeta^{i(k-l-i)+(k-l-i)l+l_j} t_{k-l-i} t_l t_j \\ &= \frac{1}{e\sqrt{e}} \zeta_{24}^{3(e-1)} \sum_k \zeta^{\frac{1}{2}(k^2+ek)+\frac{1}{2}(j^2-i^2+ej-ei)} \sum_l \zeta^{lj-li}. \end{aligned}$$

The inner sum is equal to 0 if $i \neq j$ and equal to e if $i = j$. For $i = j$ it remains to prove

$$\sum_{k=0}^{e-1} \zeta^{\frac{1}{2}(k^2+ek)} = \zeta_{24}^{-3(e-1)} \sqrt{e},$$

which is a corollary of a theorem of Gauß (or Eq. 4.11 in [9]). \square

6. Kac–Peterson matrices and exterior powers

The construction of Fourier matrices from exterior powers also shows up in a different context: let $\mathfrak{g}(A)$ be an affine Kac–Moody algebra belonging to an $n \times n$ generalized Cartan matrix A of rank l , \mathfrak{h} its Cartan subalgebra and $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{C}$ the corresponding pairing (we use the notation of [7]). Define

$$\begin{aligned} P &:= \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, i = 0, \dots, n-1 \}, \\ P_+ &:= \{ \lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0, i = 0, \dots, n-1 \}. \end{aligned}$$

Now let $\mathfrak{g}(A)$ be of arbitrary untwisted type $X_l^{(1)}$ or $A_{2l}^{(2)}$ (then $n = l + 1$). The *fundamental weights* $\Lambda_i \in P, i = 0, \dots, l$, are given by the equations

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \langle \Lambda_i, d \rangle = 0,$$

for $j = 0, \dots, l$, where $d \in \mathfrak{h}^*$ is given by $\langle \alpha_i, d \rangle = \delta_{i,0}$. The $\{ \alpha_0^\vee, \dots, \alpha_l^\vee, d \}$ form a basis of \mathfrak{h} and $\{ \alpha_0, \dots, \alpha_l, \Lambda_0 \}$ form a basis of \mathfrak{h}^* . The fundamental weights $\bar{\Lambda}_i$ of the finite-dimensional Lie algebra \mathfrak{g}° satisfy

$$\Lambda_i = \bar{\Lambda}_i + a_i^\vee \Lambda_0$$

($\bar{\Lambda}_0 = 0$ because $a_0^\vee = 1$; for a definition of a_i^\vee , see [7, 6.1]).

For each positive integer k , let $P_+^k \subseteq P_+$ be the finite set

$$P_+^k := \left\{ \sum_{j=0}^l \lambda_j \Lambda_j \mid \lambda_j \in \mathbb{Z}, \lambda_j \geq 0, \sum_{j=0}^l a_j^\vee \lambda_j = k \right\}.$$

Kac and Peterson defined a natural \mathbb{C} -representation of the group $SL_2(\mathbb{Z})$ on the subspace spanned by the affine characters of \mathfrak{g} which are indexed by P_+^k . The image of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ under

this representation is determined in Theorem 13.8 of [7]. It is the so-called *Kac–Peterson matrix*. For affine algebras of type $X_l^{(1)}$ or $A_{2l}^{(2)}$, this matrix is

$$S_{\Lambda, \Lambda'} = c \sum_{w \in W^\circ} \det(w) \exp\left(-\frac{2\pi i(\bar{\Lambda} + \bar{\rho} | w(\bar{\Lambda}' + \bar{\rho}))}{k + h^\vee}\right), \tag{4}$$

where Λ, Λ' runs through P_+^k , $(\cdot | \cdot)$ is the normalized bilinear form of Chapter 6 of [7] and W° is the Weyl group of \mathfrak{g}° . The constant c is unimportant for us, since we want to use the matrix in the formula of Verlinde (1).

Each of these matrices defines a based ring. A classification of the matrices belonging to type $X_l^{(1)}$ up to isomorphism was given by Gannon in [5]. Here we prove that the matrices of type $A_1^{(1)}$ are connected to those of type $C_l^{(1)}$ via exterior powers:

Proposition 6.1. *Let $k, l \in \mathbb{N}$, $k \geq 1$, $l \geq 2$ and S be the Kac–Peterson matrix of type $A_1^{(1)}$ and level $(k + l - 1)$. The Kac–Peterson matrix of type $C_l^{(1)}$ and level k is the exterior power $\Lambda^l S$ of S .*

Proof. We use the notation of [7]. Let A be the Cartan matrix of type $C_l^{(1)}$. Choose \bar{a}, \bar{a}^\vee elements of the kernel of A respectively A^T , say $\bar{a} := (2, \dots, 2, 1, 1)$, $\bar{a}^\vee := (1, \dots, 1)$ (arrange the matrix in such a way that α_0 is at the end). Furthermore, we have $\kappa := k + h^\vee = k + \sum_{i=0}^l a_i^\vee = k + l + 1$, $\bar{\rho} := (1, \dots, 1)$,

$$P_+^k = \left\{ \bar{\lambda} + \bar{\rho} \mid \bar{\lambda} \in \{0, \dots, k\}^l, \sum_{i=1}^l a_i^\vee \lambda_i \leq k \right\}$$

and W° the Weyl group of type C_l . Let D be the diagonal matrix with $\frac{a_1^\vee}{a_1}, \dots, \frac{a_l^\vee}{a_l}$ on its diagonal. Then Eq. (4) becomes

$$\begin{aligned} S_{\mu, \nu} &= c \sum_{w \in W^\circ} \det(w) \exp\left(-\frac{2\pi i}{\kappa} \mu D w^T A^{-1T} \nu^T\right) \\ &= c \sum_{w \in W^\circ} \det(w) \exp\left(-\frac{2\pi i}{\kappa} \mu M_w \nu^T\right), \end{aligned}$$

where $\mu, \nu \in P_+^k$ and $M_w := D w^T A^{-1T}$. Consider the set

$$\tilde{P} := \left\{ \tilde{\mu} := \left(\mu_1, \mu_1 + \mu_2, \dots, \sum_{i=1}^l \mu_i \right) \mid \mu \in P_+^k \right\}.$$

The base change $\mu \mapsto \tilde{\mu}$ transforms the M_w , $w \in W^\circ$, into monomial matrices with entries $\pm \frac{1}{2}$ (up to a factor, this is the base change $\alpha_i \mapsto \nu_i$ from [7, 6.7]). The formula for S is now

$$\begin{aligned}
 S_{\tilde{\mu}, \tilde{\nu}} &= c \sum_{\sigma \in S_l} \sum_{f \in \{\pm 1\}^l} \left(\varepsilon_{\sigma} \prod_{i=1}^l f_i \right) \zeta^{-\sum_{i=1}^l f_i \tilde{\mu}_i \tilde{\nu}_{\sigma(i)}} = c \sum_{\sigma \in S_l} \sum_{f \in \{\pm 1\}^l} \varepsilon_{\sigma} \prod_{i=1}^l f_i \zeta^{-f_i \tilde{\mu}_i \tilde{\nu}_{\sigma(i)}} \\
 &= \tilde{c} \sum_{\sigma \in S_l} \varepsilon_{\sigma} \prod_{i=1}^l (\zeta^{\tilde{\mu}_i \tilde{\nu}_{\sigma(i)}} - \zeta^{-\tilde{\mu}_i \tilde{\nu}_{\sigma(i)}}),
 \end{aligned}$$

where $\zeta := \exp(\frac{2\pi i}{2\kappa})$. In the last term, we recognize the determinants of the $l \times l$ -submatrices of the Kac–Peterson matrix of type $A_1^{(1)}$ and level $k + l - 1 = \kappa - 2$. \square

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