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Global Asymptotic Stability of Nonlinear Cascade Systems

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Abstract—In this paper, we give a new, quick proof for a known result on the global asymptotic stability of continuous-time nonlinear cascade systems. Next, we state and prove a similar result for the global asymptotic stability of discrete-time nonlinear cascade systems. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Nonlinear systems, Cascade systems, Global asymptotic stability.

1. INTRODUCTION

In this paper, we study the stability problem of a nonlinear cascade system of the form

$$\begin{aligned}\dot{x} &= f(x, \omega),\\ \dot{\omega} &= s(\omega),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$. We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $s : \mathbb{R}^m \to \mathbb{R}^m$ are both $\mathcal{C}^1$ vector fields. We also assume that $f(0,0) = 0$, $s(0) = 0$, so that $(x, \omega) = (0,0)$ is an equilibrium of the cascade system (1).

An interesting question in control systems is whether the asymptotic stability of the subsystems

$$\dot{x} = f(x, 0) \qquad \text{and} \qquad \dot{\omega} = s(\omega) \tag{2}$$

imply the asymptotic stability of the cascade system (1). Locally, this is true [1]. In [2], Seibert and Suarez derived sufficient conditions for global asymptotic stability of the cascade system (1).

THEOREM 1. *(See [2].) Suppose that $x = 0$ is a globally asymptotically stable equilibrium of the subsystem*

$$\dot{x} = f(x, 0),$$

$\omega = 0$ is a globally asymptotically stable equilibrium of the subsystem

$$\dot{\omega} = s(\omega),$$

and that all the trajectories $(x(t), \omega(t))$ of (1) are bounded for $t > 0$. Then $(x, \omega) = (0,0)$ is a globally asymptotically stable equilibrium of the cascade system (1). ∎

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Interpreting ω as an input, Theorem 1 can also be equivalently stated in the form of the *input-to-stability* (ISS) results of Sontag [3,4]. We note that Sontag's results also assume that the trajectories of the cascade system (1) are bounded for $t > 0$.

This paper is organized as follows. In Section 2, we give a new, quick proof of Theorem 1. In Section 3, we derive a similar result for the discrete-time nonlinear cascade systems.

2. PROOF OF THEOREM 1

Our proof uses Massera's converse Lyapunov theorem for asymptotic stability of nonlinear systems [5] and LaSalle's invariance principle for nonlinear autonomous systems [6].

Since $\omega = 0$ is a globally asymptotically stable equilibrium of the subsystem $\dot{\omega} = s(\omega)$, it follows from Massera's converse Lyapunov theorem [5] that there exists a $\mathcal{C}^1$ Lyapunov function $U : \mathbb{R}^m \to \mathbb{R}$ for the subsystem $\dot{\omega} = s(\omega)$.

Thus, U is a $\mathcal{C}^1$ positive definite function on $\mathbb{R}^m$ and

$$\dot{U}(\omega) = \frac{\partial U}{\partial \omega}(\omega) \cdot s(\omega)$$

is a negative definite function on $\mathbb{R}^m$.

To show that $(x, \omega) = (0,0)$ is a globally asymptotically stable equilibrium of the cascade system (1), we consider the candidate Lyapunov function

$$V(x, \omega) = U(\omega).$$

Then we have

$$\dot{V}(x, \omega) = \dot{U}(\omega) = \frac{\partial U}{\partial \omega}(\omega) \cdot s(\omega) \leq 0, \qquad \forall\, (x, \omega) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Hence, by LaSalle's invariance principle [6], as $t \to \infty$, all trajectories $(x(t), \omega(t))$ of the cascade system (1) (which are globally bounded for $t > 0$) tend to the largest invariant subset of the locus of points defined by

$$\dot{V}(x, \omega) = \dot{U}(\omega) = 0.$$

Since $\dot{U}(\omega)$ is a negative definite function, it follows that $\dot{U}(\omega) = 0 \iff \omega = 0$. Also, when $\omega = 0$, the differential equation for x reduces to

$$\dot{x} = f(x, 0),$$

which has the origin $x = 0$ as a globally asymptotically stable equilibrium.

Hence, we conclude that $(x, \omega) = (0,0)$ is a globally asymptotically stable equilibrium of the cascade system (1). ∎

3. SUFFICIENT CONDITIONS FOR THE DISCRETE-TIME NONLINEAR CASCADE SYSTEMS

In this section, we study the stability problem of a discrete-time nonlinear cascade system of the form

$$\begin{aligned} x_{k+1} &= f\left(x_k, \omega_k\right), \\ \omega_{k+1} &= s\left(\omega_k\right), \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$, $\omega \in \mathbb{R}^m$. We assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $s : \mathbb{R}^m \to \mathbb{R}^m$ are both $\mathcal{C}^1$ maps. We also assume that $f(0,0) = 0$, $s(0) = 0$, so that $(x, \omega) = (0,0)$ is an equilibrium of the cascade system (3). Let $\mathcal{Z}_+$ denote the set of all positive integers.

Now, we state and prove a theorem giving sufficient conditions for global asymptotic stability of discrete-time nonlinear cascade systems of form (3).

THEOREM 2. *Suppose that $x = 0$ is a globally asymptotically stable equilibrium of the subsystem*

$$x_{k+1} = f(x_k, 0),$$

$\omega = 0$ is a globally asymptotically stable equilibrium of the subsystem

$$\omega_{k+1} = s(\omega_k),$$

and that all the trajectories (x_k, ω_k) of (3) are bounded for $k \in \mathcal{Z}_+$. Then $(x, \omega) = (0, 0)$ is a globally asymptotically stable equilibrium of the cascade system (3).

PROOF. Since $\omega = 0$ is a globally asymptotically stable equilibrium of the subsystem $\omega_{k+1} = s(\omega_k)$, it follows from Lyapunov stability theory [7] that there exists a $\mathcal{C}^1$ Lyapunov function $U : \mathbb{R}^m \to \mathbb{R}$ for the subsystem $\omega_{k+1} = s(\omega_k)$. Thus, U is a $\mathcal{C}^1$ positive definite function on $\mathbb{R}^m$ and

$$\nabla U(\omega) = U(s(\omega)) - U(\omega)$$

is a negative definite function on $\mathbb{R}^m$.

To show that $(x, \omega) = (0, 0)$ is a globally asymptotically stable equilibrium of the cascade system (3), we consider the candidate Lyapunov function

$$V(x, \omega) = U(\omega).$$

Then we have

$$\nabla V(x, \omega) = \nabla U(\omega) = U(s(\omega)) - U(\omega) \leq 0, \qquad \forall (x, \omega) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Hence, by LaSalle's invariance principle [7], as $k \to \infty$, all trajectories (x_k, ω_k) of the cascade system (3) (which are globally bounded for $k \in \mathcal{Z}_+$) tend to the largest invariant subset of the locus of points defined by

$$\nabla V(x, \omega) = \nabla U(\omega) = 0.$$

Since $\nabla U(\omega)$ is a negative definite function, it follows that $\nabla U(\omega) = 0 \iff \omega = 0$. Also, when $\omega = 0$, the difference equation for x reduces to

$$x_{k+1} = f(x_k, 0),$$

which has the origin $x = 0$ as a globally asymptotically stable equilibrium.

Hence, we conclude that $(x, \omega) = (0, 0)$ is a globally asymptotically stable equilibrium of the cascade system (3). ∎

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