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Proof of the perpetuity equation

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ABSTRACT

A perpetuity is a perpetual annuity. Although there have been a number of different derivations, which we discuss in detail, we present what appears to be the first mathematical proof of the perpetuity equation based on the fundamental properties of the real numbers (Result (2.2.1) of Dieudonne (1960) [14]).

A perpetuity is simply a perpetual annuity. Perhaps the best example is the determination of the minimum amount, \$$P$, of an endowment that would be required to provide a scholarship of exactly \$$A$ starting at the end of one period and going on forever if the interest earned (or rate of return) on the endowment is i% per period. The perpetuity equation states that

$$P = \frac{A}{i}. \tag{1}$$

Note that the present value, P, of the perpetuity is sometimes called the capitalized cost (see [1–3]) or the capitalized worth of A (see [4,5]). There are a number of different derivations of Eq. (1) which will be discussed below. However, we will present what appears to be the first mathematical proof of the equation.

The most common derivation (see [1–3], for example) of Eq. (1) involves taking the limit as n goes to infinity of the following equation for the present value at time 0 of a finite annuity (or a uniform series of payments A that start at the end of period 1 and go to the end of period n).

$$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]. \tag{2}$$

This is, of course, easier to see if either one divides both the numerator and the denominator of Eq. (2) by $(1+i)^n$, as done in [4,5], or by separating the numerator of Eq. (2) into two terms, as done in [6,7], before taking the limit as n approaches infinity.

Another common derivation of Eq. (1) involves simply solving for P as the sum of an infinite (geometric) series of the A's as done in [8]. A more interesting approach to the derivation of Eq. (1) is loosely based on the principle of optimality of dynamic programming [9]. Specifically, if exactly A is to be provided every period, then one must have exactly $P + A$ left at the end of every period as noted in [10,11]. Therefore, we must have

$$P + iP = P + A \tag{3}$$

which when solved results in Eq. (1).

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Our proof of the perpetuity equation (1) is based on the fundamental order properties of the real numbers (which also endow them with certain topological properties [12]) that has been aptly named the Trichotomy Law of Real Numbers (Result 1.1(1) of [13]).

Lemma 0.1 (*Result (2.2.1) of [14]*)**.** *For every pair of real numbers, x and y, one and only one of the following three relations holds:*

(i) *Either $x < y$ or,*
(ii) *$x = y$ or,*
(iii) *$x > y$.*

Theorem 0.2. *The perpetuity equation (1) is valid, i.e.,*

$$P = \frac{A}{i}.$$

Proof. Consider the relationship between iP and A. If $iP < A$, then it would not even be possible to provide A at the end of period 1, much less at future periods. On the other hand, if $iP > A$, then either a payment greater than A will be provided at the end of period 1 or P is greater than the minimum required and will even continue to grow and this trend will escalate in future periods. Therefore, by the lemma, it must be true that

$$iP = A \tag{4}$$

so that $P = \frac{A}{i}$, thus completing our proof. □

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