A global stability criterion in nonautonomous delay differential equations

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Abstract

Consider the following nonautonomous nonlinear delay differential equation:

\[
\begin{cases}
    \frac{dy(t)}{dt} = -a(t)y(t) - \sum_{i=0}^{m} a_i(t)g_i(y(\tau_i(t))), & t \geq t_0, \\
    y(t) = \phi(t), & t \leq t_0,
\end{cases}
\]

where we assume that there is a strictly monotone increasing function \( f(x) \) on \((-\infty, +\infty)\) such that

\[
\begin{align*}
    f(0) &= 0, \quad 0 < \frac{g_i(x)}{f(x)} \leq 1, \quad x \neq 0, \quad 0 \leq i \leq m, \quad \text{and} \\
    \text{if } f(x) \neq x, \quad \text{then } \lim_{x \to -\infty} f(x) \text{ or } \lim_{x \to +\infty} f(x) \text{ is finite.}
\end{align*}
\]

In this paper, to the above nonautonomous nonlinear delay differential equation, we establish conditions of global asymptotic stability for the zero solution. In particular, for a special wide class of \( f(x) \) which contains two cases \( f(x) = e^x - 1 \) and \( f(x) = x \), we give more explicit conditions which are some extension of the “3/2-type criterion.” Applying these to discrete models of nonautonomous delay differential equations, we also obtain new sufficient conditions of the global asymptotic stability of the zero solution.

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1. Introduction

Let us consider the following nonautonomous nonlinear delay differential equation:

\[
\begin{aligned}
\frac{dy(t)}{dt} = -a(t)y(t) - \sum_{i=0}^{m} a_i(t) g_i(y(\tau_i(t))), & \quad t \geq t_0, \\
y(t) = \phi(t), & \quad t \leq t_0,
\end{aligned}
\]

where in this paper, for simplicity, we denote that \( y'(t) \) means the left-hand side Dini-derivative of \( y(t) \), if necessary, and we assume

\[
\begin{aligned}
a(t) \text{ is continuous and bounded on } [t_0, +\infty), \\
a_i(t), 0 \leq i \leq m, \text{ are piecewise continuous and bounded on } [t_0, +\infty), \\
\inf_{t \geq t_0} a(t) > 0, a_i(t) \geq 0, 0 \leq i \leq m, \\
\sum_{i=0}^{m} a_i(t) > 0, \quad \int_{t_0}^{\infty} \sum_{i=0}^{m} a_i(t) \, dt = +\infty, \\
\tau_i(t) \text{ is piecewise continuous on } [t_0, +\infty) \text{ and } \tau_i(t) \leq t, \quad t \geq t_0, \quad \sup_{t \geq t_0} \{t - \tau_i(t)\} \leq \hat{\tau} < +\infty, 0 \leq i \leq m, \\
\phi(t) \text{ is continuous and bounded on } [t_0 - \hat{\tau}, t_0], \\
g_i(x) \text{ is continuous on } (-\infty, +\infty) \text{ and for } x > 0, \\
g_i(-x) < g_i(0) = 0 < g_i(x), 0 \leq i \leq m,
\end{aligned}
\]

and there is a strictly monotone increasing function \( f(x) \) on \((-\infty, +\infty)\) such that

\[
\begin{aligned}
f(0) = 0, \quad 0 < \frac{g_i(x)}{f(x)} \leq 1, \quad x \neq 0, \quad 0 \leq i \leq m, \quad \text{and} \\
\text{if } f(x) \neq x, \quad \text{then } \lim_{x \to -\infty} f(x) \text{ or } \lim_{x \to +\infty} f(x) \text{ is finite.}
\end{aligned}
\]

Put

\[
\begin{aligned}
\tau^{-1}(t) &= \sup\{s \mid \tau(s) \leq t\}, \\
\lambda &= \left( \sup_{t \geq \tau^{-1}(t_0)} \sum_{i=0}^{m} a_i(t) \right) \exp\left( \sup_{t \geq \tau^{-1}(t_0)} \int_{\tau(t)}^{t} a(s) \, ds \right) - 1, \quad \text{and} \\
\mu &= \exp\left( \inf_{t \geq \tau^{-1}(t_0)} \int_{\tau(t)}^{t} a(s) \, ds \right).
\end{aligned}
\]

Note that by assumption (1.2), \( 1 < \mu < +\infty \) and \( 0 < \lambda < +\infty \).

Concerning the autonomous case of (1.1) that \( a(t) \equiv 0, a_i(t) \equiv a_i, 0 \leq i \leq m, \) and \( f(x) = x, \)

\[
\lambda = (\sum_{i=0}^{m} a_i) \hat{\tau} \text{ and } \mu = 1, \quad \text{and there are many papers on a condition } \lambda < \frac{3}{2} \text{ of global asymptotic stability for the zero solution of (1.1), for instance, Yorke \[1\], Yoneyama \[2,3\], Muroya \[4\] for } m = 1 \text{ and Krisztin \[5\] for distributed delays (see also Arino et al. \[6\], Györi and Ladas \[7\], Kulenovic et al. \[8\] and Yoneyama and Sugie \[9\]).}
On the other hand, for the autonomous case of (1.1) that $a(t) \equiv 0$ and $f(x) = e^x - 1$, by Gopalsamy et al. [10], it was obtained a condition $\lambda = r(m + 1) \leq \log 2$ of global asymptotic stability for the zero solution of (1.1), where each $a_i(t)$ is a constant $r_i$ with $\sum_{i=0}^{m} a_i = 1$ and each $\tau_i(t)$ is a piecewise constant delay $[t - i], 0 \leq i \leq m$ (cf. Gopalsamy [11]). Here, $[t]$ means the maximal integer not greater than $t$. By So and Yu [12], this condition was improved to
$$\lambda = \sup_{n \geq m} \int_{n-1}^{n+1} r(s) \, ds \leq \frac{3}{2}$$
for variable coefficients $a_i(t) = r(t) a_i$, $0 \leq i \leq m$, with $\sum_{i=0}^{m} a_i = 1$ (see also Yu [13] and Zhou and Zhang [14], and cf. Furumochi [15], Seifert [16] and Matsunaga et al. [17] for a piecewise constant delay). For a nonautonomous Lotka–Volterra competition system with distributed delays but without instantaneous negative feedbacks, Tang and Zou [18, 19] established some $3/2$-type criteria for global attractivity of the positive equilibrium of the system. Applying the similar techniques in So and Yu [12] to the above nonautonomous nonlinear delay differential equation (1.1) for the special case $a(t) \equiv 0$, Muroya [20] established conditions of the global asymptotic stability for the zero solution.

Recently, introducing the generalized York conditions, Liz, Tkachenko and Trofimchuk [21] have determined an important family of scalar functional differential equations for which a simple criterion of global attractivity was found. They consider the scalar functional differential equation

$$\dot{x}(t) = -ax(t) + f(t, x_t) \quad (x_t(s) = x(t + s), s \in [-h, 0]),$$

where it is assumed that $f : R \times C[-h, 0] \to R$ is a measurable function satisfying the following additional condition (H):

(H1) $f$ satisfies the Carathéodory condition, and for every $q \in R$, there exists $\theta : R \to R$ such that $f(t, \phi) \leq \theta(q)$ almost everywhere on $R$ for every $\phi \in C[-h, 0]$ satisfying the inequality $\phi(s) \geq q, s \in [-h, 0]$.

(H2) There are $b > 0, a < 0$ such that for $M(\phi) = \max(0, \max_{s \in [-h, 0]} \phi(s))$,

$$f(t, \phi) \geq \frac{aM(\phi)}{1 + bM(\phi)} \quad \text{for all } \phi \in C[-h, 0],$$

and

$$f(t, \phi) \leq \frac{-aM(-\phi)}{1 - bM(-\phi)} \quad \text{for all } \phi \in C[-h, 0] \quad \text{such that}$$

$$\min_{s \in [-h, 0]} \phi(s) > -b^{-1} \in [-\infty, 0).$$

Liz, Tkachenko and Trofimchuk [21] established the following nice result.

**Theorem A.** Assume that (H) holds and let $x : [\alpha, \omega) \to R$ be a solution of (1.5) defined on the maximal interval of existence. Then $\omega = +\infty$ and $x$ is bounded on $[\alpha, +\infty)$. If, additionally, the following condition holds:

$$-\frac{\delta}{a} \exp(-h\delta) > \frac{a^2 - a\delta}{\delta^2 + a^2},$$

then $\lim_{t \to +\infty} x(t) = 0$. Furthermore, condition (1.8) is sharp within the class of equations satisfying (H): for every triple $a < 0, \delta > 0, h > 0$ which does not meet (1.8), there is a nonlinearity $f$ satisfying (H) and such that the equilibrium $x(t) = 0$ of the corresponding equation (1.5) is not asymptotically stable.
Note that the rational function \( r(x) = ax/(1 + x) \) played a key role in their constructions.

In this paper, expanding the techniques in Muroya [20] to the above nonautonomous nonlinear delay differential equation (1.1) with (1.2) and (1.3), we establish conditions of the global asymptotic stability for the zero solution (see Theorem 2.1).

In particular, applying the similar techniques in Tang and Zou [18,19] and Muroya [20] to a special wide class of \( f(x) \) which contains two cases \( f(x) = e^x - 1 \) and \( f(x) \equiv x \), we give more explicit conditions which are some extension of the “3/2-type criterion” (see Theorems 1.1 and 2.2). For an application, we also offer new conditions of the global asymptotic stability for solutions of discrete models of nonautonomous delay differential equations (see Theorem 2.3).

According to Muroya [20], as a wide class of \( f(x) \) which satisfies (1.3), we take

\[
 f(x) = \int_0^x e^{\int_0^y T(z) \, dz} \, dy,
\]

(1.9)

where \( T(z) \) is continuous on \((-\infty, +\infty)\).

Then, \( f(0) = 0 \), \( f'(x) = e^{\int_0^x T(z) \, dz} > 0 \) and \( f'(0) = 1 \).

In particular, corresponding to a condition (2.10) in Lemma 2.4, we can take

\[
 T(z) = \begin{cases} 
 \gamma_1, & z \geq 0, \\
 \gamma_2, & z < 0,
\end{cases}
\]

(1.10)

where \( \gamma_1 \) and \( \gamma_2 \) are constants such that \( \gamma_1 \leq \gamma_2 \) and \( \gamma_1^2 + \gamma_2^2 > 0 \), or \( \gamma_1 > \gamma_2 \) and \( \gamma_1 \gamma_2 > 0 \).

Then, for \( x > 0 \), we obtain that

\[
 f(x) = \begin{cases} 
 \frac{e^{\gamma_1 x} - 1}{\gamma_1}, & \gamma_1 \neq 0, \\
 \gamma_1 = 0,
\end{cases}
\]

and

\[
 f^{-1}(x) = \begin{cases} 
 \frac{1}{\gamma_1} \log(1 + \gamma_1 x), & \gamma_1 \neq 0, \\
 x, & \gamma_1 = 0,
\end{cases}
\]

(1.12)

and for \( x < 0 \),

\[
 f(x) = \begin{cases} 
 \frac{e^{\gamma_2 x} - 1}{\gamma_2}, & \gamma_2 \neq 0, \\
 \gamma_2 = 0,
\end{cases}
\]

and

\[
 f^{-1}(x) = \begin{cases} 
 \frac{1}{\gamma_2} \log(1 + \gamma_2 x), & \gamma_2 \neq 0, \\
 x, & \gamma_2 = 0.
\end{cases}
\]

(1.13)

In this case, we can see that if \( f(x) \not= x \), that is, \( \gamma_1^2 + \gamma_2^2 > 0 \), then \( \lim_{x \to -\infty} f(x) \) or \( \lim_{x \to +\infty} f(x) \) is finite.

Note that for the cases \( f(x) \not= x \) defined by (1.11)–(1.13), we may restrict our attention to a special function \( f(x) = e^x - 1 \) (see Lemma 2.5). Moreover, it holds that for \( x < 1 \), \( e^x - 1 \leq r(-x) = x/(1 - x) \) with \( a = -1 \) (cf. Liz, Tkachenko and Trofimchuk [21]).

To the cases that we can choose this special wide class of \( f(x) \), we obtain our main result in this paper as follows.

**Theorem 1.1.** In (1.1)–(1.4), assume that \( f(x) \not= x \) is defined by (1.11)–(1.13). If \( \lambda \leq \hat{\lambda} = \frac{1}{2} + \mu \), then the zero solution of (1.1)–(1.3) is globally asymptotically stable.

Obtained results are direct extensions of those of Muroya [20] to nonautonomous separable nonlinear delay differential equations (1.1) with \( a(t) > 0 \). Applying these and techniques in Tang and Zou [18,19], some improved 3/2-type criteria for global asymptotic stability for solutions of nonautonomous Lotka–Volterra systems with delays is obtained (see Muroya [22]).
2. Global stability for nonautonomous nonlinear delay differential equations

In this section, we first consider global stability conditions for solutions of a nonautonomous nonlinear delay differential equation (1.1) with (1.2) and (1.3).

Applying the techniques used in So and Yu [12], we obtain the following lemmas and theorem (see Lemmas 2.1, 2.2 and Theorem 3.1 in So and Yu [12]).

**Lemma 2.1.** Let \( y(t) \) be the solution of (1.1) with (1.2) and (1.3). If \( y(t) \) is eventually greater (respectively less) than 0, then \( y(t) \) is eventually decreasing (respectively increasing), and \( \lim_{t \to \infty} y(t) \) exists and it holds \( \lim_{t \to \infty} y(t) = 0 \).

**Proof.** Assume that \( y(t) \) is eventually greater than 0. Then, by (1.1), we have eventually

\[
y'(t) \leq - \sum_{i=0}^{m} a_i(t) g_i(0) = 0,
\]

which implies that \( y(t) \) is eventually decreasing, and so \( \lim_{t \to \infty} y(t) \) exists. Set

\[
\alpha = \lim_{t \to \infty} y(t).
\]

Suppose \( \alpha > 0 \). Then, there exists \( \bar{t}_1 \geq t_0 + \bar{\tau} \) such that

\[
y(t_i(t)) \geq \alpha, \quad 0 \leq i \leq m \text{ for } t \geq \bar{t}_1,
\]

since \( y(t) \) eventually decreases to \( \alpha \). Using this and (1.1), we have

\[
\frac{dy(t)}{dt} \leq -a(t)y(t) - \sum_{i=0}^{m} a_i(t) g_i(\alpha) \quad \text{for } t \geq \bar{t}_1.
\]

Thus, we have

\[
\frac{d(e^{\int_{t_0}^{t} a(s) ds} y(t))}{dt} \leq -e^{\int_{t_0}^{t} a(s) ds} \sum_{i=0}^{m} a_i(t) g_i(\alpha) \quad \text{for } t \geq \bar{t}_1.
\]

Integrating from \( \bar{t}_1 \) to \( t \), we have

\[
y(t) \leq e^{-\int_{\bar{t}_1}^{t} a(s) ds} y(\bar{t}_1) - \left( \int_{\bar{t}_1}^{t} e^{-\int_{\tau}^{t} a(s) ds} \sum_{i=0}^{m} a_i(s) ds \right) \min_{0 \leq j \leq m} g_j(\alpha),
\]

which implies, due to \( a(t) \geq a > 0 \) and \( \int_{t_0}^{\infty} \sum_{i=0}^{m} a_i(t) dt = +\infty \),

\[
\lim_{t \to \infty} y(t) = -\infty.
\]

This contradicts \( \alpha > 0 \). Hence, \( \lim_{t \to \infty} y(t) = 0 \).

The case that \( y(t) \) is eventually less than 0, is similar, and hence the proof is complete. \( \square \)

**Lemma 2.2.** Let \( f(x) \neq x \) and \( y(t) \) be the solution of (1.1) with (1.2) and (1.3) and put (1.4). If \( y(t) \) is oscillatory about 0, then \( y(t) \) is bounded above and below.
Proof. Let us consider the case that \( \lim_{x \to -\infty} f(x) = -\beta > -\infty \). Then, by (1.1)–(1.3),
\[
\frac{d}{dt} \left( e^{\int_{t_0}^{t} a(s) ds} y(t) \right) \leq \beta e^{\int_{t_0}^{t} a(s) ds} \sum_{i=0}^{m} a_i(t) \quad \text{for } t \geq t_0 + \hat{\tau}.
\]
(2.1)

First we prove that \( y(t) \) is bounded above. Suppose \( \limsup_{t \to \infty} y(t) = +\infty \). Then, by Lemma 2.1, \( y(t) \) is both unbounded and oscillatory, and hence, there exists a strictly monotone increasing sequence \( \{\bar{t}_p\}_{p=1}^{\infty} \) such that \( \bar{t}_p \geq t_0 + \hat{\tau} \) and
\[
y(\bar{t}_p) = \max_{t_0 \leq t \leq \bar{t}_p} y(t) > 0, \quad y'(\bar{t}_p) \geq 0 \quad \text{and} \quad \lim_{p \to \infty} y(\bar{t}_p) = +\infty.
\]
Then,
\[
0 \leq y'(\bar{t}_p) = -a(\bar{t}_p)y(\bar{t}_p) - \sum_{i=0}^{m} a_i(\bar{t}_p)g_i\left(y\left(\tau_i(\bar{t}_p)\right)\right),
\]
and so \( a(\bar{t}_p)y(\bar{t}_p) + \sum_{i=0}^{m} a_i(\bar{t}_p)g_i\left(y\left(\tau_i(\bar{t}_p)\right)\right) \leq 0 \). Thus, there exists \( \xi_p \in [\tau(\bar{t}_p), \bar{t}_p) \) such that \( y(\xi_p) = 0 \) and \( y(t) > 0 \) for \( t \in (\xi_p, \bar{t}_p] \). Integrating (2.1) from \( \xi_p \) to \( \bar{t}_p \), we have
\[
y(\bar{t}_p) \leq \beta \int_{\xi_p}^{\bar{t}_p} e^{\int_{t_0}^{s} a(s) ds} \sum_{i=0}^{m} a_i(t) dt \leq \beta \lambda.
\]
Consequently, \( \limsup_{p \to \infty} y(\bar{t}_p) \leq \beta \lambda \). This contradiction shows that \( y(t) \) is bounded above. Similar to the discussion above, we also see that
\[
y(t) \leq \beta \lambda \quad \text{for } t \geq t_0 + \hat{\tau}.
\]
Thus, by (1.1)–(1.3), we have
\[
\left(e^{\int_{t_0}^{t} a(s) ds} y(t)\right)' \geq -\left(e^{\int_{t_0}^{t} a(s) ds} \sum_{i=0}^{m} a_i(t)\right) f(\beta \lambda) \quad \text{for } t \geq t_0 + 2\hat{\tau}.
\]
(2.2)

Next, we will show that \( y(t) \) is bounded below. Suppose that \( \liminf_{t \to \infty} y(t) = -\infty \). Since \( y(t) \) is oscillatory about 0, there exists a strictly monotone increasing sequence \( \{t_p\}_{p=1}^{\infty} \) such that \( t_p \geq t_0 + 3\hat{\tau} \) and
\[
y(t_p) = \min_{t_0 \leq t \leq t_p} y(t) < 0, \quad y'(t_p) \leq 0 \quad \text{and} \quad \lim_{p \to \infty} y(t_p) = -\infty.
\]
Then,
\[
0 \geq y'(t_p) = -a(t_p)y(t_p) - \sum_{i=0}^{m} a_i(t_p)g_i\left(y\left(\tau_i(t_p)\right)\right),
\]
which shows that there exists \( \eta_p \in [\tau(t_p), t_p) \) such that \( y(\eta_p) = 0 \) and \( y(t) < 0 \) for \( t \in (\eta_p, t_p] \). Integrating (2.2) from \( \eta_p \) to \( t_p \), we have
\[
y(t_p) \geq -\left(\int_{\eta_p}^{t_p} e^{\int_{t_0}^{s} a(s) ds} \sum_{i=0}^{m} a_i(t) dt\right) f(\beta \lambda) \geq -\lambda f(\beta \lambda).
\]
Consequently, \( \liminf_{t \to \infty} y(t) \geq -\lambda f(\beta \lambda) \), which is a contradiction. Thus, \( y(t) \) is bounded below.

The case that \( \lim_{x \to +\infty} f(x) < +\infty \), is similar and its proof is omitted. Hence the proof is complete. \( \square \)

**Remark 2.1.** If \( f(x) \neq x \), then by Lemma 2.2, we see that any solution \( y(t) \) of (1.1) which is oscillatory about 0, is bounded above and below.

By Lemma 2.1, hereafter, we restrict our attention to the case that a solution \( y(t) \) of (1.1) considered, is oscillatory about 0. Then, by Lemma 2.2, \( y(t) \) is bounded above and below.

For \( \lambda, \bar{u}, \bar{v} > 0 \), we define that
\[
F(u, \lambda) = \begin{cases} 
\frac{1}{\mu u} \int_{0}^{\lambda u} f(x) \, dx, & \lambda < \frac{f^{-1}(u)}{u}, \\
\mu^{-1}(\lambda u - f^{-1}(u)) + \frac{1}{\mu u} \int_{0}^{f^{-1}(u)} f(x) \, dx, & \lambda \geq \frac{f^{-1}(u)}{u},
\end{cases}
\]
and for a constant \( \lambda^* > 0 \),
\[
G(v, \lambda; \lambda^*) = \begin{cases} 
-\frac{1}{\mu v} \int_{0}^{-\lambda v} f(x) \, dx, & \lambda < \lambda^*, \\
\mu^{-1}(\lambda v - \lambda^* v) - \frac{1}{\mu v} \int_{-\lambda v}^{0} f(x) \, dx, & \lambda \geq \lambda^*.
\end{cases}
\]

The following lemma is a basic result in this paper.

**Lemma 2.3.** In (1.1)–(1.4), assume that \( f(x) \neq x \). Consider any solution \( y(t) \) of (1.1) which is oscillatory about 0. For \( x(t) = f(y(t)) \), we choose \( u, v, \bar{u} \) and \( \bar{v} \) by
\[
u = \limsup_{t \to \infty} x(t) < \bar{u} < +\infty \quad \text{and} \quad v = -\liminf_{t \to \infty} x(t) < \bar{v} < +\infty,
\]
and for a constant \( \lambda^* > 0 \) we take (2.3)–(2.4).

Then for \( \lambda > 0 \) defined by (1.4), it holds that for \( 0 < u < \bar{u} \) and \( 0 < v < \bar{v} \),
\[
\begin{align*}
&\left\{ \begin{array}{l}
u \leq f(G(v, \lambda; \lambda^*)), \\
v \leq -f(-F(u, \lambda)).
\end{array} \right.
\end{align*}
\]

**Proof.** In view of Lemma 2.1, it suffices to prove the case that a solution \( y(t) \) of (1.1) is oscillatory about 0.

Consider a solution \( y(t) \) which is oscillatory about 0 and for \( x(t) = f(y(t)) \) put (2.5).

For any \( \epsilon > 0 \), choose an integer \( T = T(\epsilon) > t_0 \) such that
\[\begin{array}{c}
u - \epsilon < x(t) < \bar{u} + \epsilon \
u - \epsilon < x(t) < \bar{u} + \epsilon \\nu - \epsilon < x(t) < \bar{u} + \epsilon \quad \text{for} \quad t \geq T.
\end{array}\]

Using (1.1)–(1.3), we have that for \( v_1 = v + \epsilon > 0 \) and \( u_1 = u + \epsilon > 0 \),
\[
\left( e^{\int_{t_0}^{t} a(s) \, ds} f^{-1}(x(t)) \right)' = \left( e^{\int_{t_0}^{t} a(s) \, ds} y(t) \right)' \leq \left( e^{\int_{t_0}^{t} a(s) \, ds} \sum_{i=0}^{m} a_i(t) \right) v_1 \quad \text{for} \quad t \geq T,
\]
and
\[
\left( e^{\int_{t_0}^{t} a(s) \, ds} f^{-1}(x(t)) \right)' = \left( e^{\int_{t_0}^{t} a(s) \, ds} y(t) \right)' \geq -\left( e^{\int_{t_0}^{t} a(s) \, ds} \sum_{i=0}^{m} a_i(t) \right) u_1 \quad \text{for} \quad t \geq T.
\]


Let \( \{T_n\}_{n=1}^{\infty} \) be an increasing sequence such that \( T_n \geq T + 2\hat{\tau} \), \( x'(T_n) = 0 \), \( x(T_n) > 0 \), \( \lim_{n \to \infty} x(T_n) = u \), and \( \lim_{n \to \infty} T_n = +\infty \). Thus, by (1.1)–(1.3), we have

\[
-a(T_n)y(T_n) - \sum_{i=0}^{m} a_i(T_n)x(\tau_i(T_n)) \geq 0,
\]

which implies that there exists \( \xi_n \in [\xi(T_n), T_n) \) such that \( y(\xi_n) = 0 \) and \( y(t) > 0 \) for \( t \in (\xi_n, T_n] \).

Then, \( x(\xi_n) = 0 \) and \( x(t) > 0 \) for \( t \in (\xi_n, T_n] \).

For \( T \leq t \leq \xi_n \), by integrating (2.7) from \( t \) to \( \xi_n \), we get

\[
f^{-1}(x(t)) = y(t) \geq \left( \int_{t}^{\xi_n} e^{\int_{s}^{T} a(u) du} \sum_{i=0}^{m} a_i(s) ds \right) v_1 \text{ for } T \leq t \leq \xi_n.
\]

Hence, for \( t \in [\xi_n, T_n] \), by (1.2), \( \tau(t) \leq \tau(T_n) \leq \xi_n \), and

\[
f^{-1}(x(\tau_i(t))) = y(\tau_i(t)) \geq \left( \int_{\tau(t)}^{\tau_i(t)} e^{\int_{s}^{T} a(u) du} \sum_{j=0}^{m} a_j(s) ds \right) v_1 \text{ for } 0 \leq i \leq m.
\]

Substituting this into (1.1), we have that for \( t \in [\xi_n, T_n] \),

\[
(e^{\int_{0}^{t} a(s) ds} f^{-1}(x(t)))' = (e^{\int_{0}^{t} a(s) ds} y(t))' \\
\leq -e^{\int_{0}^{t} a(s) ds} \sum_{i=0}^{m} a_i(t) f \left( -\left( \int_{\tau(t)}^{\xi_n} e^{\int_{s}^{T} a(u) du} \sum_{j=0}^{m} a_j(s) ds \right) v_1 \right).
\]

Thus, for \( t \in [\xi_n, T_n] \),

\[
(e^{\int_{0}^{t} a(s) ds} f^{-1}(x(t)))' = (e^{\int_{0}^{t} a(s) ds} y(t))' \\
\leq \min \left\{ \left( e^{\int_{0}^{t} a(s) ds} \sum_{i=0}^{m} a_i(t) \right) v_1, \right. \\
\left. -e^{\int_{0}^{t} a(s) ds} \sum_{i=0}^{m} a_i(t) f \left( -\left( \int_{\tau(t)}^{\xi_n} e^{\int_{s}^{T} a(u) du} \sum_{j=0}^{m} a_j(s) ds \right) v_1 \right) \right\}.
\]

Then, for any \( h_n \in [\xi_n, T_n] \),

\[
f^{-1}(x(T_n)) \leq \left( \int_{\xi_n}^{T_n} e^{-\int_{t}^{T_n} a(s) ds} \sum_{i=0}^{m} a_i(t) dt \right) v_1 \\
- \int_{h_n}^{T_n} \left\{ e^{-\int_{t}^{T_n} a(s) ds} \sum_{i=0}^{m} a_i(t) f \left( -\left( \int_{\tau(t)}^{\xi_n} e^{\int_{s}^{T} a(u) du} \sum_{j=0}^{m} a_j(s) ds \right) v_1 \right) \right\} dt
\]

and
\[
\begin{align*}
\int_{\xi_n}^{t} e^{f_{\xi(t)} a(u) du} \sum_{j=0}^{m} a_j(s) ds &= \int_{\xi_n}^{t} e^{f_{\xi(t)} a(u) du} \sum_{j=0}^{m} a_j(s) ds - \int_{\xi_n}^{t} e^{f_{\xi(t)} a(u) du} \sum_{j=0}^{m} a_j(s) ds \\
&\leq \left( \sup_{t \geq \tau^{-1}(t)} \sum_{j=0}^{m} a_j(t) \right) e^{(\sup_{t \geq \tau^{-1}(t)} f_{\xi(t)} a(u) du)} - 1 \\
&\quad - e^{(\inf_{t \geq \tau^{-1}(t)} f_{\xi(t)} a(u) du)} \int_{\xi_n}^{t} e^{-f_{\xi(t)} a(u) du} \sum_{j=0}^{m} a_j(s) ds \\
&\leq \lambda - \mu \int_{\xi_n}^{t} e^{-f_{\xi(s)} a(u) du} \sum_{j=0}^{m} a_j(s) ds.
\end{align*}
\]

Therefore, we have that

\[
\begin{align*}
f^{-1}(x(T_n)) \\
&\leq \left( \int_{\xi_n}^{t} e^{-f_{\xi(s)} a(u) du} \sum_{i=0}^{m} a_i(t) dt \right) v_1 \\
&\quad - \int_{\xi_n}^{T_n} e^{-f_{\xi(s)} a(u) du} \sum_{i=0}^{m} a_i(t) f \left( - \left( \lambda - \mu \int_{\xi_n}^{t} e^{-f_{\xi(s)} a(u) du} \sum_{j=0}^{m} a_j(s) ds \right) v_1 \right) dt \\
&= \left( \int_{\xi_n}^{t} e^{-f_{\xi(s)} a(u) du} \sum_{i=0}^{m} a_i(t) dt \right) v_1 - \frac{1}{\mu v_1} \int_{\xi_n}^{T_n} f(x) dx,
\end{align*}
\]

(2.9)

where

\[
\begin{align*}
\bar{x}_n &= - \left( \lambda - \mu \int_{\xi_n}^{T_n} e^{-f_{\xi(s)} a(u) du} \sum_{i=0}^{m} a_i(t) dt \right) v_1, \\
\bar{x}_n &= - \left( \lambda - \mu \int_{\xi_n}^{T_n} e^{-f_{\xi(s)} a(u) du} \sum_{i=0}^{m} a_i(t) dt \right) v_1.
\end{align*}
\]

Since

\[
\begin{align*}
\int_{\xi_n}^{T_n} e^{-f_{\xi(s)} a(u) du} \sum_{i=0}^{m} a_i(t) dt &\leq e^{-f_{\xi(T_n)} a(u) du} \int_{\xi(T_n)}^{T_n} e^{f_{\xi(t)} a(u) du} \sum_{i=0}^{m} a_i(t) dt \\
&\leq \mu^{-1} \int_{\xi(T_n)}^{T_n} e^{f_{\xi(t)} a(u) du} \sum_{i=0}^{m} a_i(t) dt \leq \mu^{-1} \lambda,
\end{align*}
\]
from (2.9), we obtain that
\[
\begin{aligned}
f^{-1}(x(T_n)) &\leq \left( \int_{0}^{e^{-\int_{T_n}^{\xi_n} a(s)ds}} \sum_{i=0}^{m} a_i(t) dt \right) v_1 - \frac{1}{\mu v_1} \int_{\xi_n}^{0} f(x) dx.
\end{aligned}
\]

In particular, for \( \lambda < \lambda^* \), we put \( h_n = \xi_n \). Then, by (2.3),
\[
\begin{aligned}
f^{-1}(x(T_n)) &\leq G(v_1, \lambda; \lambda^*), \\
\text{that is, } x(T_n) &\leq f(G(v_1, \lambda; \lambda^*)).
\end{aligned}
\]

On the other hand, for \( \lambda \geq \lambda^* \), since \( 0 \leq \lambda - \lambda^* \leq \lambda \), we take \( h_n \) as \( \lambda - \lambda^* = \mu \int_{\xi_n}^{e^{-\int_{T_n}^{\xi_n} a(s)ds}} \sum_{i=0}^{m} a_i(t) dt \), then by (2.4),
\[
\begin{aligned}
f^{-1}(x(T_n)) &\leq G(v_1, \lambda; \lambda^*).
\end{aligned}
\]

Thus, for \( \lambda \geq \lambda^* \),
\[
\begin{aligned}
x(T_n) &\leq f(G(v_1, \lambda; \lambda^*)).
\end{aligned}
\]

Then, letting \( n \to \infty \) and \( \epsilon \to +0 \), we obtain the first part of (2.6).

Similarly, for the increasing sequence \( \{S_n\}_{n=1}^{\infty} \) such that \( S_n \geq T + 2\hat{\tau}, x'(S_n) = 0, x(S_n) < 0 \), \( \lim_{n \to \infty} x(S_n) = -v \) and \( \lim_{n \to \infty} S_n = +\infty \), there exists \( \eta_n \in [\tau(S_n), S_n) \) such that \( y(\eta_n) = 0 \) and \( y(t) < 0 \) for \( t \in (\eta_n, S_n] \), and for \( T \leq t \leq \eta_n \), by integrating (2.8) from \( t \) to \( \eta_n \), we obtain that
\[
\begin{aligned}
f^{-1}(x(S_n)) &\geq -\left( \int_{\eta_n}^{h_n} e^{-\int_{\eta_n}^{S_n} a(s)ds} \sum_{i=0}^{m} a_i(t) dt \right) u_1 \\
&\quad + \frac{1}{\mu u_1} \int_{(\lambda - \mu \hat{p}) u_1}^{0} f(x) dx.
\end{aligned}
\]

Now, for \( \lambda > 0 \), consider the following function \( Q(p) \) of \( p \):
\[
\begin{aligned}
Q(p) &= -p u_1 + \frac{1}{\mu u_1} \int_{(\lambda - \mu \hat{p}) u_1}^{0} f(x) dx.
\end{aligned}
\]

Then, from
\[
\begin{aligned}
Q'(p) &= -u_1 + f((\lambda - \mu \hat{p}) u_1) = 0,
\end{aligned}
\]
we put
\[
\hat{\rho} = \begin{cases} 
0, & \lambda < \frac{f^{-1}(u_1)}{u_1} \\
\mu^{-1}\lambda - \frac{f^{-1}(u_1)}{\mu u_1}, & \lambda \geq \frac{f^{-1}(u_1)}{u_1}.
\end{cases}
\]

Then, we see that
\[
0 \leq \hat{\rho} < \mu^{-1}\lambda \quad \text{and} \quad Q(p) \geq Q(\hat{\rho}), \quad \text{for } 0 \leq p \leq \mu^{-1}\lambda.
\]
where
\[ Q(\hat{\lambda}) = \begin{cases} 
\frac{1}{\mu u_1} \int_0^{\lambda u_1} f(x) \, dx, & \lambda < \frac{f^{-1}(u_1)}{u_1}, \\
\mu^{-1}(-\lambda u_1 + f^{-1}(u_1)) + \frac{1}{\mu u_1} \int_{f^{-1}(u_1)}^{0} f(x) \, dx, & \lambda \geq \frac{f^{-1}(u_1)}{u_1}.
\end{cases} \]

Hence, we have
\[ f^{-1}(x(S_n)) \geq Q(\hat{\lambda}). \]
Thus, for \( \lambda > 0 \),
\[ f^{-1}(x(S_n)) \geq -F(u_1, \hat{\lambda}), \]
that is, \( x(S_n) \geq f(-F(u_1, \hat{\lambda})) \), and we obtain the second part of (2.6). This completes the proof.

By Lemma 2.3, we obtain a general result in this paper as follows.

**Theorem 2.1.** In (1.1)–(1.4), assume that \( f(x) \neq x \).
Suppose that there are constants \( \lambda^* \) and \( \hat{\lambda} > 0 \) such that for (2.3)–(2.6), system of inequalities
\[
\begin{align*}
\quad u &\leq f(G(v, \hat{\lambda}; \lambda^*)) , \\
v &\leq -f(-F(u, \hat{\lambda})) ,
\end{align*}
\]
has no solution for \( 0 < u < \bar{u} \) and \( 0 < v < \bar{v} \). Then, for \( \lambda \leq \hat{\lambda} \), the zero solution of (1.1)–(1.3) is globally asymptotically stable.

**Proof.** Similar to the proof of Lemma 2.3, we can easily prove the uniform stability of the zero solution for (1.1). Therefore, it suffices to prove that for \( \lambda \leq \hat{\lambda} \), (2.10) implies \( u = v = 0 \).
Suppose that \( u, v > 0 \). Then from (2.6), we have that for \( \lambda \leq \hat{\lambda} \),
\[
\begin{align*}
\quad u &\leq f(G(v, \lambda; \lambda^*)) \leq f(G(v, \hat{\lambda}; \lambda^*)) , \\
v &\leq -f(-F(u, \lambda)) \leq -f(-F(u, \hat{\lambda})) .
\end{align*}
\]
Thus, by assumptions, we have that \( u = 0 \) or \( v = 0 \), each of which implies \( u = v = 0 \) by assumption (2.6). Hence the proof of Theorem 2.1 is complete.

For the zero solution of (1.1) with (1.2) and (1.3), by Theorem 2.1, we provide globally asymptotically stable condition that (2.10) implies \( u = v = 0 \) which is able to investigate by computations.

Now, for a selection of \( f(x) \) in (1.3), we give a useful lemma.

**Lemma 2.4.** Assume that for \( g_i(x), \, 0 \leq i \leq n \), in (1.1), there is a constant \( \gamma > 0 \) such that
\[
\begin{align*}
g_i(x) &= O(e^\gamma), \quad 0 \leq i \leq n, \text{ as } x \to +\infty, \text{ if } \lim_{x \to +\infty} g_i(x), \, 0 \leq i \leq n, \text{ is finite} , \\
or
\quad g_i(x) &= O(e^\gamma), \quad 1 \leq i \leq n, \text{ as } x \to -\infty, \text{ if } \lim_{x \to +\infty} g_i(x), \, 1 \leq i \leq n, \text{ is finite} .
\end{align*}
\]
Then, there is a strictly monotone increasing function \( f(x) \) on \( (-\infty, +\infty) \) such that (1.3) holds and there are constant \( \gamma_1 \) and \( \gamma_2 \) such that (1.11)–(1.13) holds.
Proof. By assumptions (1.3) and (2.11), we can easily prove the existence of such constants \( \gamma_1 \) and \( \gamma_2 \) of (1.11)–(1.13).

By Lemma 2.4, if (2.11) holds, then we may restrict a selection of \( f(t) \) in (1.3) to a special wide function class (1.11)–(1.13).

Moreover, we can easily obtain the following lemma.

Lemma 2.5. In (1.1)–(1.4), assume that \( f(x) \neq x \) is defined by (1.11)–(1.13). Put

\[
\tilde{y}(t) = \begin{cases} 
\gamma_1 y(t), & y(t) \geq 0, \\
\gamma_2 y(t), & y(t) < 0
\end{cases},
\]

(2.12)

Then, (1.1) becomes

\[
\begin{cases}
\frac{d\tilde{y}(t)}{dt} = -a(t)\tilde{y}(t) - \sum_{i=0}^{m} a_i(t) \tilde{g}_i(\tilde{y}(\tau_i(t))), & t \geq t_0, \\
\tilde{y}(t) = \begin{cases} 
\gamma_1 \phi(t), & \phi(t) \geq 0, \\
\gamma_2 \phi(t), & \phi(t) < 0
\end{cases}, & t \leq t_0,
\end{cases}
\]

(2.13)

where for \( \tilde{f}(x) = e^x - 1 \),

\[
0 < \frac{\tilde{g}_i(x)}{\tilde{f}(x)} \leq 1, \quad x \neq 0, \quad 0 \leq i \leq m.
\]

(2.14)

Proof of Theorem 1.1. By Lemmas 2.1, 2.2 and 2.5, it is sufficient to prove the special case \( f(x) = e^x - 1 \) in (1.1)–(1.3) and the case that solution \( y(t) \) of (1.1) is oscillatory about 0. Then, by Lemma 2.2, \( y(t) \) is bounded above and below, and by Lemma 2.3, we have (2.6). For \( f(x) = e^x - 1 \), we may take \( \tilde{v} = 1 \) in (2.5) and put \( \lambda^* = 1 + \frac{v}{2} \) in (2.4) and \( \hat{\lambda} = \frac{3}{2} \). Since for \( 0 < v < 1 \),

\[
-\frac{1}{v} \int_{-\lambda v}^{0} (e^x - 1) \, dx = -\frac{1}{v} \left[ 1 - (e^{-\lambda v} + \lambda v) \right] < \frac{\lambda^2 v}{2!} - \frac{\lambda^3 v^2}{3!} + \frac{\lambda^4 v^3}{4!},
\]

we have that

\[
\mu G\left(v, \lambda; 1 + \frac{v}{2}\right) < \begin{cases} 
\frac{\lambda^2 v}{2!} - \frac{\lambda^3 v^2}{3!} + \frac{\lambda^4 v^3}{4!}, & \lambda < 1 + \frac{v}{2}, \\
(\lambda - (1 + \frac{v}{2})) v + (1 + \frac{v}{2})^2 \frac{v}{2!} - (1 + \frac{v}{2})^3 \frac{v^2}{3!} + (1 + \frac{v}{2})^4 \frac{v^3}{4!}, & \lambda \geq 1 + \frac{v}{2}.
\end{cases}
\]

Then, similar to Muroya [20, p. 1921], we have that for \( 0 < v < 1 \) and \( \lambda < 1 + \frac{v}{2} \),

\[
\mu G\left(v, \lambda ; 1 + \frac{v}{2}\right) < v - \frac{v^2}{6},
\]

and for \( 0 < v < 1 \) and \( 1 + \frac{v}{2} \leq \lambda \leq \frac{1}{2} + \mu \),

\[
\mu G\left(v, \lambda ; 1 + \frac{v}{2}\right) < \mu \left(v - \frac{v^2}{6\mu}\right).
\]
Thus, for $0 < v < 1$ and $\lambda \leq \frac{1}{2} + \mu$, we have

$$G(v, \lambda; 1 + \frac{v}{2}) < v - \frac{v^2}{6\mu}.$$  \hfill (2.15)

Therefore, by (1.12) and (2.5),

$$f^{-1}(u) = \log(1 + u) < v < 1 \quad \text{and} \quad u < e - 1 < 2.$$  \hfill (2.16)

So, by (2.5), we may take $\bar{u} = 2$. Similarly, for $0 < u < \bar{u}$, we obtain that

$$\frac{f^{-1}(u)}{u} = \frac{1}{u} \log(1 + u) < 1$$

and by (2.3),

$$\mu F(u, \lambda) \leq \begin{cases} 1 - \frac{1}{u} \log(1 + u), & \lambda < \frac{1}{u} \log(1 + u), \\ \lambda u - \left(1 + \frac{1}{u} \log(1 + u) - 1\right), & \lambda \geq \frac{1}{u} \log(1 + u). \end{cases}$$

Then, similar to Muroya [20, p. 1922], we have that for $0 < u < \bar{u}$ and $\lambda < \frac{1}{u} \log(1 + u)$,

$$\mu F(u, \lambda) < u + \frac{u^2}{6},$$

and for $0 < u < \bar{u}$ and $\frac{1}{u} \log(1 + u) \leq \lambda \leq \frac{1}{2} + \mu$,

$$\mu F(u, \lambda) < \left(\lambda - \frac{1}{2}\right) u + \frac{u^2}{6} \leq \mu u + \frac{u^2}{6}.$$

Hence, for $0 < u < \bar{u}$ and $\lambda \leq \frac{1}{2} + \mu$, we have

$$F(u, \lambda) < u + \frac{u^2}{6\mu}.$$  \hfill (2.17)

Suppose that for $\lambda \leq \frac{1}{2} + \mu$, $0 < u \leq v < 1$. Then for $w = u + \frac{w^2}{6\mu}$, by (2.17), $F(u, \lambda) < w$ and $u < w = (1 + \frac{u^2}{6\mu})u < \frac{7u}{6}$. Then, by (2.10), for $0 < v < 1$ and $0 < u < \bar{u}$,

$$v \leq -f(-w) < w - \frac{w^2}{2!} + \frac{w^3}{3!} < \left(u + \frac{u^2}{6\mu}\right) - \frac{u^2}{2} + \frac{\left(\frac{7u}{6}\right)^3}{6}$$

$$< u - \frac{u^2}{6} \left(2 - \left(\frac{7}{6}\right)^3 u\right) < u,$$

which is a contradiction. Hence we have $v < u$. As a result, for $0 < u < \bar{u}$ and $0 < v < 1$, we obtain the following system of inequalities:

$$\begin{cases} u \leq P(v) \equiv f\left(v - \frac{v^2}{6\mu}\right), \\ v \leq Q(u) \equiv -f\left(-\left(u + \frac{u^2}{6\mu}\right)\right). \end{cases}$$  \hfill (2.18)

Now, for $0 < v < 1$, consider the following equation of $v$:

$$v = H(v), \quad H(v) = Q(P(v)).$$
Then, similar to Muroya [20, p. 1922], for \( u = P(v) \), we have that
\[
H'(v) = Q'(u)P'(v) < \left\{ 1 + \frac{1}{3\mu} (u - v) - \frac{uv}{9\mu^2} \right\} \left/ \left\{ 1 + (u - v) + \frac{u^2 + v^2}{6\mu} \right\} \right.,
\]
and \( H'(v) < 1 \). \( H(0) = 0 \) and \( H'(v) < 1 \) imply that system (2.18) and hence (2.10) of inequalities has no solution for \( 0 < u < \bar{u} \) and \( 0 < v < \bar{v} \).

Hence by Theorem 2.1 and Lemma 2.5, we obtain the conclusions of Theorem 1.1.

**Corollary 2.1.** (Cf. So and Yu [12].) In case of \( f(x) = e^x - 1 \), the zero solution of (1.1)–(1.3) is globally asymptotically stable for \( \lambda \leq \frac{1}{2} + \mu \).

For the special case of \( f(x) \equiv x \), it can be considered a particular case that \( \gamma_1 = 0 \) and \( \gamma_2 \to +0 \) in (1.11) or \( \gamma_1 \to -0 \) and \( \gamma_2 = 0 \) in (1.12) but the proof is a little different, because \( f(x) \) is unbounded above and below, that is, \( \lim_{x \to -\infty} f(x) = -\infty \) and \( \lim_{x \to +\infty} f(x) = +\infty \).

Similar to the proof of Lemma 2.3, we have the following theorem.

**Theorem 2.2.** Assume that in (1.1)–(1.4), \( f(x) \equiv x \) and \( \int_{t_0}^t \sum_{i=1}^\infty a_i(t) \, dt = +\infty \). Then, the solution \( y(t) \) of (1.1)–(1.3) is bounded above and below for \( \lambda \leq \frac{1}{2} + \mu \), and the zero solution of (1.1)–(1.3) is globally asymptotically stable for \( \lambda < \frac{1}{2} + \mu \).

**Proof.** In view of Lemma 2.1, it suffices to prove that for a solution \( y(t) \) which is oscillatory about 0, \( y(t) \) is bounded above and below if \( \lambda \leq \frac{1}{2} + \mu \), and the zero solution is uniformly stable and \( \lim_{t \to \infty} y(t) = 0 \) holds if \( \lambda < \frac{1}{2} + \mu \).

Consider a solution \( y(t) \) which is oscillatory about 0.

Let \( S_n \) be a point such that \( \tau^2(T_n) \leq S_n \leq T_n \), \( y(S_n) \leq y(t) \) for \( \tau^2(T_n) \leq t \leq T_n \).

Since
\[
0 \leq y'(S_n) = -a(S_n) y(S_n) - \sum_{i=0}^m a_i(S_n) g_i \left( y(\tau(S_n)) \right),
\]
there exists \( \xi_n \in [\tau(T_n), S_n) \) such that \( y(\xi_n) = 0 \) and \( y(t) > 0 \) for \( t \in (\xi_n, S_n] \).

Let \( S_n \) be a point such that \( \tau^2(T_n) \leq S_n \leq T_n \), \( y(S_n) \leq 0 \) and \( y(S_n) \leq y(t) \) for \( \tau^2(T_n) \leq t \leq T_n \).

\[
0 \geq y'(S_n) = -a(S_n) y(S_n) - \sum_{i=0}^m a_i(S_n) g_i \left( y(\tau(S_n)) \right)
\]
implies that there exists \( \eta_n \in [\tau(S_n), S_n] \) such that \( y(\eta_n) = 0 \) and \( y(t) < 0 \) for \( t \in (\eta_n, S_n] \).

Since for \( v_1 = -\min_{\tau^2(T_n) \leq t \leq T_n} y(t) = 0 \), \( y(t) > 0 \),
\[
\left( e^{\int_{t_0}^t a(s) \, ds} y(t) \right)' \leq \left( e^{\int_{t_0}^t a(s) \, ds} \sum_{i=0}^m a_i(t) \right) v_1 \quad \text{for} \quad \tau(T_n) \leq t \leq T_n,
\]
by integrating the above equation from \( t \) to \( \xi_n \), we get

\[
y(t) \geq - \left( \int_t^{\xi_n} e^{\int_s^t a(u) du} \sum_{i=0}^m a_i(s) ds \right) v_1 \quad \text{for } \xi_n \leq t \leq T_n.
\]

Hence, for \( t \in [\xi_n, T_n] \),

\[
y(\tau_i(t)) \geq - \left( \int_{\xi(t)}^{\xi_n} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds \right) v_1 \quad \text{for } 0 \leq i \leq m.
\]

Substituting this into (1.1), we have

\[
\left( e^{\int_0^t a(u) du} y(t) \right)' \leq e^{\int_0^t a(u) du} \sum_{i=0}^m a_i(t) \left( \int_{\xi(t)}^{\xi_n} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds \right) v_1 \quad \text{for } t \in [\xi_n, T_n].
\]

Thus, for \( t \in [\xi_n, T_n] \),

\[
\left( e^{\int_0^t a(u) du} y(t) \right)' \leq \min \left\{ e^{\int_0^t a(u) du} \sum_{i=0}^m a_i(t) \left( \int_{\xi(t)}^{\xi_n} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds \right) v_1 \right\}.
\]

Then, for any \( h_n \in [\xi_n, T_n] \),

\[
y(T_n) \leq e^{-\int_{\xi_n}^{T_n} a(u) du} y(\xi_n) + \left( \int_{\xi_n}^{h_n} e^{\int_s^{T_n} a(u) du} \sum_{i=0}^m a_i(t) dt \right) v_1
\]

\[
+ \int_{h_n}^{T_n} \left\{ e^{-\int_s^{T_n} a(u) du} \sum_{i=0}^m a_i(t) \left( \int_{\xi(t)}^{\xi_n} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds \right) v_1 \right\} dt
\]

and similarly, we have that

\[
\int_{\xi(t)}^{\xi_n} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds = \int_t^{\xi_n} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds - \int_{\xi_n}^{t} e^{\int_s^{\tau_i(t)} a(u) du} \sum_{j=0}^m a_j(s) ds
\]

\[
\leq \lambda - \mu \int_{h_n}^{T_n} e^{\int_s^{T_n} a(u) du} \sum_{j=0}^m a_j(s) ds.
\]

Therefore, we have that

\[
y(T_n) \leq \left( \int_{\xi_n}^{h_n} e^{\int_s^{T_n} a(u) du} \sum_{i=0}^m a_i(t) dt \right) v_1
\]
\[
\begin{align*}
+ \int_{\xi_n}^{T_n} \frac{e^{-\int_{t}^{T_n} a(u) du}}{h_n} \sum_{i=0}^{m} a_i(t) \left( \lambda - \mu - \int_{\xi_n}^{t} \frac{e^{-\int_{s}^{T_n} a(u) du}}{h_n} \sum_{j=0}^{m} a_j(s) ds \right) v_1 dt \\
= \left( \int_{\xi_n}^{h_n} \frac{e^{-\int_{t}^{T_n} a(u) du}}{h_n} \sum_{i=0}^{m} a_i(t) dt \right) v_1 - \frac{1}{\mu v_1} \int_{S_2}^{\bar{\xi}_2} x \, dx \\
\leq \left( \int_{\xi_n}^{h_n} \frac{e^{-\int_{t}^{T_n} a(u) du}}{h_n} \sum_{i=0}^{m} a_i(t) dt \right) v_1 - \frac{1}{\mu v_1} \int_{S_2}^{0} x \, dx,
\end{align*}
\]

where
\[
\begin{align*}
\bar{\xi}_2 &= -\left( \lambda - \mu - \int_{\xi_n}^{T_n} \frac{e^{-\int_{t}^{T_n} a(u) du}}{h_n} \sum_{i=0}^{m} a_i(t) dt \right) v_1, \\
\bar{x}_2 &= -\left( \lambda - \mu - \int_{\xi_n}^{h_n} \frac{e^{-\int_{t}^{T_n} a(u) du}}{h_n} \sum_{i=0}^{m} a_i(t) dt \right) v_1.
\end{align*}
\]

Now, consider the following function \(Q(p; v_1, \lambda)\) of \(p\):
\[
Q(p; v_1, \lambda) = pv_1 - \frac{1}{\mu v_1} \int_{-\left(\lambda - \mu p\right) v_1}^{0} x \, dx.
\]

Then, from
\[
Q'(p; v_1, \lambda) = v_1 - (\lambda - \mu p)v_1,
\]
we put
\[
\hat{p} = \begin{cases} 0, & \lambda < 1, \\ \mu^{-1}(\lambda - 1), & \lambda \geq 1. \end{cases}
\]

Then, we have that \(0 \leq \hat{p} < \mu^{-1}\lambda\) and \(Q(p; v_1, \lambda) \geq Q(\hat{p}; v_1, \lambda)\) for \(0 \leq p \leq \mu^{-1}\lambda\), where
\[
Q(\hat{p}; v_1, \lambda) = \begin{cases} \frac{\lambda^2}{2\mu} v_1, & \lambda < 1, \\ \mu^{-1}(\lambda - \frac{1}{2}) v_1, & \lambda \geq 1. \end{cases}
\]

Hence, we have
\[
y(T_n) \leq Q(\hat{p}; v_1, \lambda) \leq \max \left( \frac{1}{2\mu}, \mu^{-1}\left(\lambda - \frac{1}{2}\right) \right) (-y(S_n)).
\]

Similarly, we obtain that \(y(S_n) \geq -\max(\frac{1}{2\mu}, \mu^{-1}(\lambda - \frac{1}{2}))y(\bar{T}_n)\), where \(t_0 \leq \bar{T}_n \leq S_n\), \(y'(\bar{T}_n) \geq 0\) and \(y(\bar{T}_n) \geq y(t)\) for \(t_0 \leq \bar{T}_2(S_n) \leq t \leq S_n\). Thus, we have that for \(u_1 = \max_{\bar{T}_2(S_n) \leq t \leq S_n} y(t) = y(\bar{T}_n) > 0\),
\[
\begin{align*}
-y(S_n) \leq \begin{cases} \frac{\lambda^2}{2\mu} u_1, & \lambda < 1, \\ \mu^{-1}(\lambda - \frac{1}{2}) u_1, & \lambda \geq 1. \end{cases}
\end{align*}
\]

Hence \(y(T_n) \leq \max(\frac{1}{2\mu}, \mu^{-1}(\lambda - \frac{1}{2})) y(\bar{T}_n)\).
These imply that for \( \lambda \leq \frac{1}{2} + \mu \), \( y(t) \) is bounded above and below, and for \( \lambda < \frac{1}{2} + \mu \), the zero solution of (1.1)–(1.3) is globally asymptotically stable. Hence, the proof is complete. \( \square \)

3. Applications

Consider the following nonautonomous difference equation (cf. Tkachenko and Trofimchuk [23]):

\[
x(n + 1) = qx(n) + \sum_{i=0}^{m} a_i(n) g_i(x(n - i)),
\]

where \( 0 < q < 1 \) and the nonlinear functions \( g_i(x), 0 \leq i \leq m, \) satisfy (1.2) and (1.3). Following an idea from Györi and Hartung [23] (see also Tkachenko and Trofimchuk [24]), with (3.1), we will associate the delay differential equation

\[
y'(t) = -(-\ln q) y(t) + \left(\frac{-\ln q}{1 - q} \sum_{i=0}^{m} a_i(t) g_i(y([t - i]))\right),
\]

where

\[
a_i(t) = a_i(n), \quad n \leq t < n + 1, \quad 0 \leq i \leq m.
\]

Since \( t_0 = 0 \), \( \tau(t) = [t - m] \) and \( a(t) = -\ln q \) in (1.1), we have that for \( t = n, \ \tau(t) \geq t_0 \) implies \( n \geq m \). Then,

\[
\sup_{t \geq t_0} \sum_{i=0}^{m} a_i(t) = \sup_{n \geq m} \sum_{i=0}^{m} a_i(n) \quad \text{and} \quad \exp\left(\sup_{t \geq \tau^{-1}(t_0)} \int_{\tau(t)}^{t} a(s) \, ds\right) = \bar{q}^{-(m+1)}.
\]

Therefore, for (1.4) corresponding to (3.2), \( \lambda \leq \frac{1}{2} + \mu \) is satisfied by

\[
\frac{\bar{q}^{-1} - 1}{(-\ln q)} \left(\frac{-\ln q}{1 - q} \left(\sup_{n \geq m} \sum_{i=0}^{m} a_i(n)\right)\right) \leq \frac{1}{2} + \bar{q}^{-1} \quad \text{and} \quad \bar{q} = q^{m+1},
\]

which is equivalent to

\[
\frac{1 - \bar{q}}{1 - q} \left(\sup_{n \geq m} \sum_{i=0}^{m} a_i(n)\right) \leq 1 + \frac{\bar{q}}{2}.
\]

Thus, by Theorem 2.2 and Corollary 2.1, we obtain the following two theorems.

**Theorem 3.1.** For the case \( f(x) = x \), suppose that

\[
\frac{1 - \bar{q}}{1 - q} \left(\sup_{n \geq m} \sum_{i=0}^{m} a_i(n)\right) < 1 + \frac{\bar{q}}{2}.
\]

Then, the zero solution of (3.1) is globally asymptotically stable.

**Theorem 3.2.** For the case \( f(x) = e^x - 1 \), suppose that

\[
\frac{1 - \bar{q}}{1 - q} \left(\sup_{n \geq m} \sum_{i=0}^{m} a_i(n)\right) \leq 1 + \frac{\bar{q}}{2}.
\]

Then, the zero solution of (3.1) is globally asymptotically stable.
For the case that \( q = 1 \), the conditions (3.4) and (3.5) as \( q \to 1 - 0 \) become the well-known \( 3/2 \)-type criterions

\[
(m + 1) \sup_{n \geq m} \left( \sum_{i=0}^{m} a_i(n) \right) < \frac{3}{2} \quad \text{for } f(x) = x,
\]

and

\[
(m + 1) \left( \sup_{n \geq m} \sum_{i=0}^{m} a_i(n) \right) \leq \frac{3}{2} \quad \text{for } f(x) = e^x - 1,
\]

respectively (see Muroya [20, Theorem 2.2 and Corollary 2.1]).

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References


