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Solvability of Some Fourth (and Higher) Order Singular Boundary Value Problems

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Existence results for a variety of singular fourth order boundary value problems of the form $y^{iv} = f(t, y, y', y'')$ are given. Here our nonlinear term f may be singular at $t = 0$, $t = 1$, $y = 0$, and/or $y'' = 0$. For example, some singularities of the type y^{-a} and $|y''|^{-b}$ are included. Also we discuss and treat the extension of these results to nth order boundary value problems. © 1991 Academic Press, Inc.

1. INTRODUCTION

This paper presents existence results for solutions to nonlinear fourth order boundary value problems of the form

$$y^{iv} = f(t, y, y', y''), \qquad 0 < t < 1; \qquad y \in B, \tag{1.1}$$

where B specifies suitable boundary conditions. In the problems discussed in this paper we allow our nonlinear term f to be singular at $t = 0$, $t = 1$, $y = 0$, and/or $y'' = 0$. In particular singularities in y of the type y^{-a} for $a > 0$ small, in y'' of the type $|y''|^{-b}$ for $b > 0$ small and in t of the form $t^{-\alpha}(1-t)^{-\beta}$ for $\alpha, \beta > 0$ small are included.

Most of the available literature on fourth order boundary value problems, for example [1, 2, 9, 12–14, 16], discuss the case when f is either continuous or a Caratheodory function. Recently the author in [19] discussed problems of the form $y^{iv} = f(t, y, y')$, where f could be singular at $t = 0$, $t = 1$, $y = 0$, and/or $y' = 0$; however, the analysis presented in this paper is quite different although as before we use the Topological Transversality of Andrzej Granas to obtain our main existence theorems. For this paper we in fact restrict B to be

(a) $y(0) = a \geq 0$, $y'(1) = b \geq 0$, $y''(0) = c \leq 0$, $y'''(1) = 0$

(b) $y(0) = a \geq 0$, $y'(0) = b \geq 0$, $y''(1) = 0$, $y'''(1) = 0$

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(c) $y(0)=a\geqslant 0$, $y'(0)=0$, $y'(1)=0$, $y'''(1)=0$

(d) $y(0)=a\geqslant 0$, $y'(1)=b\geqslant 0$, $y''(0)=0$, $y''(1)=0$.

Many other boundary conditions, for example

(e) $y(0)=a\geqslant 0$, $y'(0)=b\geqslant 0$, $y''(0)=0$, $y'''(1)=0$

(f) $y(0)=a\geqslant 0$, $y'(1)=b\geqslant 0$, $y''(1)=0$, $y'''(1)=0$

(g) $y(0)=a\geqslant 0$, $y'(0)=b\geqslant 0$, $y''(0)=0$, $y''(1)=0$

(h) $y(0)=a\geqslant 0$, $y'(1)=b\geqslant 0$, $y''(1)=0$, $y'''(0)=0$

(i) $y(0)=a\geqslant 0$, $y'(1)=b\geqslant 0$, $y''(0)=c\leqslant 0$, $y''(1)=c\leqslant 0$,

could be considered, in fact each of the above boundary conditions has a natural dual version when 0 and 1 are interchanged. However, the analysis of these other boundary conditions is quite similar to that given in this paper so for simplicity in reading and writing we omit the details. This paper is divided into four main sections: the first considers the case when our nonlinear term f is singular at $y=0$ but not at $y''=0$, the second part when f is singular at $y''=0$ but not at $y=0$, and the third part examines the problem when f is singular at *both* $y=0$ and $y''=0$. Also throughout these sections our nonlinear term may be singular at $t=0$ and/or $t=1$ as well. For the purposes of this paper we examine the nonlinear differential equation $y^{iv}=f(t, y, y'')$. It should be noted here however that all the results of this paper could be extended to include equations of the form $y^{iv}=f(t, y, y', y'')$, where f has bounded dependence on its y' variable for any fixed values of the other arguments. In the last section we discuss nth order singular boundary value problems of the form

$$y^{(n)}=f(t, y, y', y''), \qquad 0<t<1; \qquad y\in B$$

and obtain existence of solutions for a certain class of problems. Finally we summarize briefly the methods used to deduce the existence of a solution to (1.1).

(i) We first examine approximating problems

$$y^{iv}=f(t, y, y', y''), \qquad 0<t<1; \qquad y\in B_n. \tag{1.2}$$

The signs of y, y', y'', and y''' are deduced only from the properties of f and the boundary condition B_n. Then as a result problems of the form (1.2) do not involve singularities in y and/or y''.

(ii) Existence of solutions to (1.2) is then deduced from the Topological Transversality theorem of A. Granas. Here the key idea is to obtain a priori bounds on solutions and their first four derivatives to (1.2).

(iii) To show the existence of a solution to (1.1) we pass to the limit in n. To apply this step we first need additional estimates independent of n on "some" of the a priori bounds obtained in step (ii). Also we need to show that the limit function y(and/or y'') has no zeros on $(0, 1)$.

2. Singularities at $y=0$ but Not at $y''=0$

Each boundary condition has its own set of ideas so for simplicity we discuss them individually.

A. $y(0)=0,\ y'(1)=b\geq 0,\ y''(0)=c\leq 0,\ y'''(1)=0$

Here we examine the two point boundary value problem

$$\begin{gathered} y^{iv}=f(t, y, y''), \qquad 0<t<1 \\ y(0)=0, \qquad y'(1)=b\geq 0, \qquad y''(0)=c\leq 0, \qquad y'''(1)=0 \end{gathered} \tag{2.1}$$

with the following conditions being satisfied:

f is continuous on $[0, 1]\times(0, \infty)\times(-\infty, 0]$ with $f\geq 0$ on $(0, 1)\times(0, \infty)\times(-\infty, \infty)$ and $\lim_{y\to 0^+} f(t, y, q)=\infty$ uniformly on compact subsets of $(0, 1)\times(-\infty, \infty)$ (2.2)

$0<f(t, y, q)\leq g(y)\,\phi(|q|)$ on $(0, 1)\times(0, \infty)\times(-\infty, 0]$, where $g>0$ is continuous and nonincreasing on $(0, \infty)$ and ϕ is continuous on $[0, \infty)$ (2.3)

$$\frac{u}{\phi(u)} \text{ is nondecreasing on } (0, \infty) \tag{2.4}$$

Suppose there exists constants $A\geq 0$, $B\geq 0$, $0\leq r<1$ such that for all $z\in[0, \infty)$

$$\int_0^z g(u)\,du\leq\int_0^{Az^r+B}\frac{u}{\phi(u)}\,du. \tag{2.5}$$

First by a solution to (2.1) we mean a function $y\in C^2[0, 1]\cap C^3(0, 1]\cap C^4(0, 1)$ that satisfies the differential equation and boundary conditions. To establish the existence of a solution to (2.1) we first consider for $n\in N^+=\{1, 2, ...\}$ the problems

$$\begin{gathered} y^{iv}=f(t, y, y''), \qquad 0<t<1 \\ y(0)=\frac{1}{n}, \qquad y'(1)=b\geq 0, \qquad y''(0)=c\leq 0, \qquad y'''(1)=0. \end{gathered} \tag{2.6^n}$$

The strategy is to show (2.6^n) has a solution for each n and then we use a compactness argument via the Arzela–Ascoli Theorem to show that (2.1) has a solution.

THEOREM 2.1. *Suppose* (2.2), (2.3), (2.4), *and* (2.5) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$\begin{gathered} y^{iv} = \lambda f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = \frac{1}{n}, \qquad y'(1) = b \geq 0, \qquad y''(0) = c, \qquad y'''(1) = 0 \end{gathered} \tag{2.7_λ^n}$$

for fixed $n \in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that for* $t \in [0, 1]$

$$\frac{1}{n} \leq y(t) \leq M_0, \qquad b \leq y'(t) \leq M_1, \qquad -M_2 \leq y''(t) \leq c,$$

$$-M_3 \leq y'''(t) \leq 0, \qquad 0 \leq y^{iv}(t) \leq M_4$$

for each solution $y \in C^4[0, 1]$ *to* (2.7_λ^n).

Proof. The case $\lambda = 0$ is trivial so assume $0 < \lambda \leq 1$. Now condition (2.2) implies $y > 0$ on (0, 1) and as a result we have $y^{iv} > 0$, $y''' < 0$ on (0, 1); thus $y'' < c$ is strictly decreasing on (0, 1) and as a result $y' > b$ on (0, 1) which in turn implies $y > 1/n$ is strictly increasing on (0, 1). In addition we have from assumption (2.4)

$$\frac{(-y'' + c)}{\phi(-y'' + c)} y^{iv} \leq \frac{-y'' y^{iv}}{\phi(-y'')} \leq \lambda g(y)(-y'') \leq g(y)(-y'')$$

so integrating from t to 1 using condition (2.4) yields

$$\begin{aligned} \frac{[-y''(t) + c][-y'''(t)]}{\phi(-y''(t) + c)} &= \frac{[-y''(t) + c]}{\phi(-y''(t) + c)} \int_t^1 y^{iv}(s)\, ds \\ &\leq g(y(t)) \int_t^1 [-y''(s)]\, ds \end{aligned}$$

since y and $-y''$ are strictly increasing on (0, 1) and g is nonincreasing on $(0, \infty)$. Thus

$$\frac{-[-y''(t) + c]\, y'''(t)}{\phi(-y''(t) + c)} \leq g(y(t))[y'(t) - b] \leq g(y(t))\, y'(t)$$

and so integration from 0 to t yields

$$\int_0^{-y''(t) + c} \frac{u}{\phi(u)}\, du \leq \int_0^{y(t)} g(u)\, du.$$

Define $I(z)=\int_0^z (u/\phi(u))\,du$ so I is an increasing map from $[0,\infty)$ onto $[0,\infty)$ and therefore has an increasing inverse I^{-1}. So we have

$$-y''(t)\leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right)-c \tag{2.8}$$

$$y'(t)\leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right)+b-c. \tag{2.9}$$

Finally integration from 0 to 1 will give

$$y(1)\leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right)+b+1-c\leqslant A[y(1)]^r+B+b+1-c$$

using assumption (2.5). Thus there exists a constant $M_0>0$ such that $y(1)\leqslant M_0$. In addition (2.8) and (2.9) yield

$$-y''(t)\leqslant I^{-1}\left(\int_0^{M_0} g(u)\,du\right)=M_2$$

and

$$y'(t)\leqslant I^{-1}\left(\int_0^{M_0} g(u)\,du\right)+b=M_1.$$

Remark. Note M_0, M_1, and M_2 are independent of n.

Now returning to the inequality $y^{iv}\leqslant\lambda g(y)\,\phi(-y'')$ we have $0\leqslant y^{iv}(t)\leqslant g(1/n)\sup_{[-c,M_2]}\phi(q)=M_4$ and integration yields M_3. ∎

THEOREM 2.2. *Suppose* (2.2), (2.3), (2.4), *and* (2.5) *are satisfied. Then a* $C^4[0,1]$ *solution of* (2.6^n) *exists.*

Proof. Consider the family of problems

$$\begin{gathered} y^{iv}=\lambda f^*(t,y,y''),\qquad 0<t<1\\ y(0)=\frac{1}{n},\qquad y'(1)=b\geqslant 0,\qquad y''(0)=c,\qquad y'''(1)=0, \end{gathered} \tag{2.10^n}$$

where $f^*>0$ is any continuous extension of f from $y\geqslant 1/n$, $y''\leqslant 0$. Now every solution u of (2.10$^n_\lambda$) satisfies $u\geqslant 1/n$, $u''\leqslant c$ and hence is a solution of (2.7$^n_\lambda$); also the conclusions of Theorem 2.1 remain valid for solutions to (2.10$^n_\lambda$). Let $C^4_B[0,1]=\{u\in C^4[0,1]: u(0)=1/n,\ u'(1)=b,\ u''(0)=c,$

$u'''(1)=0\}$, $C^4_{B_0}[0,1]=\{u\in C^4[0,1]: u(0)=0,\ u'(1)=0,\ u''(0)=0,\ u'''(1)=0\}$ and

$$U=\{u\in C^4_B[0,1]: |u|_0<M_0+1, |u'|_0<M_1+1, |u''|_0<M_2+1, |u'''|_0<M_3+1, |u^{iv}|_0<M_4+1\},$$

where $|u|_0=\sup_{[0,1]}|u(t)|$. Define mapping $F_\lambda: C^2[0,1]\to C[0,1]$, $j: C^4_B[0,1]\to C^2[0,1]$ and $L: C^4_B[0,1]\to C[0,1]$ by $F_\lambda v(t)=\lambda f^*(t, v(t), v''(t))$, $ju=u$ and $Lv(t)=v^{iv}(t)$. F_λ is continuous from the continuity of f^* and j is completely continuous by the Arzela–Ascoli Theorem. Now define $N: C^4_{B_0}[0,1]\to C[0,1]$ by $Nv(t)=v^{iv}(t)$, so N^{-1} is a continuous linear operator by the Bounded Inverse Theorem. Thus L^{-1} exists and is given by

$$(L^{-1}g)(x)=\frac{1}{n}+(b-c)x+\frac{cx^2}{2}+(N^{-1}g)(x)$$

and so is continuous. Now the map $H_\lambda: \bar{U}\to C^4[0,1]$ given by $H_\lambda u=L^{-1}F_\lambda ju$ is a compact homotopy with the fixed points of H_λ being precisely the solutions to $(2.10''_\lambda)$. The choice of U guarantees that this homotopy is fixed point free on the boundary of U. Since the constant map $H_0(u)=1/n+(b-c)t+ct^2/2\in U$ is essential [8] the Topological Transversality Theorem [8] assures that H_1 has a fixed point; i.e., $(2.10''_1)$ has a solution and therefore $(2.6'')$ has a solution. ▮

Now Theorem 2.2 implies $(2.6'')$ has a solution y_n for each n. In addition we showed that there are constants M_0, M_1, M_2 independent of n such that

$$\frac{1}{n}\leqslant|y|_0\leqslant M_0,\qquad b\leqslant|y'|_0\leqslant M_1,\qquad |y''|_0\leqslant M_2$$

for each solution y to $(2.6'')$. The next argument is broken into three cases, when $b>0$, $b=0$ and $c<0$, and finally $b=0$ and $c=0$.

Case (1). $b>0$.

Then we claim there is a constant M_3 independent of n such that $|y'''|_0\leqslant M_3$; to see this note $y^{iv}\leqslant g(y)\,\phi(-y'')\leqslant Dg(y)$, where $D=\sup_{[-c,M_2]}\phi(q)$. Integration from t to 1 with the fact that $y\geqslant bt$ for $t\in[0,1]$ yields

$$-y'''(t)\leqslant D\int_t^1 g(y(s))\,ds\leqslant D\int_0^1 g(bt)\,dt=M_3.$$

Case (2). $b=0$ and $c<0$.

Suppose we have

$$\int_0^1 g(u)u^{1/2}\,du<\infty. \tag{2.11}$$

Then we claim that there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$; to see this note $y(t) \geqslant -ct^2/2$ and so $y^{iv} \leqslant g(y)\,\phi(-y'') \leqslant g(-ct^2/2)\,D$ and integration from t to 1 proves the claim.

Case (3). $b=0$ and $c=0$.

$$\text{Suppose there exists } p>3 \text{ with } \int_0^1 g^p(u)\,du<\infty. \tag{2.12}$$

Then we claim there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$; to see this note

$$(-y''')^{1/m}\,y^{iv} \leqslant Dg(y)(y')^{1/p}\,(y')^{-1/p}\,(-y'')^{1/q}\,(-y'')^{-1/q}\,(-y''')^{1/m},$$

where $1/q=(p-1)/(2p)-\varepsilon$, $1/m=(p-1)/(2p)+\varepsilon$, with $\varepsilon<(p-3)/(2p)$. Also note $1/p+1/q+1/m=1$ and $p>q>m$. Now integrate from t to 1 using the Generalized Holders integral inequality to obtain

$$\frac{m}{m+1}\,[-y'''(t)]^{(m+1)/m} \leqslant D\left\{\int_0^{M_0} g^p(u)\,du\right\}^{1/p} \times\left\{\int_0^{M_1} u^{-q/p}\,du\right\}^{1/q}\left\{\int_0^{M_2} u^{-m/q}\,du\right\}^{1/m}$$

and our claim is established.

THEOREM 2.3. (i) *Let $b>0$ and suppose* (2.2), (2.3), (2.4), *and* (2.5) *are satisfied. Then a $C^2[0,1] \cap C^3(0,1] \cap C^4(0,1)$ solution of* (2.1) *exists.*

(ii) *Let $b=0$ and $c<0$ and suppose* (2.2), (2.3), (2.4), (2.5), *and* (2.11) *are satisfied. Then a $C^2[0,1] \cap C^3(0,1] \cap C^4(0,1)$ solution of* (2.1) *exists.*

(iii) *Let $b=0$ and $c=0$ and suppose* (2.2), (2.3), (2.4), (2.5), *and* (2.12) *are satisfied. In addition assume*

For any constants $M>0$, $K>-c$ there exists $\eta(t)$ continuous on $[0,1]$ and positive on $(0,1)$ such that $f(t,y,q) \geqslant \eta(t)$ on $(0,1)\times(0,M]\times[-K,c]$. (2.13)

Then a $C^2[0,1] \cap C^3(0,1] \cap C^4(0,1)$ solution of (2.1) *exists.*

Proof. Theorem 2.2 implies (2.6^n) has a solution y_n for each n. Moreover by the above arguments there exists constants M_0, M_1, M_2, and M_3 independent of n such that

$$\frac{1}{n} \leqslant |y_n|_0 \leqslant M_0, \qquad b \leqslant |y_n'|_0 \leqslant M_1, \qquad |y_n''|_0 \leqslant M_2, \qquad |y_n'''|_0 \leqslant M_3.$$

It follows that $\{y_n\}$, $\{y_n'\}$, $\{y_n''\}$ are uniformly bounded and equicontinuous on $[0, 1]$. Now the Arzela–Ascoli Theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some twice continuously differentiable function y, i.e., $|y_{n'} - y|_2 \to 0$ for some $y \in C^2[0, 1]$. Clearly $y \geqslant 0$, $y' \geqslant b$, $y'' \leqslant 0$ on $[0, 1]$ with $y(0) = 0$, $y'(1) = b$ and $y''(0) = c$. In fact $y > 0$ on $(0, 1]$. To see this we need consider three cases, when $b > 0$, and $b = 0$ and $c < 0$ the result is trivial, whereas in the case $b = 0$ and $c = 0$ assumption (2.13) implies $y_n^{iv}(t) \geqslant \eta(t)$ so either integrating four times and interchanging the order of integration or equivalently using the Green's function (which is positive on $(0, 1) \times (0, 1)$) of the operator y^{iv} with the homogeneous boundary conditions corresponding to (2.1) we deduce that

$$\begin{aligned} y_n(t) \geqslant \frac{1}{n} &+ \frac{1}{2} \int_0^t [(1-s)^2 - (1-t)^2]\, s\eta(s)\, ds + \int_0^t (1-s)\, s^2 \eta(s)\, ds \\ &+ \int_t^1 t(1-s)\, s\eta(s)\, ds + \int_0^t \frac{s^3}{3} \eta(s)\, ds \\ &+ \frac{1}{2} \int_t^1 t\left(s^2 - \frac{t^2}{3}\right) \eta(s)\, ds. \end{aligned}$$

Now $y_{n'}$ satisfies the integral equation

$$\begin{aligned} y_{n'}(t) = y_{n'}(1) - y_{n'}'(1)(1-t) + y_{n'}''(1) \frac{(1-t)^2}{2} \\ + \int_t^1 \frac{(s-t)^3}{6} f(s, y_{n'}(s), y_{n'}''(s))\, ds, \end{aligned}$$

so for $t \in (0, 1]$ and $s \in [t, 1]$ we have $f(s, y_{n'}(s), y_{n'}''(s)) \to f(s, y(s), y''(s))$ uniformly since f is uniformly continuous on compact subsets of $[0, 1] \times (0, M_0] \times [-M_2, 0]$. From $f(s, y_{n'}(s), y_{n'}''(s)) \to f(s, y(s), y''(s))$ uniformly in $[t, 1]$ if $t > 0$ it follows that $y \in C^4(0, 1]$. Thus letting $n' \to \infty$ yields

$$\begin{aligned} y(t) = y(1) - y'(1)(1-t) + y''(1) \frac{(1-t)^2}{2} \\ + \int_t^1 \frac{(s-t)^3}{6} f(s, y(s), y''(s))\, ds. \end{aligned}$$

From the integral equation we see that $y \in C^4(0, 1]$, $y^{iv}(t) = f(t, y(t), y''(t))$ and $y'''(1) = 0$. ∎

Remark. For the case $b = 0$ it is possible to replace (2.12) by the following assumption and existence of a solution to (2.1) is guaranteed again:

$$\text{Suppose } \int_0^1 g(\theta(t))\,dt < \infty, \text{ where}$$

$$\theta(t) = \frac{1}{2}\int_0^t [(1-s)^2 - (1-t)^2]\, s\eta(s)\,ds + \int_0^t (1-s)\,s^2\eta(s)\,ds$$

$$+ \int_t^1 t(1-s)\,s\eta(s)\,ds + \int_0^t \frac{s^3}{3}\eta(s)\,ds$$

$$+ \frac{1}{2}\int_t^1 t\left(s^2 - \frac{t^2}{3}\right)\eta(s)\,ds.$$

The proof follows from the arguments above once we show that there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$ for each solution to (2.6^n). To see this note (2.13) with the fact that M_0 and M_2 are independent of n yields $y^{iv}(t) \geqslant \eta(t)$ so integration with the boundary conditions yields $y(t) \geqslant \theta(t)$. Thus $y^{iv} \leqslant g(y(t))\,\phi(-y''(t)) \leqslant g(\theta(t))\,D$, where $D = \sup_{[0, M_2]} \phi(q)$, and integration gives the result.

EXAMPLE. Consider the two point boundary value problem

$$y^{iv} = y^{-\alpha}(|y''|^\beta + 1), \qquad 0 < t < 1;$$

$$y(0) = y''(0) = y'''(1) = 0, \qquad y'(1) = b \geqslant 0$$

with $0 < \alpha$, $\beta < 1$. In addition if $b = 0$ assume $\alpha < \frac{1}{3}$.

Take $g(y) = y^{-\alpha}$ and $\phi(|q|) = |q|^\beta + 1$ and so (2.2) and (2.3) are satisfied. In addition (2.4) holds since $0 < \beta < 1$ and (2.5) is immediate also since $0 < \alpha < 1$. Thus if $b > 0$ a $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ solution exists by Theorem 2.3(i). Now if $b = 0$, then (2.13) holds with $\eta(t) = M^{-\alpha}$. Also since $\alpha < \frac{1}{3}$, (2.12) is true. Thus a $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ solution exists by Theorem 2.3(iii).

B. $y(0) = 0$, $y'(0) = b \geqslant 0$, $y''(1) = 0$, $y'''(1) = 0$

We begin by examining the two point boundary value problem

$$\begin{gathered} y^{iv} = f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = 0, \qquad y'(0) = b \geqslant 0, \qquad y''(1) = 0, \qquad y'''(1) = 0 \end{gathered} \tag{2.14}$$

with the following assumptions satisfied:

f is continuous on $[0, 1] \times (0, \infty) \times [0, \infty)$ with $f \geqslant 0$ on $(0, 1) \times (0, \infty) \times (-\infty, \infty)$ and $\lim_{y \to 0^+} f(t, y, q) = \infty$ uniformly on compact subsets of $(0, 1) \times (-\infty, \infty)$ (2.15)

$0 < f(t, y, q) \leqslant g(y)\, \phi(q)$ on $(0, 1) \times (0, \infty) \times [0, \infty)$ where $g > 0$ is continuous and nonincreasing on $(0, \infty)$ and ϕ is continuous and nondecreasing on $[0, \infty)$ (2.16)

Suppose there exist constants $A \geqslant 0$, $B \geqslant 0$, $0 \leqslant r < 2$ such that for all $z \in [0, \infty)$, with $J(z) = \int_0^z (du/\phi(u))$,

$$\int_0^z J^{-1}[g(u)]\, du \leqslant Az^r + B \tag{2.17}$$

Suppose there exist constants $C \geqslant 0$, $D \geqslant 0$, $0 \leqslant q < 2$ such that for all $z \in [0, \infty)$

$$\phi(z) \leqslant Cz^q + D. \tag{2.18}$$

To establish the existence of a solution to (2.14) we first consider for $n \in N^+$ the problems

$$\begin{gathered} y^{iv} = f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = \frac{1}{n}, \qquad y'(0) = b \geqslant 0, \qquad y''(1) = 0, \qquad y'''(1) = 0. \end{gathered} \tag{2.19^n}$$

THEOREM 2.4. *Suppose* (2.15), (2.16), (2.17), *and* (2.18) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$\begin{gathered} y^{iv} = \lambda f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = \frac{1}{n}, \qquad y'(0) = b, \qquad y''(1) = 0, \qquad y'''(1) = 0 \end{gathered} \tag{2.20$^n_\lambda$}$$

for fixed $n \in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that for* $t \in [0, 1]$

$$\frac{1}{n} \leqslant y(t) \leqslant M_0, \qquad b \leqslant y'(t) \leqslant M_1, \qquad 0 \leqslant y''(t) \leqslant M_2,$$

$$-M_3 \leqslant y'''(t) \leqslant 0, \qquad 0 \leqslant y^{iv}(t) \leqslant M_4$$

for each solution $y \in C^4[0, 1]$ *to* (2.20^n_λ).

Proof. Let $0 < \lambda \leqslant 1$. Now condition (2.15) implies $y > 0$ on $(0, 1)$ and as a result we have $y^{iv} > 0$, $y''' < 0$ on $(0, 1)$; thus $y'' > 0$ is strictly decreasing on $(0, 1)$ and as a result $y' > b$ on $(0, 1)$ which in turn implies $y > 1/n$ is strictly increasing on $(0, 1)$. In addition $-y'''(t) \leq \int_t^1 g(y(s))\,\phi(y''(s))\,ds \leq g(y(t))\,\phi(y''(t))$ since y is strictly increasing on $(0, 1)$, y'' is strictly decreasing on $(0, 1)$, g is nonincreasing on $(0, \infty)$, and ϕ is nondecreasing on $(0, \infty)$. Thus

$$\int_0^{y''(t)} \frac{du}{\phi(u)} \leq \int_t^1 g(y(s))\,ds \leq g(y(t)).$$

Define $J(z) = \int_0^z (du/\phi(u))$ so J is an increasing map from $[0, \infty)$ onto $[0, \infty)$ and therefore has an increasing inverse J^{-1}. Thus $y''(t) \leq J^{-1}(g(y(t)))$ and so multiplying by y' and integrating from 0 to t yields

$$y'(t) \leq \left\{2 \int_0^{y(1)} J^{-1}(g(u))\,du + b^2\right\}^{1/2}. \tag{2.21}$$

Finally integration from 0 to 1 together with (2.17) yields

$$y(1) \leq \{2A[y(1)]^r + 2B + b^2\}^{1/2} + 1.$$

Thus there exists a constant $M_0 > 0$ such that $y(1) \leq M_0$. In addition (2.21) implies M_1.

Remark. Note M_0 and M_1 are independent of n.

Now integrate $y'y^{iv} \leq g(y)\,\phi(y'')\,y'$ from t to 1 to obtain

$$-y'(t)\,y'''(t) + \frac{[y''(t)]^2}{2} \leq \phi(y''(t)) \int_0^{M_0} g(u)\,du.$$

Also since $y'(t)\,y'''(t) \leq 0$ we have

$$y''(t) \leq \left\{2\phi(y''(t)) \int_0^{M_0} g(u)\,du\right\}^{1/2}$$

$$\leq \left\{2 \int_0^{M_0} g(u)\,du(C[y''(t)]^q + D)\right\}^{1/2}.$$

Thus there exists a constant $M_2 > 0$ such that $y''(t) \leq M_2$ for $t \in [0, 1]$.

Remark. Note M_2 is independent of n.

Remark. Note (2.17) implies $\int_0^z g(u)\,du < \infty$ for all $z > 0$. To see this note for all $z > 0$ that $J(\phi(0)z) \leq z$ and so $\int_0^z J^{-1}(g(u))\,du \geq \phi(0) \int_0^z g(u)\,du$.

Finally $0 \leq y^{iv}(t) \leq g(1/n)\,\phi(M_2) = M_4$ and integration yields M_3. ∎

Essentially the same reasoning as in Theorem 2.2 establishes.

THEOREM 2.5. *Suppose* (2.15), (2.16), (2.17), *and* (2.18) *are satisfied. Then a* $C^4[0, 1]$ *solution of* (2.19″) *exists.*

In addition Theorem 2.4 implies there exists constants M_0, M_1, and M_2 independent of n such that $1/n \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $|y''|_0 \leqslant M_2$ for each solution y to (2.19″). The next argument is broken into two cases, when $b = 0$ and $b > 0$.

Case (i). $b > 0$.

The exact same argument as that in Case (1), part A, implies there exists a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$.

Case (ii). $b = 0$.

Suppose (2.12) holds. Then we claim there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$. To see this note $y^{iv} \leqslant g(y)\,\phi(M_2)$ so

$$(-y''')^{1/m}\, y^{iv} \leqslant \phi(M_2)\, g(y)(y')^{1/p}\, (y')^{-1/p}\, (y'')^{1/q}\, (y'')^{-1/q}\, (-y''')^{1/m}.$$

Now integration from t to 1 along with the Generalized Holders integral inequality proves the claim.

Essentially the same reasoning as in Theorem 2.3 establishes

THEOREM 2.6. (i) *Let* $b > 0$ *and suppose* (2.15), (2.16), (2.17), *and* (2.18) *are satisfied. Then a* $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ *solution of* (2.14) *exists.*

(ii) *Let* $b = 0$ *and suppose* (2.12), (2.15), (2.16), (2.17), *and* (2.18) *are satisfied. In addition assume*

For any constants $M > 0$, $K > 0$ *there exists* $\eta(t)$ *continuous and positive on* $(0, 1)$ *such that* $f(t, y, q) \geqslant \eta(t)$ *on* $(0, 1) \times (0, M] \times [0, K]$. (2.22)

Then a $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ *solution of* (2.14) *exists.*

We now discuss briefly the case where our nonlinear term may in addition be singular at $t = 0$ and/or $t = 1$. Consider

$$\begin{gathered} y^{iv} = \psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = 0, \qquad y'(0) = b \geqslant 0, \qquad y''(1) = 0, \qquad y'''(1) = 0 \end{gathered} \tag{2.23}$$

with assumptions (2.15) and (2.16) being satisfied. In addition assume the following hold:

$1/\psi\colon [0, 1] \to [0, \infty)$ is continuous with $\psi > 0$ on $(0, 1)$ and $\int_0^1 \psi(s)\, ds < \infty$ (2.24)

Suppose there exist constants $A \geq 0$, $B \geq 0$, $0 \leq r < 2$ such that for all $z \in [0, \infty)$, with $J(z) = \int_0^z (du/\phi(u))$, $\int_0^z J^{-1}(g(u))\, du \leq Az^r + B$ (2.25)

Suppose there is a constant $p > 2$ with $\int_0^1 g^p(u)\, du < \infty$. (2.26)

Once again to establish the existence of a solution to (2.23) we first consider for $n \in N^+$ the problems

$$\begin{gathered} y^{iv} = \psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = \frac{1}{n}, \qquad y'(0) = b \geq 0, \qquad y''(1) = 0, \qquad y'''(1) = 0. \end{gathered} \tag{2.27^n}$$

THEOREM 2.7. *Suppose* (2.15), (2.16), (2.24), (2.25), *and* (2.26) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$\begin{gathered} y^{iv} = \lambda\psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = \frac{1}{n}, \qquad y'(0) = b \geq 0, \qquad y''(1) = 0, \qquad y'''(1) = 0 \end{gathered} \tag{2.28$^n_\lambda$}$$

for fixed $n \in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that*

$$\frac{1}{n} \leq y(t) \leq M_0, \qquad b \leq y'(t) \leq M_1, \qquad 0 \leq y''(t) \leq M_2,$$

$$-M_3 \leq y'''(t) \leq 0; \qquad t \in [0, 1]$$

and

$$0 \leq \frac{y^{iv}(t)}{\psi(t)} \leq M_4; \qquad t \in (0, 1)$$

for each solution $y \in C^3[0, 1] \cap C^4(0, 1)$ *to* (2.28^n_λ).

Proof. Let $0 < \lambda \leq 1$. As before, condition (2.15) implies $y > 1/n$, $y' > b$, $y'' > 0$, $y''' < 0$, $y^{iv} > 0$ on (0, 1) with y'' strictly decreasing on (0, 1) and y strictly increasing on (0, 1). In addition (2.16) and (2.24) yield

$$\begin{aligned} -y'''(t) &\leq \int_t^1 g(y(s))\, \phi(y''(s))\, \psi(s)\, ds \leq g(y(t))\, \phi(y''(t)) \int_t^1 \psi(s)\, ds \\ &\leq g(y(t))\, \phi(y''(t)) K^*, \end{aligned}$$

where $K^* = \int_0^1 \psi(s)\, ds$. Proceeding exactly as in the proof of Theorem 2.4 (with assumption (2.25) replacing (2.17)) we deduce the existence of constants M_0 and M_1 independent of λ (and also of n) such that $1/n \leqslant |y|_0 \leqslant M_0$ and $b \leqslant |y'|_0 \leqslant M_1$. Now returning to the inequality $-y'''(t) \leqslant g(y(t))\,\phi(y''(t))K^*$ multiply by $(y'')^{1/q}$, where $q = p/(p-1) < p$ to obtain

$$\frac{-(y'')^{1/q}}{\phi(y'')}\,y''' \leqslant g(y)(y')^{1/q}\,(y')^{-1/p}\,(y'')^{1/q}\,K^*.$$

Integration from t to 1 using Holders integral inequality yields

$$\int_0^{y''(t)} \frac{u^{1/q}}{\phi(u)}\,du \leqslant K^* \left\{\int_0^{M_0} g^p(u)\,du\right\}^{1/p} \left\{\int_0^{M_0} u^{-q/p}\,du\right\}^{1/q} = \tilde{M}$$

using assumption (2.26). Define $V(z) = \int_0^z (u^{1/q}/\phi(u))\,du$ so V is an increasing map from $[0, \infty)$ onto $[0, \infty)$ and therefore has an increasing inverse V^{-1}. Thus $y''(t) \leqslant V^{-1}(\tilde{M}) = M_2$ for $t \in [0, 1]$.

Remark. Note M_2 is independent of n.

Finally $0 \leqslant y^{iv}(t)/\psi(t) \leqslant g(1/n)\,\phi(M_2) = M_4$, $t \in (0, 1)$, and integration yields M_3. ∎

For our next theorem we need the following notation. Let $K = C(0, 1)$ be the Banach space of function ω continuous on $(0, 1)$ and for which $\|\omega\|_\infty = \sup_{(0,1)} |\omega(t)| < \infty$. Also let

$$K^4 = \{u \in C^3[0, 1] \cap C^4(0, 1) : \|u\|_4 < \infty\},$$

where

$$\|u\|_4 = \max\left\{|u|_0, |u'|_0, |u''|_0, |u'''|_0, \left\|\frac{u^{iv}}{\psi}\right\|_\infty\right\}$$

with $|u|_0 = \sup_{[0,1]} |u(t)|$ which is a Banach space [11] and define $K^4_B = \{u \in K^4 : u(0) = 1/n,\ u'(0) = b,\ u''(1) = 0,\ u'''(1) = 0\}$ with $K^4_{B0} = \{u \in K^4 : u(0) = 0,\ u'(0) = 0,\ u''(1) = 0,\ u'''(1) = 0\}$.

THEOREM 2.8. *Suppose* (2.15), (2.16), (2.24), (2.25), *and* (2.26) *are satisfied. Then a* $C^3[0, 1] \cap C^4(0, 1)$ *solution of* (2.27^n) *exists.*

Proof. This follows immediately via the ideas of Theorem 2.2 (see also [11]) with the only major changes being that $F_\lambda : C^2[0, 1] \to K$, $j: K^4_B \to C^2[0, 1]$ and $L: K^4_B \to K$ are defined by $F_\lambda v(t) = \lambda f^*(t, v(t), v''(t))$, $ju = u$ and $Lv(t) = v^{iv}(t)/\psi(t)$. Also define

$$V = \left\{ u \in K_B^4 : |u|_0 < M_0 + 1,\ |u'|_0 < M_1 + 1,\ |u''|_0 < M_2 + 1, \right.$$

$$\left. |u'''|_0 < M_3 + 1,\ \left\| \frac{u^{iv}}{\psi} \right\|_\infty < M_4 + 1 \right\}$$

and it is easy to show that $H_\lambda : \bar{V} \to K_B^4$ defined by $H_\lambda = L^{-1} F_\lambda ju$ is a compact homotopy of admissible maps joining the essential map H_0 with H_1. Thus the Topological Transversality Theorem [11] implies H_1 is essential and as a consequence this implies a $C^3[0, 1] \cap C^4(0, 1)$ solution of (2.27^n) exists. ∎

In addition Theorem 2.7 implies there exists constants M_0, M_1, and M_2 independent of n such that $1/n \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $|y''|_0 \leqslant M_2$ for each solution y to (2.27^n). The next argument is broken into two cases, when $b = 0$ and $b > 0$.

Case (1). $b > 0$.

Suppose we have

$$\int_0^1 g(bt)\, \psi(t)\, dt < \infty. \tag{2.29}$$

Then we claim that there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$; to see this note $y^{iv} \leqslant g(y)\, \phi(y'')\psi \leqslant g(y)\, \phi(M_2)\psi$. Integrating from t to 1 with the fact that $y \geqslant bt$ for $t \in [0, 1]$ yields

$$-y'''(t) \leqslant \phi(M_2) \int_t^1 g(bs)\, \psi(s)\, ds \leqslant \phi(M_2) \int_0^1 g(bt)\, \psi(t)\, dt = M_3.$$

Case (2). $b = 0$.

Suppose there are constants $p > 3$, $r > 1$ with $1/r < (p-3)/(2p)$ and with $\int_0^1 g^p(u)\, du < \infty$, $\int_0^1 \psi^r(t)\, dt < \infty$. (2.30)

Then we claim that there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$; to see this note $y^{iv} \leqslant g(y)\, \phi(M_2)\psi$ so

$$(-y''')^{1/m}\, y^{iv} \leqslant \phi(M_2)\, g(y)(y')^{1/p}\, (\,\cdot')^{-1/p}\, (y'')^{1/q}\, (y'')^{1/q}\, (-y''')^{1/m}\, \psi,$$

where $1/q = (p-1)/(2p) - \varepsilon$, $1/m = (p-1)/(2p) + \mu$ with $\mu = \varepsilon - 1/r$ and $1/r < \varepsilon < (p-3)/(2p)$. Also note $1/p + 1/q + 1/m + 1/r = 1$ and $p > q > m$. Now integration from t to 1 along with the generalized Holders integral inequality proves the claim.

Essentially the same reasoning as in Theorem 2.3 establishes.

THEOREM 2.9. (i) *Let* $b>0$ *and suppose* (2.15), (2.16), (2.24), (2.25), (2.26), *and* (2.29) *are satisfied. Then a* $C^2[0,1]\cap C^3(0,1]\cap C^4(0,1)$ *solution of* (2.23) *exists.*

(ii) *Let* $b=0$ *and suppose* (2.15), (2.16), (2.22), (2.24), (2.25), (2.26), *and* (2.30) *are satisfied. Then a* $C^2[0,1]\cap C^3(0,1]\cap C^4(0,1)$ *solution of* (2.23) *exists.*

C. $y(0)=0,\ y'(0)=0,\ y'(1)=0,\ y'''(1)=0$

In this case we examine the two point boundary value problem

$$\begin{gathered} y^{iv}=f(t,y,y''), \qquad 0<t<1 \\ y(0)=0, \qquad y'(0)=0, \qquad y'(1)=0, \qquad y'''(1)=0 \end{gathered} \tag{2.31}$$

with the following assumptions being satisfied:

$f\geqslant 0$ is continuous on $[0,1]\times(0,\infty)\times(-\infty,\infty)$ and $\lim_{y\to 0^+} f(t,y,q)=\infty$ uniformly on compact subsets of $(0,1)\times(-\infty,\infty)$ (2.32)

$0<f(t,y,q)\leqslant g(y)\,\phi(|q|)$ on $(0,1)\times(0,\infty)\times(-\infty,\infty)$, where $g>0$ is continuous and nonincreasing on $(0,\infty)$ and ϕ is continuous and nondecreasing on $[0,\infty)$. (2.33)

As usual we begin by examining for $n\in N^+$ the problems

$$\begin{gathered} y^{iv}=f(t,y,y''), \qquad 0<t<1 \\ y(0)=\frac{1}{n}, \qquad y'(0)=0, \qquad y'(1)=0, \qquad y'''(1)=0. \end{gathered} \tag{2.34^n}$$

THEOREM 2.10. *Suppose* (2.4), (2.5), (2.18), (2.32), *and* (2.33) *are satisfied. For* $\lambda\in[0,1]$ *consider the family of problems*

$$\begin{gathered} y^{iv}=\lambda f(t,y,y''), \qquad 0<t<1 \\ y(0)=\frac{1}{n}, \qquad y'(0)=0, \qquad y'(1)=0, \qquad y'''(1)=0 \end{gathered} \tag{2.35$^n_\lambda$}$$

for fixed $n\in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that for* $t\in[0,1]$

$$\frac{1}{n}\leqslant y(t)\leqslant M_0, \qquad 0\leqslant y'(t)\leqslant M_1, \qquad |y''(t)|\leqslant M_2,$$

$$-M_3\leqslant y'''(t)\leqslant 0, \qquad 0\leqslant y^{iv}(t)\leqslant M_4$$

for each solution $y\in C^4[0,1]$ *to* (2.35^n_λ).

Proof. Let $0<\lambda\leqslant 1$. Now condition (2.32) implies $y>0$ on (0, 1) and as a result we have $y^{iv}>0$, $y'''<0$ on (0, 1), thus y'' is strictly decreasing on (0, 1), also $y'>0$ on (0, 1) which in turn implies $y>1/n$ is strictly increasing on (0, 1). Let $y'_{\max}$ be the maximum of $y'(t)$ on [0, 1] and suppose $y'_{\max}$ occurs at $t_0\in(0,1)$. Then $y''(t_0)=0$ with $y''(t)\geqslant 0$ for $t\leqslant t_0$ and $y''(t)\leqslant 0$ for $t\geqslant t_0$. Now for $t\geqslant t_0$ we have $y^{iv}\leqslant\lambda g(y)\,\phi(-y'')$ so

$$\frac{-y''y^{iv}}{\phi(-y'')}\leqslant g(y)(-y'').$$

Integration from $t(t\geqslant t_0)$ to 1 using assumption (2.4) yields

$$\frac{-y''(t)}{\phi(-y''(t))}\left[-y'''(t)\right]\leqslant g(y(t))\int_t^1(-y''(s))\,ds=g(y(t))\,y'(t)$$

since y'' is strictly decreasing on (0, 1) and y is strictly increasing on (0, 1). Now integrate from t_0 to t to obtain (with I as defined in Theorem 2.1)

$$-y''(t)\leqslant I^{-1}\left(\int_0^{y(1)}g(u)\,du\right)\qquad\text{for}\quad t\geqslant t_0. \tag{2.36}$$

Integrate from t to 1 to obtain $y'(t)\leqslant I^{-1}(\int_0^{y(1)}g(u)\,du)$ for $t\geqslant t_0$ and since the maximum of $y'(t)$ occurs at t_0 we have

$$y'(t)\leqslant y'(t_0)\leqslant I^{-1}\left(\int_0^{y(1)}g(u)\,du\right),\qquad t\in[0,1]. \tag{2.37}$$

Finally integration from 0 to 1 yields $y(1)\leqslant I^{-1}(\int_0^{y(1)}g(u)\,du)+1$ and assumption (2.5) implies there is a constant $M_0>0$ such that $y(1)\leqslant M_0$. In addition (2.37) yields M_1.

Remark. Note M_0 and M_1 are independent of n.

Also for $t\geqslant t_0$, (2.36) yields $-y''(t)\leqslant I^{-1}(\int_0^{M_0}g(u)\,du)=M_1$. To bound $|y''(t)|=y''(t)$ for $t\leqslant t_0$ we first need to obtain a bound for $-y'(t_0)\,y'''(t_0)$. Considering $t\geqslant t_0$ we have $y^{iv}(t)\leqslant g(y(t))\,\phi(-y''(t))\leqslant g(y)\sup_{[0,M_1]}\phi(q)=g(y)\,\phi(M_1)$. Multiply by y' and integrate from t_0 to 1 to obtain

$$-y'(t_0)\,y'''(t_0)-\frac{[y''(1)]^2}{2}\leqslant\phi(M_1)\int_0^{M_0}g(u)\,du$$

so we have

$$-y'(t_0)\,y'''(t_0)\leqslant\phi(M_1)\int_0^{M_0}g(u)\,du+\frac{M_1^2}{2}=M^*. \tag{2.38}$$

Now for the case $t \leqslant t_0$ we have $y^{iv} \leqslant g(y)\,\phi(y'')$; so multiply by y' and integrate from 0 to t_0 to obtain

$$y'(t_0)\,y'''(t_0) + \frac{[y''(0)]^2}{2} \leqslant \phi(y''(0)) \int_0^{M_0} g(u)\,du$$

so we have with (2.38)

$$\frac{[y''(0)]^2}{2} \leqslant \phi(y''(0)) \int_0^{M_0} g(u)\,du + M^*.$$

Thus (2.18) implies there exists a constant $M_2^* > 0$ such that $y''(0) \leqslant M_2^*$. In particular for $t \leqslant t_0$ we have $y''(t) \leqslant y''(0) \leqslant M_2^*$. Hence $|y''(t)| \leqslant M_2 = \max\{M_1, M_2^*\}$.

Remark. Note M_2 is independent of n.

The existence of M_4 and M_3 follows easily. ∎

Essentially the same reasoning as in Theorem 2.2 establishes

THEOREM 2.11. *Suppose* (2.4), (2.5), (2.18), (2.32), *and* (2.33) *are satisfied. Then a* $C^4[0, 1]$ *solution of* (2.34^n) *exists.*

In addition Theorem 2.10 implies there exists constants M_0, M_1, and M_2 independent of n such that $1/n \leqslant |y|_0 \leqslant M_0$, $|y'|_0 \leqslant M_1$, $|y''|_0 \leqslant M_2$ for each solution y to (2.34^n). Now suppose (2.12) holds. Then we claim that there is a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$. To see this consider first the case $t \geqslant t_0$, where we have $y^{iv} \leqslant g(y)\,\phi(M_2)$; so with $1/q = (p-1)/(2p) - \varepsilon$, $1/m = (p-1)/(2p) + \varepsilon$, and $\varepsilon < (p-3)/(2p)$ we have

$$(-y''')^{1/m}\,y^{iv} \leqslant \phi(M_2)\,g(y)(y')^{1/p}\,(y')^{-1/p}\,(-y'')^{1/q}\,(-y'')^{-1/q}\,(-y''')^{1/m}.$$

Now integration from $t (t \geqslant t_0)$ to 1 along with the Generalized Holders integral inequality implies there exists a constant M_3^* independent of n such that $|y'''(t)| \leqslant M_3^*$ for $t \geqslant t_0$. On the other hand for $t \leqslant t_0$ we have

$$(-y''')^{1/m}\,y^{iv} \leqslant \phi(M_2)\,g(y)(y')^{1/p}\,(y')^{-1/p}\,(y'')^{1/q}\,(y'')^{-1/q}\,(-y''')^{1/m};$$

so integration from t to t_0 together with $|y'''(t_0)| \leqslant M_3^*$ yields the claim. Essentially the same reasoning as in Theorem 2.3 establishes

THEOREM 2.12. *Suppose* (2.4), (2.5), (2.12), (2.18), (2.32), *and* (2.33) *are satisfied. In addition assume*

> *For any constants* $M > 0$, $K > 0$ *there exists* $\eta(t)$ *continuous on* $[0, 1]$ *and positive on* $(0, 1)$ *such that* $f(t, y, q) \geqslant \eta(t)$ *on* $[0, 1] \times (0, M] \times [-K, K]$. (2.39)

Then a $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ *solution of* (2.31) *exists.*

D. $y(0)=0,\ y'(1)=b\geqslant 0,\ y''(0)=0,\ y''(1)=0$

To show the existence of a solution to the two point boundary value problems

$$\begin{gathered} y^{iv}=f(t, y, y''), \qquad 0<t<1 \\ y(0)=0, \qquad y'(1)=b\geqslant 0, \qquad y''(0)=0, \qquad y''(1)=0 \end{gathered} \tag{2.40}$$

we begin by examining for $n\in N^+$ the problems

$$\begin{gathered} y^{iv}=f(t, y, y''), \qquad 0<t<1 \\ y(0)=\frac{1}{n}, \qquad y'(1)=b, \qquad y''(0)=0, \qquad y''(1)=0. \end{gathered} \tag{2.41^n}$$

THEOREM 2.13. *Suppose* (2.2), (2.3), (2.4), *and* (2.5) *are satisfied. For* $\lambda\in[0, 1]$ *consider the family of problems*

$$\begin{gathered} y^{iv}=\lambda f(t, y, y''), \qquad 0<t<1 \\ y(0)=\frac{1}{n}, \qquad y'(1)=b, \qquad y''(0)=0, \qquad y''(1)=0 \end{gathered} \tag{2.42$^n_\lambda$}$$

for fixed $n\in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that for* $t\in[0, 1]$

$$\frac{1}{n}\leqslant y(t)\leqslant M_0, \qquad b\leqslant y'(t)\leqslant M_1, \qquad -M_2\leqslant y''(t)\leqslant 0,$$

$$|y'''(t)|\leqslant M_3, \qquad 0\leqslant y^{iv}(t)\leqslant M_4$$

for each solution $y\in C^4[0, 1]$ *to* (2.42$^n_\lambda$).

Proof. Let $0<\lambda\leqslant 1$. Now condition (2.2) implies $y>0$ on $(0, 1)$ and as a result we have $y^{iv}>0$, $y''<0$, $y'>b$ on $(0, 1)$, which in turn implies $y>1/n$ is strictly increasing on $(0, 1)$. Let $-y''_{\max}$ be the maximum of $-y''(t)$ on $[0, 1]$ and suppose $-y''_{\max}$ occurs at $t_0\in(0, 1)$. Then $y'''(t_0)=0$ with $y'''(t)\geqslant 0$ for $t\geqslant t_0$ and $y'''(t)\leqslant 0$ for $t\leqslant t_0$. Now for $t\leqslant t_0$ we have y'' is strictly decreasing on $(0, t_0)$ and $y^{iv}\leqslant\lambda g(y)\,\phi(-y'')$. Now this together with assumption (2.4) yields for $t\leqslant t_0$

$$\begin{aligned} \frac{-y''(t)}{\phi(-y''(t))}\int_t^{t_0}[y^{iv}(s)]\,ds &\leqslant\int_t^{t_0} g(y(s))[-y''(s)]\,ds \\ &\leqslant g(y(t))\int_t^{t_0}[-y''(s)]\,ds \end{aligned}$$

and as a result we have

$$\frac{-y''(t)}{\phi(-y''(t))}[-y'''(t)]\leqslant g(t)[-y'(t_0)+y'(t)]\leqslant g(y(t))\,y'(t)$$

since $y'(t_0)>0$. Now integration from 0 to $t(t\leqslant t_0)$ with I as defined in Theorem 2.1 yields for $t\leqslant t_0$, $-y''(t)\leqslant I^{-1}(\int_0^{y(1)} g(u)\,du)$; $t\leqslant t_0$ and since the maximum of $-y''(t)$ occurs at t_0 we have

$$-y''(t)\leqslant -y''(t_0)\leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right), \qquad t\in[0,1]. \tag{2.43}$$

Also integration from t to 1 yields

$$y'(t)\leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right)+b, \qquad t\in[0,1] \tag{2.44}$$

and finally integration from 0 to 1 will give $y(1)\leqslant I^{-1}(\int_0^{y(1)} g(u)\,du)+b+1$. Assumption (2.5) implies there is a constant $M_0>0$ such that $y(1)\leqslant M_0$. In addition (2.43) and (2.44) yields M_2 and M_1 respectively.

Remark. Note M_0, M_1, and M_2 are independent of n.

The differential equation now yields M_4 and M_3. ∎

Essentially reasoning the same as in Theorem 2.2 establishes

THEOREM 2.14. *Suppose* (2.2), (2.3), (2.4), *and* (2.5) *are satisfied. Then a* $C^4[0,1]$ *solution of* (2.41″) *exists.*

In addition Theorem 2.13 implies there exist constants M_0, M_1, and M_2 independent of n such that $1/n\leqslant |y|_0\leqslant M_0$, $b\leqslant |y'|_0\leqslant M_1$, $|y''|_0\leqslant M_2$ for each solution y to (2.41″). The next argument is broken into two cases, when $b=0$ and $b>0$.

Case (i). $b>0$.

We claim there is a constant M_3 independent of n such that $|y'''|_0\leqslant M$; to see this note $y\geqslant bt$ on $[0,1]$ and $y^{iv}\leqslant g(y)\,\phi(-y'')\leqslant g(bt)\sup_{[0,M_2]}\phi(q)$. Integrating from t to t_0 yields $|y'''(t)|=|\int_{t_0}^t y^{iv}(s)\,ds|\leqslant \sup_{[0,M_2]}\phi(q)$ $\int_0^1 g(bt)\,dt=M_3$.

Case (ii). $b=0$.

Suppose (2.12) holds. Then we claim again that there is a constant M_3 independent of n such that $|y'''|_0\leqslant M_3$. To see this note for $t\geqslant t_0$

$$(y''')^{1/m}\,y^{iv}\leqslant \{\sup_{[0,M_2]}\phi(q)\}\;g(y)(y')^{1/p}\,(y')^{-1/p}$$
$$\times(-y'')^{1/q}\,(-y'')^{-1/q}\,(+y''')^{1/m}$$

while for $t \leqslant t_0$

$$(-y''')^{1/m}\, y^{iv} \leqslant \{\sup_{[0,M_2]} \phi(q)\}\, g(y)(y')^{1/p}\,(y')^{-1/p} \times (-y'')^{1/q}\,(-y'')^{-1/q}\,(-y''')^{1/m}.$$

Now integrate from t to t_0 using the Generalized Holders integral inequality to deduce the claim.

Essentially reasoning the same as in Theorem 2.3 establishes

THEOREM 2.15. (i) *Let* $b>0$ *and suppose* (2.2), (2.3), (2.4), *and* (2.5) *are satisfied. Then a* $C^2[0,1]\cap C^4(0,1)$ *solution of* (2.40) *exists.*

(ii) *Let* $b=0$ *and suppose* (2.2), (2.3), (2.4), (2.5), (2.12), *and* (2.13) *are satisfied. Then a* $C^2[0,1]\cap C^4(0,1)$ *solution of* (2.40) *exists.*

3. SINGULARITIES AT $y''=0$ BUT NOT AT $y=0$

A. $y(0)=a\geqslant 0,\ y'(1)=b\geqslant 0,\ y''(0)=0,\ y'''(1)=0$

In this case we examine the problem

$$\begin{gathered} y^{iv}=\psi(t)\, f(t,y,y''), \qquad 0<t<1 \\ y(0)=a\geqslant 0, \qquad y'(1)=b\geqslant 0, \qquad y''(0)=0, \qquad y'''(1)=0 \end{gathered} \tag{3.1}$$

with the following conditions being satisfied:

f is continuous on $[0,1]\times[a,\infty)\times(-\infty,0)$ with $\lim_{q\to 0^-} f(t,y,q)=\infty$ uniformly on compact subsets of $(0,1)\times(-\infty,\infty)$ (3.2)

$0<f(t,y,q)\leqslant g(y)\,\phi(|q|)$ on $(0,1)\times(a,\infty)\times(-\infty,0)$, where $\phi>0$ is continuous and nonincreasing on $(0,\infty)$ and g is continuous and nondecreasing on $[a,\infty)$ (3.3)

Suppose there exist constants $A\geqslant 0$, $B\geqslant 0$, $0\leqslant r<1$ such that for all $z\in[0,\infty)$, $g(z)\leqslant\int_0^{Az^r+B}(du/\phi(u))$. (2.25)*

In addition suppose assumption (2.24) also holds. To establish the existence of a solution to (3.1) we first consider for $n\in N^+$ the problems

$$\begin{gathered} y^{iv}=\psi(t)\, f(t,y,y''), \qquad 0<t<1 \\ y(0)=a\geqslant 0, \qquad y'(1)=b\geqslant 0, \qquad y''(0)=-\frac{1}{n}, \qquad y'''(1)=0. \end{gathered} \tag{3.4^n}$$

THEOREM 3.1. *Suppose* (2.24), (2.25)*, (3.2), *and* (3.3) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$y^{iv} = \lambda\psi(t)\, f(t, y, y''), \qquad 0 < t < 1$$
$$y(0) = a \geqslant 0, \qquad y'(1) = b \geqslant 0, \qquad y''(0) = -\frac{1}{n}, \qquad y'''(1) = 0 \tag{3.5$^n_\lambda$}$$

for fixed $n \in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that*

$$a \leqslant y(t) \leqslant M_0, \qquad b \leqslant y'(t) \leqslant M_1, \qquad -M_2 \leqslant y''(t) \leqslant \frac{-1}{n},$$
$$-M_3 \leqslant y'''(t) \leqslant 0; \qquad t \in [0, 1]$$

and

$$0 \leqslant \frac{y^{iv}(t)}{\psi(t)} \leqslant M_4; \qquad t \in (0, 1)$$

for each solution $y \in C^3[0, 1] \cap C^4(0, 1)$ *to* (3.5$^n_\lambda$).

Proof. Let $0 < \lambda \leqslant 1$. Now condition (3.2) implies $y'' < 0$ on $(0, 1)$ which implies $y' > b$ on $(0, 1)$ and as a result $y > a$ is strictly increasing on $(0, 1)$. Also condition (3.3) implies $y^{iv} > 0$, $y''' < 0$ on $(0, 1)$ which in turn implies y'' is strictly decreasing on $(0, 1)$. In addition we have $y^{iv} \leqslant \psi(t)\, g(y)\, \phi(-y'')$ so integrating from t to 1 yields

$$-y'''(t) \leqslant \int_t^1 g(y(s))\, \phi(-y''(s))\, \psi(s)\, ds \leqslant g(y(1))\, \phi(-y''(t)) \int_t^1 \psi(s)\, ds$$
$$\leqslant K^* g(y(1))\, \phi(-y''(t)) \leqslant K^* g(y(1))\, \phi\left(-y''(t) - \frac{1}{n}\right),$$

where $K^* = \int_0^1 \psi(s)\, ds$, since ϕ is nonincreasing on $(0, \infty)$. Thus integration from 0 to t with J as defined in Theorem 2.4 yields

$$-y''(t) \leqslant J^{-1}(K^* g(y(1))) + 1, \; t \in [0, 1]. \tag{3.5}$$

Now integrate from t to 1 to obtain

$$y'(t) \leqslant J^{-1}(K^* g(y(1))) + 1 + b \tag{3.6}$$

and finally integration from 0 to 1 yields $y(1) \leqslant J^{-1}(K^* g(y(1))) + 1 + b + a$. Assumption (2.25)* implies there exists a constant $M_0 > 0$ such that $y(1) \leqslant M_0$. In addition (3.5) and (3.6) yield $-y''(t) \leqslant J^{-1}(K^* g(M_0)) + 1 = M_2$ and $y'(t) \leqslant J^{-1}(K^* g(M_0)) + 1 + b = M_1$ since g is nondecreasing on $[a, \infty)$.

Remark. Note M_0, M_1, and M_2 are independent of n.

The differential equation yields M_4 and M_3. ∎

Essentially reasoning the same as in Theorem 2.8 establishes

THEOREM 3.2. *Suppose* (2.24), (2.25)*, (3.2), *and* (3.3) *are satisfied. Then a* $C^3[0, 1] \cap C^4(0, 1)$ *solution of* (3.4") *exists.*

In addition Theorem 3.1 implies there exist constants M_0, M_1, and M_2 independent of n such that $a \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $|y''|_0 \leqslant M_2$ for each solution y to (3.4").

Suppose ϕ satisfies

$$q\phi(q) \text{ is nondecreasing on } (0, \infty). \tag{3.7}$$

Then we claim that there is a constant M_3 independent of n such that $\|y'''\|_{L^2} \leqslant M_3$ for each solution y to (3.4"). To see this note

$$-y''y^{iv} \leqslant \psi(t)\, g(y)(-y'')\, \phi(-y'') \leqslant \psi(t)\, g(M_0)\, M_2\phi(M_2)$$

and integration from 0 to 1 yields

$$-\frac{1}{n}\, y'''(0) + \int_0^1 [y'''(s)]^2\, ds \leqslant K^* g(M_0)\, M_2\phi(M_2),$$

where $K^* = \int_0^1 \psi(s)\, ds$. Now since $y'''(0) \leqslant 0$ our claim is established.

THEOREM 3.3. *Suppose* (2.24), (2.25)*, (3.2), (3.3), *and* (3.7) *are satisfied. In addition assume*

$$\begin{aligned}&\textit{For any constants } M > a,\ K > 0 \textit{ there exists } \eta(t) \textit{ continuous}\\ &\textit{on } [0, 1] \textit{ and positive on } (0, 1) \textit{ such that } f(t, y, q) \geqslant \eta(t) \textit{ on}\\ &[0, 1] \times [a, M] \times [-K, 0),\end{aligned} \tag{3.8}$$

and a $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ *solution of* (3.1) *exists.*

Proof. Theorem 3.2 implies (3.4") has a solution y_n for each n and moreover there exist constants M_0, M_1, M_2, and M_3 independent of n such that

$$a \leqslant |y|_0 \leqslant M_0,\ b \leqslant |y'|_0 \leqslant M_1,\ |y''|_0 \leqslant M_2,\ \|y'''\|_{L^2} \leqslant M_3.$$

It follows that $\{y_n\}$, $\{y_n'\}$, $\{y_n''\}$ are uniformly bounded and equicontinuous (Holders integral inequality with $p = q = 2$) on $[0, 1]$. Essentially reasoning the same as in Theorem 2.3 concludes the proof, observing that $y > 0$ on $(0, 1]$ implies $y'' < 0$ on $(0, 1]$. ∎

EXAMPLE. Consider the two point boundary value problem

$$\begin{aligned} &y^{iv} = t^{-\gamma}(1-t)^{-\rho}(-y'')^{-\alpha}(y^{\beta}+1), \qquad 0<t<1 \\ &y(0)=a\geqslant 0, \qquad y'(1)=b\geqslant 0, \qquad y''(0)=0, \qquad y'''(1)=0 \end{aligned} \tag{3.9}$$

with $0 \leqslant \gamma, \rho, \alpha < 1$, $\beta \geqslant 0$, and $\beta < \alpha + 1$.

To show (3.9) has a solution using the results of this section we consider first

$$\begin{aligned} &y^{iv} = t^{-\gamma}(1-t)^{-\rho}|-y''|^{-\alpha}(|y|^{\beta}+1), \qquad 0<t<1 \\ &y(0)=a\geqslant 0, \qquad y'(1)=b\geqslant 0, \qquad y''(0)=0, \qquad y'''(1)=0. \end{aligned} \tag{3.10}$$

Here $f(t, y, q) = |q|^{-\alpha}(|y|^{\beta}+1)$, $\psi(t) = t^{-\gamma}(1-t)^{-\rho}$ so clearly (2.24), (3.2), and (3.3) are satisfied with $g(u) = |u|^{\beta}+1$ and $\phi(u) = |u|^{-\alpha}$. In addition with $\eta(t) = (a^{\beta}+1)K^{-\alpha}$ we see that (3.8) is also satisfied. It is also easy to check that (2.25)* holds since $\beta < \alpha + 1$. Thus a $C^2[0, 1] \cap C^3(0, 1] \cap C^4(0, 1)$ solution y of (3.10) exists by Theorem 3.3. In addition since $y > 0$ and $y'' < 0$ on $(0, 1)$ we see that y is also a solution to (3.9).

B. $y(0)=a\geqslant 0$, $y'(0)=b\geqslant 0$, $y''(1)=0$, $y'''(1)=0$

Consider the two point boundary value problem

$$\begin{aligned} &y^{iv} = \psi(t)\, f(t, y, y''), \qquad 0<t<1 \\ &y(0)=a\geqslant 0, \qquad y'(0)=b\geqslant 0, \qquad y''(1)=0, \qquad y'''(1)=0 \end{aligned} \tag{3.11}$$

with f having bounded dependence on its y variable for any fixed values of the other arguments. Assume (2.24) holds and in addition

> f is continuous on $[0, 1] \times [a, \infty) \times (0, \infty)$ with $\lim_{q\to 0^+} f(t, y, q) = \infty$ uniformly on compact subsets of $(0, 1) \times (-\infty, \infty)$ (3.12)
>
> $0 < f(t, y, q) \leqslant \phi(q)$ on $(0, 1) \times [a, \infty) \times (0, \infty)$, where ϕ is continuous and nonincreasing on $(0, \infty)$. (3.13)

Our examination of (3.11) begins by considering for $n \in N^+$ the problems

$$\begin{aligned} &y^{iv} = \psi(t)\, f(t, y, y''), \qquad 0<t<1 \\ &y(0)=a\geqslant 0, \qquad y'(0)=b, \qquad y''(1)=\frac{1}{n}, \qquad y'''(1)=0. \end{aligned} \tag{3.14^n}$$

THEOREM 3.4. *Suppose* (2.24), (3.12), *and* (3.13) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$\begin{aligned} &y^{iv} = \lambda\psi(t)\, f(t, y, y''), \qquad 0<t<1 \\ &y(0)=a\geqslant 0, \qquad y'(0)=b, \qquad y''(1)=\frac{1}{n}, \qquad y'''(1)=0 \end{aligned} \tag{3.15$^n_\lambda$}$$

for fixed $n \in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that*

$$a \leqslant y(t) \leqslant M_0, \qquad b \leqslant y'(t) \leqslant M_1,$$

$$\frac{1}{n} \leqslant y''(t) \leqslant M_2, \qquad -M_3 \leqslant y'''(t) \leqslant 0; \qquad t \in [0, 1]$$

and

$$0 \leqslant \frac{y^{iv}(t)}{\psi(t)} \leqslant M_4; \qquad t \in (0, 1)$$

for each solution $y \in C^3[0, 1] \cap C^4(0, 1)$ *to* (3.15^n_λ).

Proof. Let $0 < \lambda \leqslant 1$. Now condition (3.12) implies $y'' > 0$ on (0, 1) which implies $y' > b$, $y > a$ on (0, 1). Also condition (3.13) implies $y^{iv} > 0$, $y''' < 0$ on (0, 1) and as a result y'' is strictly decreasing on (0, 1). In addition we have

$$\frac{y^{iv}(t)}{\psi(t)} \leqslant \lambda\phi(y'') \leqslant \phi\left(\frac{1}{n}\right) = M_4$$

and integration yields M_3, M_2, M_1, and M_0. ∎

Essentially reasoning the same as that in Theorem 2.8 establishes

THEOREM 3.5. *Suppose* (2.24), (3.12), *and* (3.13) *are satisfied. Then a* $C^3[0, 1] \cap C^4(0, 1)$ *solution of* (3.14^n) *exists.*

Now suppose the following conditions are satisfied

$$\psi \text{ is nonincreasing on } (0, 1) \tag{3.16}$$

$$\text{For any } c \in (0, \infty), \int_0^c \phi(u)\, du < \infty \tag{3.17}$$

$$\text{There exists a constant } m > 1 \text{ with } \int_0^1 [\psi(s)]^{m/2}\, ds < \infty. \tag{3.18}$$

Then we claim that there are constants M_0, M_1, M_2, and M_3 independent of n such that $a \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $1/n \leqslant |y''|_0 \leqslant M_2$, $\|y'''\|_{L^m} \leqslant M_3$ for each solution y to (3.14^n). To see this multiply $y^{iv} \leqslant \psi(t)\, \phi(y'')$ by $-y'''$ and integrate from t to 1 to obtain

$$\frac{[y'''(t)]^2}{2} \leqslant \int_t^1 \psi(s)\, \phi(y''(s))(-y''(s))\, ds \leqslant \psi(t) \int_0^{y''(0)} \phi(u)\, du$$

using assumption (3.16). Thus

$$-y'''(t) \leqslant \left\{2\psi(t) \int_0^{y''(0)} \phi(u)\, du\right\}^{1/2} \tag{3.19}$$

and integration from 0 to 1 yields $y''(0) \leqslant \tilde{K}\{2\int_0^{y''(0)} \phi(u)\,du\}^{1/2} + 1$, where $\tilde{K} = \int_0^1 [\psi(s)]^{1/2}\,ds$. Assumption (3.17) implies there exists a constant M_2 (independent of n) such that $y''(t) \leqslant y''(0) \leqslant M_2$. Now integration yields M_1 and M_0. Returning to (3.19) we have for $t \in [0, 1]$, $-y'''(t) \leqslant \{2\psi(t)\int_0^{M_2} \phi(u)\,du\}^{1/2} = L[\psi(t)]^{1/2}$, where $L = \{2\int_0^{M_0} \phi(u)\,du\}^{1/2}$. Then

$$\|y'''\|_{L^m} = \left(\int_0^1 |y'''(t)|^m\,dt\right)^{1/m} \leqslant L\left\{\int_0^1 [\psi(s)]^{m/2}\,ds\right\}^{1/m} = M_3$$

and our claim is established.

THEOREM 3.6. *Suppose* (2.24), (3.12), (3.13), (3.16), (3.17), *and* (3.18) *are satisfied. In addition assume*

> *For any constants $M > a$, $K > 0$ there exists $\eta(t)$ continuous on $[0, 1]$ and positive on $(0, 1)$ such that $f(t, y, q) \geqslant \eta(t)$ on $[0, 1] \times [a, M] \times (0, K]$,* (3.20)

$$\int_0^1 \phi\left(\int_t^1 (s-t)\,\psi(s)\,\eta(s)\,ds\right)\psi(t)\,dt < \infty. \qquad (3.20)^*$$

Then a $C^3[0, 1] \cap C^4(0, 1)$ solution of (3.11) *exists.*

Proof. Essentially the same reasoning as in Theorem 3.3 guarantees the existence of a subsequence $\{y_{n'}\}$ converging uniformly on $[0, 1]$ to some $y \in C^2[0, 1]$. In addition $y(0) = a$, $y'(0) = b$, $y''(1) = 0$ with $y \in C^4(0, 1) \cap C^3[0, 1)$ and $y^{iv}(t) = f(t, y(t), y''(t))\,\psi(t)$ on $(0, 1)$. It remains to show $y'''(1) = 0$. Now $y_{n'}^{iv}(t) \geqslant \psi(t)\,\eta(t)$ so integration yields $y''_{n'}(t) \geqslant \int_t^1 (s-t)\,\psi(s)\,\eta(s)\,ds = \theta(t)$. Thus

$$0 = \lim_{n' \to \infty} y'''_{n'}(1) = \lim_{n' \to \infty}\left[y'''_{n'}(0) + \int_0^1 \psi(t)\,f(t, y_{n'}(t), y''_{n'}(t))\,dt\right]$$

$$= y'''(0) + \int_0^1 \psi(t)\,f(t, y(t), y''(t))\,dt = y'''(1)$$

by the Lebesgue dominated convergence theorem since $\psi(t)\,f(t, y_{n'}(t), y''_{n'}(t)) \leqslant \psi(t)\,\phi(\theta(t)) \in L^1$ by (3.20)*. This also proves that $y \in C^3[0, 1]$. ∎

C. $y(0) = a \geqslant 0$, $y'(1) = b \geqslant 0$, $y''(0) = 0$, $y''(1) = 0$.

To show the existence of a solution to the two point boundary value problem

$$\begin{gathered} y^{iv} = \psi(t)\,f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = a \geqslant 0, \qquad y'(1) = b \geqslant 0, \qquad y''(0) = 0, \qquad y''(1) = 0 \end{gathered} \qquad (3.21)$$

we begin by examining for $n \in N^+$ the problems

$$\begin{gathered} y^{iv} = \psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = a \geqslant 0, \qquad y'(1) = b \geqslant 0, \qquad y''(0) = -\frac{1}{n}, \qquad y''(1) = -\frac{1}{n}. \end{gathered} \tag{3.22''}$$

THEOREM 3.7. *Suppose* (2.24), (2.25)*, (3.2), *and* (3.3) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$\begin{gathered} y^{iv} = \lambda\psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = a, \qquad y'(1) = b, \qquad y''(0) = -\frac{1}{n}, \qquad y''(1) = -\frac{1}{n} \end{gathered} \tag{3.23$^n_\lambda$}$$

for fixed $n \in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that*

$$a \leqslant y(t) \leqslant M_0, \qquad b \leqslant y'(t) \leqslant M_1,$$

$$-M_2 \leqslant y''(t) \leqslant -\frac{1}{n}, \qquad |y'''(t)| \leqslant M_3; \qquad t \in [0, 1]$$

and

$$0 \leqslant \frac{y^{iv}(t)}{\psi(t)} \leqslant M_4;\ t \in (0, 1)$$

for each solution $y \in C^3[0, 1] \cap C^4(0, 1)$ *to* (3.23$^n_\lambda$).

Proof. Let $0 < \lambda \leqslant 1$. Now condition (3.2) implies $y'' < 0$ on (0, 1) which implies $y' > b$ on (0, 1) and as a result $y > a$ is strictly increasing on (0, 1). Also condition (3.3) implies $y^{iv} > 0$ on (0, 1). Let $-y''_{\max}$ occurs at $t_0 \in (0, 1)$. Then $y'''(t_0) = 0$ with $y'''(t) \geqslant 0$ for $t \geqslant t_0$ and $y'''(t) \leqslant 0$ for $t \leqslant t_0$. Now for $t \geqslant t_0$ we have y'' is strictly increasing on $(t_0, 1)$ and $y^{iv} \leqslant \lambda\psi(t)\, g(y)\, \phi(-y'')$. Integrate from t_0 to $t(t \geqslant t_0)$ and we obtain

$$\begin{aligned} y'''(t) &\leqslant \int_{t_0}^{t} g(y(s))\, \phi(-y''(s))\, \psi(s)\, ds \leqslant g(y(t))\, \phi(-y''(t)) \int_{t_0}^{t} \psi(s)\, ds \\ &\leqslant g(y(1))\, \phi(-y''(t)) K^* \leqslant g(y(1))\, \phi\left(-y''(t) - \frac{1}{n}\right) K^*, \end{aligned}$$

where $K^* = \int_0^1 \psi(s)\, ds$, since ϕ is nonincreasing on $(0, \infty)$ and g is nondecreasing on $[a, \infty)$. Thus integration from $t(t \geqslant t_0)$ to 1 with J as defined in Theorem 2.4 yields $-y''(t) \leqslant J^{-1}(K^* g(y(1))) + 1$; $t \leqslant t_0$ and since the maximum of $-y''(t)$ occurs at t_0 we have

$$-y''(t) \leqslant -y''(t_0) \leqslant J^{-1}(K^* g(y(1))) + 1,\ t \in [0, 1]. \tag{3.24}$$

Now integration from t to 1 yields

$$y'(t) \leqslant J^{-1}(K^* g(y(1))) + 1 + b, \; t \in [0, 1] \tag{3.25}$$

and finally integration from 0 to 1 will give $y(1) \leqslant J^{-1}(K^* g(y(1))) + 1 + b + a$. Assumption (2.25)* implies there is a constant $M_0 > 0$ such that $y(1) \leqslant M_0$. In addition (3.24) and (3.25) yield M_2 and M_1 respectively since g is nondecreasing on $[a, \infty)$.

Remark. Note M_0, M_1, and M_2 are independent of n.

The existence of M_4 and M_3 follows as before. ∎

Essentially the same reasoning as in Theorem 2.8 establishes

THEOREM 3.8. *Suppose* (2.24), (2.25)*, (3.2), *and* (3.3) *are satisfied. Then a* $C^3[0, 1] \cap C^4(0, 1)$ *solution of* (3.22^n) *exists.*

In addition Theorem 3.7 implies there exist constants M_0, M_1, and M_2 independent of n such that $a \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $|y''|_0 \leqslant M_2$ for each solution y to (3.22^n). Now suppose (3.7) is satisfied. Multiply $y^{iv} \leqslant \psi(s)\, g(y)\, \phi(-y'')$ by $-y''$ to obtain $-y'' y^{iv} \leqslant \psi(s)\, g(M_0)\, M_2 \phi(M_2)$. Integration from 0 to 1 gives

$$\frac{1}{n} y'''(1) - \frac{1}{n} y'''(0) + \int_0^1 [y'''(s)]^2 \, ds \leqslant K^* g(M_0)\, M_2 \phi(M_2),$$

where $K^* = \int_0^1 \psi(s)\, ds$. Now since $y'''(1) \geqslant 0$ and $y'''(0) \leqslant 0$ we have

$$\|y'''\|_{L^2} \leqslant \{K^* g(M_0)\, M_2 \phi(M_2)\}^{1/2} = M_3,$$

where M_3 is independent of n.

THEOREM 3.9. *Suppose* (2.24), (2.25)*, (3.2), (3.3), (3.7), *and* (3.8) *are satisfied. Then a* $C^2[0, 1] \cap C^4(0, 1)$ *solution of* (3.21) *exists.*

Proof. The proof more or less follows the argument in Theorem 3.3 with the following modification. Now $y > 0$ on $(0, 1]$ implies $y'' < 0$ on $(0, 1)$ and so $y_{n'}^{iv} \to y^{iv}$ uniformly on $[\varepsilon, 1 - \varepsilon]$ for each $\varepsilon \in (0, 1)$. Thus $y \in C^4(0, 1)$ and $y^{iv} = f(t, y, y'')\, \psi(t)$ on $(0, 1)$. ∎

4. SINGULARITIES AT BOTH $y = 0$ AND $y'' = 0$

Again we discuss the individual boundary conditions separately.

A. $y(0) = 0$, $y'(1) = b \geqslant 0$, $y''(0) = 0$, $y'''(1) = 0$

In this case we examine the two point boundary value problem

$$\begin{gathered} y^{iv}=f(t, y, y''), \quad 0<t<1 \\ y(0)=0, \quad y'(1)=b\geqslant 0, \quad y''(0)=0, \quad y'''(1)=0 \end{gathered} \tag{4.1}$$

f is continuous on $[0, 1]\times(0, \infty)\times(-\infty, 0)$ with $\lim_{y\to 0^+} f(t, y, q)=\infty$ uniformly on compact subsets of $(0, 1)\times(-\infty, \infty)\setminus\{0\}$ and $\lim_{q\to 0^-} f(t, y, q)=\infty$ uniformly on compact subsets of $(0, 1)\times(0, \infty)$ (4.2)

$0<f(t, y, q)\leqslant g(y)\,\phi(|q|)$ on $(0, 1)\times(0, \infty)\times(-\infty, 0)$ where $\phi>0$ and g are continuous and nonincreasing on $(0, \infty)$. (4.3)

To establish the existence of a solution to (4.1) we first consider for $n\in N^+$ the problems

$$\begin{gathered} y^{iv}=f(t, y, y''), \quad 0<t<1 \\ y(0)=\frac{1}{n}, \quad y'(1)=b\geqslant 0, \quad y''(0)=-\frac{1}{n}, \quad y'''(1)=0. \end{gathered} \tag{4.4^n}$$

THEOREM 4.1. *Suppose* (4.2) *and* (4.3) *are satisfied. For* $\lambda\in[0, 1]$ *consider the family of problems*

$$\begin{gathered} y^{iv}=\lambda f(t, y, y''), \quad 0<t<1 \\ y(0)=\frac{1}{n}, \quad y'(1)=b\geqslant 0, \quad y''(0)=-\frac{1}{n}, \quad y'''(1)=0 \end{gathered} \tag{4.5$^n_\lambda$}$$

for fixed $n\in N^+$. *Then there exist constants* M_0, M_1, M_2, M_3, *and* M_4 *independent of* λ *such that for* $t\in[0, 1]$

$$\frac{1}{n}\leqslant y(t)\leqslant M_0, \quad b\leqslant y'(t)\leqslant M_1, \quad -M_2\leqslant y''(t)\leqslant -\frac{1}{n},$$

$$-M_3\leqslant y'''(t)\leqslant 0, \quad 0\leqslant y^{iv}(t)\leqslant M_4$$

for each solution $y\in C^4[0, 1]$ *to* (4.5$^n_\lambda$).

Proof. Let $0<\lambda\leqslant 1$. Now condition (4.2) implies $y>0$, $y''<0$ on $(0, 1)$ and as a result we have $y^{iv}>0$, $y'''<0$ on $(0, 1)$; thus $y''<1/n$ is strictly decreasing on $(0, 1)$ which in turn implies $y'>b$ on $(0, 1)$ and so $y>1/n$ is strictly increasing on $(0, 1)$. In addition we have $y^{iv}\leqslant\lambda g(y)\,\phi(-y'')\leqslant g(1/n)\,\phi(1/n)=M_4$ and integration yields M_3, M_2, M_1, and M_0. ∎

Essentially reasoning the same as that in Theorem 2.2 establishes

THEOREM 4.2. *Suppose* (4.2) *and* (4.3) *are satisfied. Then a* $C^4[0,1]$ *solution of* (4.4″) *exists.*

Now suppose (2.4) and (2.5) are satisfied. Then we claim that there are constants M_0, M_1, and M_2 independent of n such that $1/n \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $|y''|_0 \leqslant M_2$ for each solution y to (4.4″). The proof of the claim follows more or less the proof of Theorem 2.1 —we provide a few details. Assumption (2.4) implies

$$\frac{(-y''-1/n)\,y^{iv}}{\phi(-y''-1/n)} \leqslant \frac{-y''y^{iv}}{\phi(-y'')} \leqslant \lambda g(y)(-y'') \leqslant g(y)(-y'')$$

so integration from t to 1 yields

$$\frac{(-y''(t)-1/n)[-y'''(t)]}{\phi(-y''(t)-1/n)} \leqslant g(y(t))\,y'(t).$$

Now with I as defined in Theorem 2.1, integrate from 0 to t to obtain

$$-y''(t) \leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right) + 1. \tag{4.6}$$

Next integration from t to 1 yields

$$y'(t) \leqslant I^{-1}\left(\int_0^{y(1)} g(u)\,du\right) + 1 + b \tag{4.7}$$

and finally integration from 0 to 1 yields $y(1) \leqslant I^{-1}(\int_0^{y(1)} g(u)\,du) + 2 + b$. Assumption (2.5) implies there exists a constant $M_0 > 0$ such that $y(1) \leqslant M_0$. Also (4.6) and (4.7) yields M_2 and M_1, respectively. Thus our claim is established. The next argument is broken into two cases, when $b = 0$ and $b > 0$.

Case (1). $b > 0$.

The exact same argument as that in Case (1), part A, of Section 2 implies there exists a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$.

Case (2). $b = 0$.

Suppose (2.12). Then the exact same argument as that in Case 3, part A of Section 2 implies there exists a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$.

Essentially the same argument as in Theorem 2.3 establishes.

THEOREM 4.3. (i) *Let* $b > 0$ *and suppose* (2.4), (2.5), (4.2), *and* (4.3) *are satisfied. Then a* $C^2[0,1] \cap C^3(0,1] \cap C^4(0,1)$ *solution of* (4.1) *exists.*

(ii) *Let* $b=0$ *and suppose* (2.4), (2.5), (2.12), (4.2), *and* (4.3) *are satisfied. In addition assume*

> *For any constants* $M>0$, $K>0$ *there exists* $\eta(t)$ *continuous on* $[0,1]$ *and positive on* $(0,1)$ *such that* $f(t,y,q)\geqslant\eta(t)$ *on* $(0,1)\times(0,M]\times[-K,0)$. (4.8)

Then a $C^2[0,1]\cap C^3(0,1]\cap C^4(0,1)$ *solution of* (4.1) *exists.*

EXAMPLE. Consider the boundary value problem

$$y^{iv}=y^{-\alpha}\,|y''|^{-\beta},\qquad 0<t<1;\qquad y(0)=y'(1)=y''(0)=y'''(1)=0$$

with $\beta>0$ and $0<\alpha<\frac{1}{3}$.

To see the above has a solution take $g(y)=y^{-\alpha}$ and $\phi(|q|)=|q|^{-\beta}$. Clearly (4.2), (4.3), (2.4), and (2.5) are satisfied. In addition (2.12) is true since $\alpha<\frac{1}{3}$ and also (4.8) holds with $\eta(t)=M^{-\alpha}K^{-\beta}$. Thus a $C^2[0,1]\cap C^3(0,1]\cap C^4(0,1)$ solution exists by Theorem 4.3(ii).

B. $y(0)=0,\ y'(1)=b\geqslant 0,\ y''(0)=0,\ y''(1)=0$

Finally in this section we discuss the two point boundary value problem

$$\begin{gathered} y^{iv}=f(t,y,y''),\qquad 0<t<1\\ y(0)=0,\qquad y'(1)=b\geqslant 0,\qquad y''(0)=0,\qquad y''(1)=0 \end{gathered}\tag{4.9}$$

with assumptions (4.2) and (4.3) being satisfied. Then by reasoning more or less the same as that in Theorems 4.1 and 2.2 we have that

$$\begin{gathered} y^{iv}=f(t,y,y''),\qquad 0<t<1\\ y(0)=\frac{1}{n},\qquad y'(1)=b\geqslant 0,\qquad y''(0)=-\frac{1}{n},\qquad y''(1)=-\frac{1}{n} \end{gathered}\tag{4.10^n}$$

has a solution y_n for each $n\in N^+$. Now suppose (2.4) and (2.5) are satisfied. Then we claim that there are constants M_0, M_1, and M_2 independent of n such that $1/n\leqslant|y|_0\leqslant M_0$, $b\leqslant|y'|_0\leqslant M_1$, $1/n\leqslant|y''|_0\leqslant M_2$ for each solution y to (4.10^n). The proof of the claim follows more or less the proof of Theorem 2.13—we provide here a few details. Now condition (4.2) implies $y>0$, $y''<0$ on $(0,1)$, and as a result we have $y^{iv}>0$, $y''<-1/n$ on $(0,1)$ which in turn implies $y'>b$ on $(0,1)$ and this $y>1/n$ is strictly increasing on $(0,1)$. Let $-y''_{\max}$ be the maximum of $-y''(t)$ on $[0,1]$ and suppose $-y''_{\max}$ occurs at $t_0\in(0,1)$. Now for $t\leqslant t_0$ we have $y'''(t)\leqslant 0$ so y'' is strictly decreasing on $(0,t_0)$. Also assumption (2.4) yields

$$\frac{(-y''-1/n)}{\phi(-y''-1/n)}\,y^{iv}\leqslant\frac{-y''y^{iv}}{\phi(-y'')}\leqslant g(y)(-y''),$$

so just as in Theorem 2.13 integration from $t(t \leqslant t_0)$ to t_0 yields

$$\frac{(-y''(t)-1/n)[-y'''(t)]}{\phi(-y''(t)-1/n)} \leqslant g(y(t))\, y'(t).$$

Now integration from 0 to $t(t \leqslant t_0)$ gives $-y''(t) \leqslant I^{-1}(\int_0^{y(1)} g(u)\, du) + 1$; $t \leqslant t_0$ and since the maximum of $-y''(t)$ occurs at t_0 we have

$$-y''(t) \leqslant I^{-1}\left(\int_0^{y(1)} g(u)\, du + 1\right), \qquad t \in [0, 1].$$

The proof of the claim now follows just as in Theorem 2.13. The next argument is broken into two cases, when $b = 0$ and $b > 0$.

Case (i). $b > 0$.

The exact same argument as that in Case (i) part D of Section 2 implies there exists a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$.

Case (2). $b = 0$.

Suppose (2.12) holds. Then the exact same argument as that in Case (ii) part D of Section 2 implies there exists a constant M_3 independent of n such that $|y'''|_0 \leqslant M_3$.

Essentially the same proof as in Theorem 2.3 establishes

THEOREM 4.4. (i) *Let $b > 0$ and suppose* (2.4), (2.5), (4.2), *and* (4.3) *are satisfied. Then a $C^2[0, 1] \cap C^4(0, 1)$ solution of* (4.9) *exists.*

(ii) *Let $b = 0$ and suppose* (2.4), (2.5), (2.12), (4.2), (4.3), *and* (4.8) *are satisfied. Then a $C^2[0, 1] \cap C^4(0, 1)$ solution of* (4.9) *exists.*

5. HIGHER ORDER EQUATIONS

In this section we give a brief treatment of two point boundary value problems for higher order equations. There are many possible permutations of boundary conditions that the ideas of this paper can handle; however we restrict our discussion to two sets of such conditions. Again for problems discussed here our nonlinear term may be singular at $t = 0$, $t = 1$, $y = 0$, and/or $y'' = 0$.

Our first problem is to consider for $n > 4$ even the two point boundary value problem

$$\begin{gathered} y^{(n)} + \psi(t)\, f(t, y, y'') = 0, \qquad 0 < t < 1 \\ y(0) = a \geqslant 0, \qquad y'(0) = b \geqslant 0, \\ y''(0) = 0, \qquad y^{(j)}(1) = 0; \qquad j = 3, ..., n-1 \end{gathered} \tag{5.1}$$

with f satisfying the following conditions:

f is continuous on $[0, 1] \times [a, \infty) \times (0, \infty)$ with $f > 0$ on $(0, 1) \times (0, \infty) \times (0, \infty)$ and $\lim_{q \to 0^+} f(t, y, q) = \infty$ uniformly on compact subsets of $(0, 1) \times (0, \infty)$ (5.2)

$f(t, y, q) \leqslant g(y)\,\phi(q)$ on $[0, 1] \times [a, \infty) \times (0, \infty)$, where $\phi > 0$ is continuous and nonincreasing on $(0, \infty)$ and g is continuous and nondecreasing on $[a, \infty)$. (5.3)

To establish the existence of a solution to (5.1) we first consider for $m \in N^+$ the problems

$$
\begin{gathered}
y^{(n)} + \psi(t)\, f(t, y, y'') = 0, \qquad 0 < t < 1 \\
y(0) = a \geqslant 0, \qquad y'(0) = b \geqslant 0, \\
y''(0) = \frac{1}{m}, \qquad y^{(j)}(1) = 0, \qquad j = 3, \ldots, n-1
\end{gathered}
\tag{5.4^m}
$$

THEOREM 5.1. *Suppose* (2.24), (2.25)*, (5.2), *and* (5.3) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$
\begin{gathered}
y^{(n)} + \lambda\psi(t)\, f(t, y, y'') = 0, \qquad 0 < t < 1 \\
y(0) = a, \qquad y'(0) = b, \qquad y''(0) = \frac{1}{m}, \\
y^{(j)}(1) = 0, \qquad j = 3, \ldots, n-1
\end{gathered}
\tag{5.5$^m_\lambda$}
$$

for fixed $m \in N^+$. *Then there exist constants* M_i, $i = 0, \ldots, n$, *independent of* λ *such that*

$$
\begin{gathered}
a \leqslant y(t) \leqslant M_0; \qquad b \leqslant y'(t) \leqslant M_1; \qquad \frac{1}{m} \leqslant y''(t) \leqslant M_2; \\
0 \leqslant y^{(i)}(t) \leqslant M_i, \qquad i = 3, 5, \ldots, n-1; \\
-M_i \leqslant y^{(i)}(t) \leqslant 0, \qquad i = 4, 6, \ldots, n-2;
\end{gathered}
$$

for $t \in [0, 1]$ *and*

$$
-M_n \leqslant \frac{y^{(n)}(t)}{\psi(t)} \leqslant 0; \qquad t \in (0, 1)
$$

for each solution $y \in C^{n-1}[0, 1] \cap C^n(0, 1)$ *to* (5.5$^m_\lambda$).

Proof. Let $0 < \lambda \leqslant 1$. Now condition (5.2) implies $y > 0$ on $(0, 1)$ and so we have $y' > b$ on $(0, 1)$ which in turn implies $y > a$ is strictly increasing on (0, 1). Also we have $y^{(n)} < 0$ on (0, 1) so $y^{(i)} < 0$; $i = n-2, \ldots, 6, 4$ and $y^{(i)} > 0$; $i = n-1, \ldots, 5, 3$ on (0, 1). In particular $y'' > 1/m$ is strictly increasing on (0, 1). In addition we have $-y^{(n)} \leqslant \psi(t)\, g(y)\, \phi(y'')$; so integrate from t to 1 to obtain

$$\begin{aligned} y^{(n-1)}(t) &\leqslant \int_t^1 g(y(s))\, \phi(y''(s))\, \psi(s)\, ds \\ &\leqslant g(y(1))\, \phi(y''(t)) \int_t^1 \psi(s)\, ds \leqslant K^* g(y(1))\, \phi(y''(t)), \end{aligned}$$

where $K^* = \int_0^1 \psi(s)\, ds$, since g is nondecreasing on $[a, \infty)$ and ϕ is nonincreasing on $(0, \infty)$. Proceeding with this we obtain in general

$$(-1)^{j+1}\, y^{(n-j)}(t) \leqslant K^* g(y(1))\, \phi(y''(t)), \qquad j = 1, \ldots, n-3. \tag{5.6}$$

In particular we have

$$\frac{y'''(t)}{\phi\left(y''(t) - \dfrac{1}{m}\right)} \leqslant K^* g(y(1))$$

and integration from 0 to t with J as defined in Theorem 2.4 yields

$$y''(t) \leqslant J^{-1}(K^* g(y(1))) + 1 \qquad \text{for} \quad t \in [0, 1]. \tag{5.7}$$

Now integrate from 0 to t to obtain

$$y'(t) \leqslant J^{-1}(K^* g(y(1))) + 1 + b, \qquad t \in [0, 1] \tag{5.8}$$

and finally integration from 0 to 1 gives $y(1) \leqslant J^{-1}(K^* g(y(1))) + 1 + b + a$. Assumption (2.25)* implies there exists a constant $M_0 > 0$ such that $y(1) \leqslant M_0$. In addition (5.7) and (5.8) yield M_2 and M_1, respectively.

Remark. Note M_0, M_1, and M_2 are independent of m.

The differential equation now yields $M_n, M_{n-1}, \ldots, M_3$. ∎

THEOREM 5.2. *Suppose* (2.24), (2.25)*, (5.2), *and* (5.3) *are satisfied. Then a* $C^{n-1}[0, 1] \cap C^n(0, 1)$ *solution of* (5.4^m) *exists.*

Proof. This follows immediately via the ideas of Theorem 2.8, where in this case $F_\lambda: C^2[0, 1] \to K$, $j: K_B^n \to C^2[0, 1]$, $L: K_B^n \to K$ are defined by $F_\lambda v(t) = -\lambda F^*(t, v(t))$, $ju = u$ and $Lv(t) = v^{(n)}(t)/\psi(t)$. ∎

In addition there exist constants M_0, M_1, and M_2 independent of m such that $a \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $1/m \leqslant |y''|_0 \leqslant M_2$ for each solution y to

(5.4‴). Now suppose (3.7) holds. Then we claim that there is a constant M_3 independent of m such that $\|y'''\|_{L^2} \leqslant M_3$. To see this note that (5.6) implies $-y^{iv}(t) \leqslant K^* g(y(1))\, \phi(y''(t))$ so

$$-y''(t)\, y^{iv}(t) \leqslant K^* g(y(1))\, y''(t)\, \phi(y''(t)) \leqslant K^* g(M_0)\, M_2 \phi(M_2).$$

Integrate from 0 to 1 to obtain

$$\frac{1}{m}\, y'''(0) + \int_0^1 [y'''(s)]^2\, ds \leqslant K^* g(M_0)\, M_2 \phi(M_2).$$

Now since $y'''(0) \geqslant 0$ our claim is established. Essentially reasoning the same as in that Theorem 3.3 established

THEOREM 5.3. *Suppose* (2.24), (2.25)*, (3.7), (3.20), (5.2), *and* (5.3) *are satisfied. Then a* $C^2[0, 1] \cap C^{n-1}(0, 1] \cap C^n(0, 1)$ *solution of* (5.1) *exists.*

Remark. It should be noted here that the exact analogue of Theorem 5.3 holds with $n = 4$.

Remark. With the above ideas we can obtain an analogue of Theorem 5.3 for the two part boundary value problem

$$\begin{gathered} y^{(n)} = \psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = a \geqslant 0, \qquad y'(0) = b \geqslant 0, \qquad y''(0) = 0, \\ y^{(j)}(1) = 0, \qquad j = 3, ..., n-1 \end{gathered}$$

with $n > 3$ odd.

Finally in this paper we examine for $n > 4$ even the two point boundary value problem

$$\begin{gathered} y^{(n)} = \psi(t)\, f(t, y, y''), \qquad 0 < t < 1 \\ y(0) = 0, \qquad y'(1) = b \geqslant 0, \qquad y''(0) = c \leqslant 0, \\ y^{(j)}(1) = 0; \qquad j = 3, ..., n-1 \end{gathered} \tag{5.9}$$

with assumptions (2.2), (2.3), (2.4), and (2.24) being satisfied. In addition assume

$$\psi \text{ is nondecreasing on } (0, \infty) \tag{5.10}$$

Suppose there exist constants $A \geqslant 0$, $B \geqslant 0$, $0 \leqslant r < 1$ such that for all $z \in [0, \infty)$, $\int_0^z g(u)\, du \leqslant \int_0^{Az^r + B} (u/\phi(u))\, du$. (5.11)

To establish the existence of a solution to (5.9) we first consider for $m \in N^+$ the problems

$$y^{(n)} = \psi(t)\, f(t, y, y''), \qquad 0 < t < 1$$
$$y(0) = \frac{1}{m}, \qquad y'(1) = b, \qquad y''(0) = c, \tag{5.12^m}$$
$$y^{(j)}(1) = 0; \qquad j = 3, ..., n-1.$$

THEOREM 5.4. *Suppose* (2.2), (2.3), (2.4), (2.24), (5.10), *and* (5.11) *are satisfied. For* $\lambda \in [0, 1]$ *consider*

$$y^{(n)} = \lambda\psi(t)\, f(t, y, y''), \qquad 0 < t < 1$$
$$y(0) = \frac{1}{m}, \qquad y'(1) = b, \qquad y''(0) = c, \tag{5.13$^m_\lambda$}$$
$$y^{(j)}(1) = 0, \qquad j = 3, ..., n-1$$

for fixed $m \in N^+$. *Then there exist constants* M_i, $i = 0, ..., n$, *independent of* λ *such that*

$$\frac{1}{m} \leqslant y(t) \leqslant M_0; \qquad b \leqslant y'(t) \leqslant M_1; \qquad -M_2 \leqslant y''(t) \leqslant c;$$
$$0 \leqslant y^{(i)}(t) \leqslant M_i, \qquad i = 4, 6, ..., n-2;$$
$$-M_i \leqslant y^{(i)}(t) \leqslant 0, \qquad i = 3, 5, ..., n-1;$$

for $t \in [0, 1]$ *and*

$$0 \leqslant \frac{y^{(n)}(t)}{\psi(t)} \leqslant M_n; \qquad t \in (0, 1)$$

for each solution $y \in C^{n-1}[0, 1] \cap C^n(0, 1)$ *to* (5.13$^m_\lambda$).

Proof. Let $0 < \lambda \leqslant 1$. Now condition (2.2) implies $y > 0$ on $(0, 1)$ and as a result we have $y^{(i)} > 0$, $i = n, n-2, ..., 4$ and $y^{(i)} < 0$, $i = n-1, n-3, ..., 3$ on $(0, 1)$; thus $y'' < c$ is strictly decreasing on $(0, 1)$ which in turn implies $y' > b$ on $(0, 1)$ so $y > 1/m$ is strictly increasing on $(0, 1)$. In addition we have

$$\frac{(-y'')\, y^{(n)}}{\phi(-y'')} \leqslant \psi(t)\, g(y)(-y'')$$

so integration from t to 1 using assumptions (2.4) and (5.10) yields

$$\frac{[-y''(t)][-y^{(n-1)}(t)]}{\phi(-y''(t))}$$

$$\leqslant \frac{-y''(t)}{\phi(-y''(t))} \int_t^1 y^{(n)}(s)\,ds \leqslant \psi(t)\, g(y(t)) \int_t^1 [-y''(s)]\,ds$$

$$\leqslant \psi(t)\, g(y(t))[-b+y'(t)] \leqslant \psi(t)\, g(y(t))\, y'(t)$$

since g is nonincreasing on $(0, \infty)$. Thus integration from t to 1 yields

$$\frac{[-y''(t)][y^{(n-2)}(t)]}{\phi(-y''(t))} \leqslant \psi(t) \int_0^{y(1)} g(u)\,du.$$

Continuing this process we obtain in general

$$\frac{[-y''(t)][(-1)^j y^{(n-j)}(t)]}{\phi(-y''(t))} \leqslant K^* \int_0^{y(1)} g(u)\,du; \qquad j=3, ..., n-3,$$

where $K^* = \int_0^1 \psi(s)\,ds$. In particular we have

$$\frac{[-y''(t)][-y'''(t)]}{\phi(-y''(t))} \leqslant K^* \int_0^{y(1)} g(u)\,du,$$

so this together with (2.4) yields

$$\frac{[-y''(t)+c]}{\phi(-y''(t)+c)} [-y'''(t)] \leqslant K^* \int_0^{y(1)} g(u)\,du.$$

Integration from 0 to t with I as defined in Theorem 2.1 yields

$$-y''(t) \leqslant I^{-1}\left(K^* \int_0^{y(1)} g(u)\,du\right) - c, \qquad t \in [0, 1]. \tag{5.14}$$

Now integrate from t to 1 to obtain

$$y'(t) \leqslant I^{-1}\left(K^* \int_0^{y(1)} g(u)\,du\right) + b - c, \qquad t \in [0, 1] \tag{5.15}$$

and finally integration from 0 to 1 gives $y(1) \leqslant I^{-1}(K^* \int_0^{y(1)} g(u)\,du) + b + 1 - c$. Assumption (5.11) implies there exists a constant $M_0 > 0$ such that $y(1) \leqslant M_0$. In addition (5.14) and (5.15) yield M_2 and M_1, respectively.

Remark. Note M_0, M_1, and M_2 are independent of n.

The differential equation now yields $M_n, M_{n-1}, ..., M_3$. ∎

Essentially reasoning the same as that in Theorem 5.2 establishes

THEOREM 5.5. *Suppose* (2.2), (2.3), (2.4), (2.24), (5.10), *and* (5.11) *are satisfied. Then a* $C^{n-1}[0, 1] \cap C^n(0, 1)$ *solution of* (5.12^m) *exists.*

Moreover there exist constants M_0, M_1, and M_2 independent of m such that $1/m \leqslant |y|_0 \leqslant M_0$, $b \leqslant |y'|_0 \leqslant M_1$, $-c \leqslant |y''|_0 \leqslant M_2$ for each solution y to (5.12^m). In fact we claim that there is a constant M_3 independent of m such that $\| y''' \|_{L^2} \leqslant M_3$. To see this note $y^{(n)}(t) \leqslant g(y(t))\, \psi(t) \sup_{[-c, M_2]} \phi(q) = Eg(y(t))\, \psi(t)$, where $E = \sup_{[-c, M_2]} \phi(q)$. Integrate from t to 1 to obtain

$$-y^{(n-1)}(t) \leqslant E \int_t^1 g(y(t))\, \psi(s)\, ds$$

$$\leqslant Eg(y(t)) \int_t^1 \psi(s)\, ds \leqslant EK^* g(y(t))$$

and continuing this process we obtain in general

$$(-1)^j\, y^{(n-j)}(t) \leqslant EK^* g(y(t)), \qquad j = 1, ..., n-5.$$

In particular we have $-y^{(5)}(t) \leqslant EK^* g(y(t))$, $t \in [0, 1]$; so multiply by $-y''$ and integrate from t to 1 to obtain

$$-y''(t)\, y^{(n-1)}(t) + \frac{[y'''(s)]^2}{2}$$

$$\leqslant EK^* g(y(t)) \int_t^1 [-y''(s)]\, ds.$$

$$= EK^* g(y(t))[-b + y'(t)] \leqslant EK^* g(y(t))\, y'(t)$$

and so $-y''(t)\, y^{iv}(t) \leqslant EK^* g(y(t))\, y'(t)$. Now integrate from 0 to 1 to obtain

$$y''(0)\, y'''(0) + \int_0^1 [y'''(s)]^2\, ds \leqslant EK^* \int_0^{M_0} g(u)\, du$$

and since $y''(0)\, y'''(0) \geqslant 0$ our claim is established. Essentially reasoning the same as that in Theorem 3.3 establishes

THEOREM 5.6. *Suppose* (2.2), (2.3), (2.4), (2.13), (2.24), (5.10), *and* (5.11) *are satisfied. Then a* $C^2[0, 1] \cap C^{n-1}(0, 1] \cap C^n(0, 1)$ *solution of* (5.9) *exists.*

Remark. It should be noted here that the results of the above case could be extended to include equations of the form $y^{(n)} = \psi(t)\, f(t, y, y', y'')$, where f has a bounded dependence on its y' variable for any fixed values of the other arguments.

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