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On critical Fujita exponents for the porous medium equation with a nonlinear boundary condition ☆

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Abstract

We establish the critical Fujita exponents for the solution of the porous medium equation $u_t = \Delta u^m$, $x \in R_+^N$, $t > 0$, subject to the nonlinear boundary condition $-\partial u^m/\partial x_1 = u^p$, $x_1 = 0$, $t > 0$, in multi-dimension.

1. Introduction

In this paper we determine the critical Fujita exponent concerned with the following initial-boundary value problem:

$$u_t = \Delta u^m, \quad x \in R_+^N,\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in R_+^N, \tag{1.2}$$

$$-\frac{\partial u^m}{\partial x_1} = u^p, \quad x_1 = 0,\ t > 0, \tag{1.3}$$

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where $R_+^N = \{(x_1, x') \mid x' \in R^{N-1},\ x_1 > 0\}$, $m > 1,\ p > 0$, and $u_0(x)$ is a nonnegative bounded function satisfying the compatibility condition

$$-\frac{\partial u_0^m(x)}{\partial x_1} = u_0^p(x), \quad x_1 = 0, \tag{1.4}$$

and is locally supported near some point, namely, for some $x^0 \in R_+^N$, $\operatorname{supp} u_0 \subset B_R(x^0) \cap R_+^N$ and $u_0(x) \not\equiv 0$. However, the last assumption is not a real requirement for deriving our results. What we want to show is that even for the initial datum $u_0(x)$ vanishing except for a small ball, the solutions may still blow up in a finite time.

The concept of critical Fujita exponents was proposed by Fujita before 1970's in discussing the heat conduction equation with nonlinear source; see, for example, [1]. Following the idea of Fujita, we may define similar concepts for problem (1.1)–(1.3). We call p_0 the critical global existence exponent if it has the following property: if $p > p_0$, there always exist nonglobal solutions of problem (1.1)–(1.3) while if $0 < p < p_0$, every solution of problem (1.1)–(1.3) is global. p_c is called the critical Fujita exponent if for $p_0 < p < p_c$ any nontrivial solutions of problem (1.1)–(1.3) blows up in a finite time; for $p > p_c$ small data solutions exist globally in time while large data solutions are nonglobal.

The problem of determining critical Fujita exponent is an interesting one in the general theory of blowing-up solutions to different nonlinear evolution equations of mathematical physics. Over the past few years there have been a number of extensions of Fujita result in various directions; see [2–6]. Recently, it was Galaktionov and Levine [2] who first studied the one-dimensional case for the nonlinear boundary-value problem (1.1)–(1.3) with $u_0(x)$ having compact support. They showed that $p_0 = (m+1)/2$, $p_c = m+1$. As for the similar questions with positive initial data $u_0(x)$, we refer to [7–9].

This paper can be thought of as a natural continuation of [2] to multi-dimensional case. Because we are interested in the phenomena that local initial perturbation may cause blow up of solutions, we need not to consider the domain rather than the half-space R_+^N. In fact, for a general domain, we may localize and flatten the boundary, and then make small modification for the arguments presented in this paper.

The main result of this paper is that $p_0 = (m+1)/2$, $p_c = m + 1/N$. The idea of the proof is to construct super-solutions and sub-solutions inspired by [2]. However, the sub-solutions are quite different from those adopted in [2], since they should be chosen to have compact support in any spatial direction. For the process of verifying such kind of sub-solutions, we use the discriminant of cubic algebraic equations.

2. The main results and their proofs

We need the following simple lemma.

Lemma 2.1 (The discriminant of cubic algebraic equations). *For the cubic equation*

$$x^3 + px + q = 0, \tag{2.1}$$

there exist three roots

$$x_1 = A + B, \qquad x_{2,3} = -\frac{A+B}{2} \pm i\frac{A-B}{2}\sqrt{3},$$

where

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}}, \qquad B = \sqrt[3]{-\frac{q}{2} - \sqrt{Q}},$$

and $Q = (p/3)^3 + (q/2)^2$.

- *If* $Q = 0$, (2.1) *has three real roots and* $x_2 = x_3$;
- *If* $Q > 0$, (2.1) *has one real root and two conjugate complex roots*;
- *If* $Q < 0$, (2.1) *has three unequal real roots.*

Theorem 2.2. *If* $p > m + 1/N$, *then any nontrivial nonnegative solution of the problem* (1.1)–(1.3) *blows up in finite time for "large"* u_0.

Proof. We begin with the construction of a nonglobal sub-solution of the self-similar form

$$\underline{u}(x,t) = (T-t)^{-\frac{1}{2p-(m+1)}}\theta(\eta), \tag{2.2}$$

where $T > 0$ is a given constant,

$$\eta = |\zeta|, \quad \zeta = \frac{x}{(T-t)^{\frac{p-m}{2p-(m+1)}}}.$$

We can see that $\underline{u}(x,t)$ is a sub-solution of (1.1)–(1.3) if the function $\theta(\eta)$ satisfies

$$\frac{1}{\eta^{N-1}}\big(\eta^{N-1}(\theta^m)'\big)' - \frac{p-m}{2p-(m+1)}\eta\theta' - \frac{1}{2p-(m+1)}\theta \geqslant 0 \tag{2.3}$$

for $\eta \in \{\eta > 0 \mid \theta(\eta) > 0\}$ and

$$-\frac{\partial\theta^m}{\partial\zeta_1} \leqslant \theta^p, \quad \zeta_1 = 0. \tag{2.4}$$

We claim that (2.3) and (2.4) admits a solution of the form

$$\theta(\eta) = A(a-\eta)_+^{\frac{1}{m-1}}(\eta-b)_+^{\frac{1}{m-1}}, \quad 0 < b < \eta < a,$$

for some positive constants A, a, b specified later. First, such $\theta(\eta)$ satisfies (2.4), since $x_1 = 0$ implies $\zeta_1 = 0$ and

$$-\frac{\partial\theta^m}{\partial\zeta_1} = -\frac{\partial\theta^m}{\partial\eta}\frac{\partial\eta}{\partial\zeta_1} = -\frac{\partial\theta^m}{\partial\eta}\frac{\zeta_1}{\eta}$$

implies

$$-\frac{\partial\theta^m}{\partial\zeta_1}\bigg|_{\zeta_1=0} = 0 \leqslant \theta^p.$$

To check (2.1), by a direct calculation we get

$$\theta' = -A\frac{1}{m-1}(a-\eta)_+^{\frac{1}{m-1}-1}(\eta-b)_+^{\frac{1}{m-1}} + A\frac{1}{m-1}(a-\eta)_+^{\frac{1}{m-1}}(\eta-b)_+^{\frac{1}{m-1}-1},$$

$$
\begin{aligned}
(\theta^m)'' = A^m \frac{m}{m-1}\bigg[&\frac{1}{m-1}(a-\eta)_+^{\frac{1}{m-1}-1}(\eta-b)_+^{\frac{m}{m-1}} \\
&- \frac{m}{m-1}(a-\eta)_+^{\frac{1}{m-1}}(\eta-b)_+^{\frac{1}{m-1}} \\
&- \frac{m}{m-1}(a-\eta)_+^{\frac{1}{m-1}}(\eta-b)_+^{\frac{1}{m-1}} \\
&+ \frac{1}{m-1}(a-\eta)_+^{\frac{m}{m-1}}(\eta-b)_+^{\frac{1}{m-1}-1}\bigg].
\end{aligned}
$$

Substituting those into (2.3) and multiplying (2.3) by $(a-\eta)_+^{1-1/(m-1)}(\eta-b)^{1-1/(m-1)}$, we have

$$
\begin{aligned}
&\frac{A^m m}{(m-1)^2}(\eta-b)_+^2 - \frac{2A^m m^2}{(m-1)^2}(a-\eta)_+(\eta-b)_+ + \frac{A^m m}{(m-1)^2}(a-\eta)_+^2 \\
&\quad - \frac{N-1}{\eta}\frac{A^m m}{(m-1)}(a-\eta)_+(\eta-b)_+^2 + \frac{N-1}{\eta}\frac{A^m m}{(m-1)}(a-\eta)_+^2(\eta-b)_+ \\
&\quad - \frac{p-m}{2p-(m+1)}\eta\left[-\frac{A}{m-1}(\eta-b)_+ + \frac{A}{m-1}(a-\eta)_+\right] \\
&\quad - \frac{A}{2p-(m+1)}(a-\eta)_+(\eta-b)_+ \geqslant 0.
\end{aligned}
$$

Set

$$
\begin{aligned}
e_1 &= \frac{2A^m m}{(m-1)^2} + \frac{2A^m m^2}{(m-1)^2} + (N-1)\frac{2A^m m}{m-1} + \frac{A}{m-1}, \\
e_2 &= -\frac{2A^m m}{(m-1)^2}(a+b) - \frac{2A^m m^2}{(m-1)^2}(a+b) - (N-1)\frac{A^m m}{m-1}3(a+b) \\
&\quad - \frac{p-m}{2p-(m+1)}\frac{A}{m-1}(a+b) - \frac{A}{2p-(m+1)}(a+b), \\
e_3 &= \frac{A^m m}{(m-1)^2}(a^2+b^2) + \frac{2A^m m^2}{(m-1)^2}ab + \frac{A}{2p-(m+1)}ab \\
&\quad + (N-1)\frac{A^m m}{m-1}(a^2+b^2+4ab), \\
e_4 &= -(N-1)\frac{A^m m}{m-1}ab(a+b).
\end{aligned}
$$

We observe that (2.3) holds if

$$
e_1\eta^3 + e_2\eta^2 + e_3\eta + e_4 \geqslant 0, \quad 0 < b < \eta < a. \tag{2.5}
$$

If we choose A large enough, it reduces to show

$$
\tilde{e}_1\eta^3 + \tilde{e}_2\eta^2 + \tilde{e}_3\eta + \tilde{e}_4 \geqslant 0, \quad 0 < b < \eta < a, \tag{2.6}
$$

where

$$\tilde{e}_1 = \frac{2+2m}{m-1} + 2(N-1),$$
$$\tilde{e}_2 = -\frac{2+2m}{m-1}(a+b) - 3(N-1)(a+b),$$
$$\tilde{e}_3 = \frac{1}{m-1}(a^2+b^2) + \frac{2m}{m-1}ab + (N-1)(a^2+b^2+4ab),$$
$$\tilde{e}_4 = -(N-1)ab(a+b).$$

To do this, set

$$y(\eta) = \tilde{e}_1\eta^3 + \tilde{e}_2\eta^2 + \tilde{e}_3\eta + \tilde{e}_4.$$

Notice that $\tilde{e}_1 > 0$ implies $\lim_{\eta\to+\infty} y(\eta) = +\infty$.

Letting $a = cb$, $c > 1$, we want to show that if $c \to 1^+$, (2.6) holds. From Lemma 2.1 we know that when $c \to 1^+$, the equation $y(\eta) = 0$ has one real root and two conjugate complex roots. We only need to show that if $c \to 1^+$, $y(b) \geqslant 0$, namely,

$$\begin{aligned} y(b) &= \left[\frac{2+2m}{m-1} + 2(N-1)\right]b^3 + \left[-\frac{(2+2m)(1+c)}{m-1} - 3(N-1)(1+c)\right]b^3 \\ &\quad + \left[\frac{1}{m-1}(1+c^2) + (N-1)(c^2+1+4c) + \frac{2m}{m-1}c\right]b^3 \\ &\quad - \left[(N-1)c(c+1)\right]b^3 \geqslant 0. \end{aligned}$$

Let

$$\begin{aligned} g(c) &= \left[\frac{2+2m}{m-1} + 2(N-1)\right] + \left[-\frac{(2+2m)(1+c)}{m-1} - 3(N-1)(1+c)\right] \\ &\quad + \left[\frac{1}{m-1}(1+c^2) + (N-1)(c^2+1+4c) + \frac{2m}{m-1}c\right] \\ &\quad - \left[(N-1)c(c+1)\right]; \end{aligned}$$

then $y(b) = g(c)b^3$, where $g(c) = (1-c)^2/(m-1) \geqslant 0$. So $y(b) \geqslant 0$.

Thus we have verified that $\underline{u}(x,t)$ is a weak sub-solution of (1.1)–(1.3) and that $\underline{u}(x,t)$ blows up in a finite time.

If for any given T, A, a, b, and c satisfying (2.3) and (2.4), the initial function u_0 is large enough such that

$$u_0(x) \geqslant \underline{u}(x,0), \quad x \in R_+^N,$$

then from the comparison principle (see [10]), $u(x,t) \geqslant \underline{u}(x,t)$ in $R_+^N \times (0,T)$ and hence $u(x,t)$ blows up in a finite time which is not larger than T. The proof is complete. □

Theorem 2.3. *If* $(m+1)/2 < p < m + 1/N$, *then any nontrivial nonnegative solution of problem* (1.1)–(1.3) *blows up in finite time.*

Proof. We now use the idea used for a different problem in [11]. We first notice that (1.1)–(1.3) admits the following well-known self-similar solution

$$u_B(x,t) = (\tau+t)^{-\frac{N}{N(m-1)+2}}\theta(\eta), \tag{2.7}$$

where

$$\eta = |\zeta|, \quad \zeta = \frac{x}{(\tau + t)^{\frac{1}{N(m-1)+2}}},$$

with $\tau > 0$ an arbitrary constant. We can see that $\underline{u}(x,t)$ is a sub-solution of (1.1)–(1.3) if $\theta(\eta)$ satisfies

$$\frac{1}{\eta^{N-1}}\big(\eta^{N-1}(\theta^m)'\big)' + \frac{1}{N(m-1)+2}\eta\theta' + \frac{N}{N(m-1)+2}\theta = 0 \tag{2.8}$$

and

$$\frac{\partial \theta^m}{\partial \zeta_1}\bigg|_{\zeta_1=0} = 0. \tag{2.9}$$

Here $\eta \in \{\eta > 0 \mid \theta(\eta) \geqslant 0\}$.

By a simple calculation, we see that $\theta(\eta) = A(c^2 - \eta^2)_+^{1/(m-1)}$, $0 < \eta < c$, satisfies (2.8) and (2.9), where

$$A = \left\{\frac{m-1}{2m[N(m-1)+2]}\right\}^{\frac{1}{m-1}}.$$

Thus u_B is a sub-solution to problem (1.1)–(1.3).

By using the properties of weak solutions of problem (1.1)–(1.3), we deduce that there exist $t_0 \geqslant 0$ such that

$$u(0, t_0) > 0.$$

Since $u(x,t_0)$ is a continuous function, there exist $\tau > 0$ large enough and small $c > 0$ such that

$$u(x,t_0) \geqslant u_B(x,t_0), \quad x \in R_+^N.$$

Then by comparison principle we deduce that

$$u(x,t) \geqslant u_B(x,t), \quad t \geqslant t_0,\ x \in R_+^N. \tag{2.10}$$

We now prove that there exist $t_* \geqslant t_0$ and T large enough so that

$$u_B(x,t_*) \geqslant \underline{u}(x,0), \quad x \in R_+^N, \tag{2.11}$$

where $\underline{u}(x,t)$ is the sub-solution given by (2.2). By using the space–time structure of both functions u_B and $\underline{u}$, we choose suitable constants a, b such that $0 < a - b < 1$. If

$$(\tau + t_*)^{-\frac{N}{N(m-1)+2}} \gg T^{-\frac{1}{2p-(m+1)}} \tag{2.12}$$

and

$$(\tau + t_*)^{\frac{N}{N(m-1)+2}} \gg T^{\frac{p-m}{2p-(m+1)}} \tag{2.13}$$

are satisfied, (2.11) is valid. We can see from (2.12) and (2.13) that such t_* and T exist if

$$T^{\frac{1}{2p-(m+1)}} \gg T^{\frac{N(p-m)}{2p-(m+1)}}$$

for arbitrarily large T. This implies that

$$\frac{1}{2p-(m+1)} > \frac{N(p-m)}{2p-(m+1)},$$

namely,

$$p < m + \frac{1}{N}.$$

Hence, from (2.10) and (2.11), using the comparison principle we have that if $p_0 < p < p_c$, $u(x,t)$ blows up in a finite time. The proof is complete. □

Theorem 2.4. *If $p > m + 1/N$, then any nontrivial nonnegative solution of problem* (1.1)–(1.3) *is global in time for "small" u_0.*

Proof. We shall seek a global super-solution of the self-similar form

$$\bar{u}(x,t) = (T+t)^{-\frac{1}{2p-(m+1)}} Bg(\eta), \tag{2.14}$$

where

$$\eta = |\zeta|, \qquad \zeta_1 = \frac{x_1 + b}{(T+t)^{\frac{p-m}{2p-(m+1)}}}, \qquad \zeta_i = \frac{x_i}{(T+t)^{\frac{p-m}{2p-(m+1)}}} \quad (i = 2, \ldots, N),$$

$T > 0$ is a given positive constant. We can see that $\bar{u}(x,t)$ is a super-solution of (1.1)–(1.3) if $g(\eta) \geqslant 0$ satisfies

$$\frac{1}{\eta^{N-1}}\bigl(\eta^{N-1}(g^m)'\bigr)' + \frac{p-m}{2p-(m+1)}\eta g' + \frac{1}{2p-(m+1)} g \leqslant 0 \tag{2.15}$$

and

$$-B^m \frac{\partial g^m}{\partial \zeta_1}\bigg|_{\zeta_1 = b} \geqslant B^p g^p. \tag{2.16}$$

Here $\zeta \in \{\eta > 0 \mid g(\eta) \geqslant 0\}$, $B > 0$.

Now we show that $g(\eta) = A(c^2 - \eta^2)_+^{1/(m-1)}$, where

$$A = \left\{\frac{m-1}{2m[N(m-1)+2]}\right\}^{\frac{1}{m-1}},$$

satisfies (2.15) and (2.16), where $b \in (0,c)$. Using

$$\frac{1}{\eta^{N-1}}\bigl(\eta^{N-1}(g^m)'\bigr)' = -\frac{1}{N(m-1)+2}\eta g' - \frac{N}{N(m-1)+2} g,$$

we see that g satisfies

$$B^{m-1}\left\{-\frac{1}{N(m-1)+2}\eta g' - \frac{N}{N(m-1)+2} g\right\} + \frac{p-m}{2p-(m+1)}\eta g' + \frac{1}{2p-(m+1)} g \leqslant 0,$$

namely,

$$\left\{\frac{2}{m-1}\left[\frac{B^{m-1}}{N(m-1)+2}-\frac{p-m}{2p-(m+1)}\right]\right.$$
$$\left.-\left[\frac{1}{2p-(m+1)}-\frac{NB^{m-1}}{N(m-1)+2}\right]\right\}\eta^2$$
$$+\left[\frac{1}{2p-(m+1)}-\frac{NB^{m-1}}{N(m-1)+2}\right]c^2\leqslant 0.$$

Since $p > m + 1/N$, we can choose a suitable constant B such that

$$\frac{B^{m-1}}{N(m-1)+2}<\frac{p-m}{2p-(m+1)},\qquad \frac{1}{2p-(m+1)}<\frac{NB^{m-1}}{N(m-1)+2}.$$

Thus (2.15) is valid. Finally, we notice that, for g, inequality (2.16) is equivalent to

$$(BA)^{p-m}(c^2-\eta^2)^{\frac{p-1}{m-1}}\leqslant\frac{2mb}{m-1},\quad b<\eta<c. \tag{2.17}$$

If the inequality

$$(BA)^{p-m}(c^2-b^2)^{\frac{p-1}{m-1}}\leqslant\frac{2mb}{m-1},\quad b<\eta<c,$$

holds, (2.17) is true. Setting $c = ab$, $\alpha \geqslant 1$, by choosing B small enough, we have

$$(BA)^{p-m}(\alpha^2-1)^{\frac{p-1}{m-1}}b^{\frac{2p-(m+1)}{m-1}}\leqslant\frac{2m}{m-1},$$

which implies that (2.17) is valid.

Thus, for $p > p_c$, there exists a nontrivial global super-solution, and hence a class of small global solutions. We have thus completed the proof. □

Theorem 2.5. *If* $0 < p < p_0$, *then any nontrivial nonnegative solution of problem* (1.1)–(1.3) *is global in time.*

Proof. If $p = p_0 = (m + 1)/2$, we can construct a global super-solution of the self-similar form

$$u^*(x_1,x',t)=u^*(x_1,0,t)=e^{\alpha(T+t)}h(\zeta),$$

where $\zeta = x_1/e^{\alpha(T+t)/2}$, $T > 0$ is a given positive constant, and $\alpha > 0$. We can see that $u^*(x,t)$ is a solution of (1.1)–(1.3) if $h(\zeta)$ satisfies

$$(h^m)''(\zeta)+\frac{\alpha(m-1)}{2}\zeta h'(\zeta)-\alpha h(\zeta)=0 \tag{2.18}$$

and

$$-(h^m)'(0)=h^{p_0}(0). \tag{2.19}$$

From [12] we see that there exists a unique solution $h \not\equiv 0$ which has compact support on x_1 such that (2.18) and (2.19) hold. Thus, we can choose T large enough such that

$$u_0(x)\leqslant u^*(x,0),\quad x\in R^N_+.$$

Using comparison principle, we have

$$u(x,t) \leqslant u^*(x,t).$$

From the global existence of $u^*(x,t)$, we see that $u(x,t)$ is also global in time.

If $p < p_0$, $u^*(x,t)$ is a global super-solution of (1.1)–(1.3) whenever $u^*(0,t) \geqslant 1$. Hence, using comparison principle again, we can get the global existence of $u(x,t)$. The proof is complete. □

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