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# Generalizations of Eulerian partially ordered sets, flag numbers, and the Möbius function

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## Abstract

A partially ordered set is $r$-*thick* if every nonempty open interval contains at least $r$ elements. This paper studies the flag vectors of graded, $r$-thick posets and shows the smallest convex cone containing them is isomorphic to the cone of flag vectors of all graded posets. It also defines a $k$-analogue of the Möbius function and $k$-Eulerian posets, which are $2k$-thick. Several characterizations of $k$-Eulerian posets are given. The generalized Dehn–Sommerville equations are proved for flag vectors of $k$-Eulerian posets. A new inequality is proved to be valid and sharp for rank 8 Eulerian posets.

## Resume

Un ensemble partiellement ordonné est $r$-*épais* si chacun de ses intervalles ouverts non-vides contient au moins $r$ éléments. Dans cet article nous étudions les vecteurs drapeaux des ensembles partiellement ordonnés gradués $r$-épais. Nous démontrons que le cône le plus petit contenant ces vecteurs est isomorphe au cône des vecteurs drapeaux des ensembles partiellement ordonnés gradués quelconques. Nous définissons aussi un $k$-analogue de la fonction de Möbius et des ensembles partiellement ordonnés $k$-eulériens qui sont $2k$-épais. Nous caractérisons les ensembles partiellement ordonnés $k$-eulériens de plusieurs manières, et généralisons les équations de Dehn–Sommerville pour le vecteur drapeaux d'un ensemble partiellement ordonné $k$-eulérien. Nous démontrons une nouvelle inégalité optimale pour les ensembles partiellement ordonnés eulériens de rang 8. 

*Keywords:* Eulerian poset; Flag vector; Möbius function; Dehn–Sommerville equations

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## 1. Introduction

In this paper we study certain classes of graded partially ordered sets (posets), defined by conditions on the sizes of rank sets in intervals. We are concerned with numerical parameters of the posets, in particular, flag vectors and the Möbius function.

A *graded poset* $P$ is a finite partially ordered set with a unique minimum element $\hat{0}$, a unique maximum element $\hat{1}$, and a *rank function* $\rho : P \to \mathbf{N}$ satisfying $\rho(\hat{0}) = 0$, and $\rho(y) - \rho(x) = 1$, whenever $y \in P$ covers $x \in P$. The *rank* $\rho(P)$ *of a graded poset* $P$ is the rank of its maximum element. Given a graded poset $P$ of rank $n+1$ and a subset $S$ of $\{1, 2, \ldots, n\}$ (which we abbreviate as $[1,n]$), define the *S-rank-selected subposet of* $P$ to be the poset

$$P_S = \{x \in P:\ \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

Denote by $f_S(P)$ the number of maximal chains of $P_S$. Equivalently, $f_S(P)$ is the number of chains $x_1 < \cdots < x_{|S|}$ in $P$ such that $\{\rho(x_1), \ldots, \rho(x_{|S|})\} = S$. (Call such a chain an *S-chain* of $P$.) The vector $(f_S(P):\ S \subseteq [1,n])$ is called the *flag f-vector* of $P$. Whenever it does not cause confusion, we write $f_{s_1 \ldots s_j}$ rather than $f_{\{s_1,\ldots,s_j\}}$; in particular, $f_{\{i\}}$ is always denoted $f_i$.

In the last 20 years there has grown a body of work on numerical conditions on flag vectors of posets and complexes, especially those arising in geometric contexts. A major recent contribution is the determination of the closed cone of flag vectors of all graded posets by Billera and Hetyei [5]. In [3] the authors study the closed cone of flag vectors of *Eulerian* posets. These are graded posets for which every (closed) interval has the same number of elements of even rank and of odd rank.

A poset is *r-thick* if every nonempty open interval has at least $r$ elements. Thus, every poset is 1-thick, and Eulerian posets are 2-thick. In the first part of this paper we show that the closed cone of flag vectors of $r$-thick posets is linearly equivalent to the Billera–Hetyei cone, the closed cone of flag vectors of all graded posets.

The second part of the paper defines a $k$-analogue of the Möbius function and $k$-Eulerian posets (which are $2k$-thick). We show that the generalized Dehn–Sommerville equations of [1] transfer to $k$-Eulerian posets. These equations have a particularly nice representation in terms of the $L^k$-vector, introduced here as a relative of the *cd*-index. The results of this paper can be used to find inequalities valid for flag vectors of Eulerian posets. In the last section we give as an example a new, sharp inequality for rank 8 Eulerian posets.

# Part I: *r*-thick posets

## 2. Flag vectors of arbitrary graded posets

We describe first the cone of flag vectors of all graded posets. This is due to Billera and Hetyei [5].

An *interval system on* $[1,n]$ is any set of subintervals of $[1,n]$ that form an antichain (that is, no interval is contained in another). A set $S \subseteq [1,n]$ *blocks* the interval system $\mathscr{I}$ if it has a nonempty intersection with every $I\in\mathscr{I}$. The family of all subsets of $[1,n]$ blocking $\mathscr{I}$ is denoted by $\mathbf{B}_{[1,n]}(\mathscr{I})$. The main result of [5] is the following.

**Theorem 2.1** (Billera and Hetyei [5]). *An expression $\sum_{S\subseteq[1,n]} a_S f_S(P)$ is nonnegative for all graded posets $P$ of rank $n+1$ if and only if*

$$\sum_{S\in\mathbf{B}_{[1,n]}(\mathscr{I})} a_S \geqslant 0 \quad \text{for every interval system } \mathscr{I} \text{ on } [1,n]. \tag{1}$$

Here is an outline of the proof from [5]. The proof of the necessity of the condition (1) involves constructing for every interval system $\mathscr{I}$ on $\{1,2,\ldots,n\}$ a family of posets $\{P(n,\mathscr{I},N)\colon\ N\in\mathbf{N}\}$ of rank $n+1$ such that

$$\lim_{N\to\infty}\frac{1}{f_{[1,n]}(P(n,\mathscr{I},N))}\sum_{S\subseteq[1,n]} a_S f_S(P(n,\mathscr{I},N)))=\sum_{S\in\mathbf{B}_{[1,n]}(\mathscr{I})} a_S.$$

For the other implication, let $P$ be an arbitrary graded poset, and assume that its Hasse-diagram is drawn in the plane. Given an interval $[x,y]$ of $P$, let $\phi(x,y)$ denote the leftmost atom in $[x,y]$. (If $y$ covers $x$ then set $\phi(x,y)=y$.) The operation $\phi$ has the following crucial property:

$$\text{if } p\in[x,y]\subseteq[x,z] \text{ and } p=\phi([x,z]) \text{ then } p=\phi([x,y]). \tag{2}$$

For every $S\subseteq[1,n]$ and $i\in[1,n]$ define $M_S(i)$ to be the smallest $j\in[i,n+1]$ such that $j\in S\cup\{n+1\}$. Consider the set of maximal chains

$$F_S=\{\hat{0}=p_0<p_1<\cdots<p_n<p_{n+1}=\hat{1}\colon\ \forall i\in[1,n],\, p_i=\phi([p_{i-1},p_{M_S(i)}])\}.$$

It is easy to verify that $F_S$ contains exactly $f_S(P)$ elements. Moreover, there is a way of associating a family of intervals $\mathscr{I}_C$ to every maximal chain $C=\{\hat{0}=p_0<p_1<\cdots<p_n<p_{n+1}=\hat{1}\}$ such that $C$ belongs to $F_S$ if and only if $S$ blocks $\mathscr{I}_C$. The fact that one may find such a family of intervals is a direct consequence of property (2).

## 3. Flag vectors of $r$-thick posets

It is easy to expand any graded poset to obtain an $r$-thick poset. Let $P$ be a graded poset of rank $n+1$. Write $D^rP$ for the poset obtained from $P$ by replacing every $x\in P\backslash\{\hat{0},\hat{1}\}$ with $r$ elements $x_1,x_2,\ldots,x_r$, such that $\hat{0}$ and $\hat{1}$ remain the minimum and maximum elements of the partially ordered set, and $x_i<y_j$ if and only if $x<y$ in $P$. The poset $D^rP$ is an $r$-thick graded poset of rank $n+1$. Clearly $f_S(D^rP)=r^{|S|}f_S(P)$.

**Theorem 3.1.** *For every positive integer $r$, $\sum_{S\subseteq[1,n]} a_S f_S(P)\geqslant 0$ for every graded poset $P$ of rank $n+1$ if and only if $\sum_{S\subseteq[1,n]} a_S r^{n-|S|} f_S(Q)\geqslant 0$ for every $r$-thick poset $Q$ of rank $n+1$.*

**Proof.** First assume $\sum_{S\subseteq[1,n]} a_S r^{n-|S|} f_S(Q) \geqslant 0$ for every $r$-thick poset $Q$ of rank $n+1$. Let $P$ be any graded poset of rank $n+1$. Since $D^r P$ is $r$-thick,

$$\begin{aligned} 0 &\leqslant \sum_{S\subseteq[1,n]} a_S r^{n-|S|} f_S(D^r P) \\ &= \sum_{S\subseteq[1,n]} a_S r^{n-|S|} r^{|S|} f_S(P) \\ &= \sum_{S\subseteq[1,n]} a_S r^n f_S(P). \end{aligned}$$

Dividing by $r^n$ gives the desired inequality for all graded posets.

Now assume $\sum_{S\subseteq[1,n]} a_S f_S(P) \geqslant 0$ for every graded poset $P$ of rank $n+1$. Let $Q$ be an $r$-thick poset of rank $n+1$. For each rank $i$, fix a total order of the elements of $Q$ of rank $i$. Given an interval $[x,y]$ of $Q$ of rank at least 2, let $\phi(x,y)$ denote the set of the first $r$ atoms in $[x,y]$. (If $y$ covers $x$, set $\phi(x,y)=\{y\}$.)

The operation $\phi$ satisfies the following:

$$\text{if } p\in[x,y]\subseteq[x,z] \text{ and } p\in\phi([x,z]) \text{ then } p\in\phi([x,y]). \tag{3}$$

Let

$$F_S = \{\hat{0} = p_0 < p_1 < \cdots < p_n < p_{n+1} = \hat{1}\colon\ \forall i\in[1,n],\ \ p_i\in\phi([p_{i-1}, p_{M_S(i)}])\}.$$

How many sequences are in the set $F_S$? Given any $S$-chain of $Q$, extend it to sequences in $F_S$ one rank at a time. Having fixed $p_0$ through $p_{i-1}$ $(1\leqslant i\leqslant n)$, if $i\notin S$, then there are exactly $r$ choices for $p_i$. Thus $|F_S| = r^{n-|S|} f_S(Q)$.

To each maximal chain $C$: $\hat{0} = p_0 < p_1 < \cdots < p_n < p_{n+1} = \hat{1}$ of $Q$ is assigned an interval system as follows. For $1\leqslant i\leqslant n$, let $\psi(C,i)$ be the largest $j$ such that $p_i\in\phi(p_{i-1}, p_j)$. Let $\mathscr{I}'_C = \{[i,\psi(C,i)]\colon\ 1\leqslant i\leqslant n,\ \ \psi(C,i) \neq n+1\}$, and let $\mathscr{I}_C$ be the interval system consisting of minimal intervals in $\mathscr{I}'_C$. We show $C$ belongs to $F_S$ if and only if $S$ blocks $\mathscr{I}_C$. Suppose $C$: $\hat{0} = p_0 < p_1 < \cdots < p_n < p_{n+1} = \hat{1}$ is in $F_S$. Then for all $i$, $p_i\in\phi([p_{i-1}, p_{M_S(i)}])$, so by the maximality of $\psi(C,i)$, $\psi(C,i)\geqslant M_S(i)$. So for all $i$ the interval $[i,\psi(C,i)]$ contains the element $M_{S(i)}$ of $S$. Thus, $S$ blocks $\mathscr{I}_C$. For the reverse implication, suppose, $C$ is a maximal chain of $Q$ and $S$ blocks $\mathscr{I}_C$. Let $1\leqslant i\leqslant n$ and $[i,\psi(C,i)]\in\mathscr{I}_C$. Since $S$ blocks $\mathscr{I}_C$, $S\cap[i,\psi(C,i)]$ contains an element $s$. So $M_{S(i)}\leqslant s\leqslant\psi(C,i)$. Apply condition (3): $p_i\in[p_{i-1}, p_{M_S(i)}]\subseteq[p_{i-1}, p_{\psi(C,i)}]$ and $p_i\in\phi([p_{i-1}, p_{\psi(C,i)}])$, so $p_i\in\phi([p_{i-1}, p_{M_{S(i)}}])$. Thus, $C$ is in $F_S$.

Given a system of intervals $\mathscr{I}$ denote by $f_{\mathscr{I}}$ the number of those maximal chains $C$ of $Q$ for which $\mathscr{I}_C = \mathscr{I}$. (Note that $f_{\mathscr{I}}$ depends not only on $Q$ but also on the ordering of the elements of each rank.) Then

$$\begin{aligned} \sum_{S\subseteq[1,n]} a_S r^{n-|S|} f_S(Q) &= \sum_{S\subseteq[1,n]} a_S |F_S| = \sum_{S\subseteq[1,n]} a_S \sum_{S\in\mathbf{B}_{[1,n]}(\mathscr{I})} f_{\mathscr{I}} \\ &= \sum_{\mathscr{I}} f_{\mathscr{I}} \sum_{S\in\mathbf{B}_{[1,n]}(\mathscr{I})} a_S. \end{aligned}$$

By Theorem 2.1 the sums $\sum_{S\in \mathbf{B}_{[1,n]}(\mathscr{I})} a_S$ are all nonnegative, and so

$$\sum_{S\subseteq[1,n]} a_S r^{n-|S|} f_S(Q)\geqslant 0. \qquad \square$$

Let $\mathscr{C}_{r,n+1}$ be the smallest closed convex cone containing the flag vectors of all $r$-thick posets of rank $n+1$.

**Corollary 3.2.** *For all positive integers $q$ and $r$, the invertible linear transformation $\alpha_{q,r}:\mathbf{Q}^{2^n}\to\mathbf{Q}^{2^n}$ defined by $\alpha_{q,r}((x_S))=((r/q)^{|S|}x_S)$ maps $\mathscr{C}_{q,n+1}$ onto $\mathscr{C}_{r,n+1}$.*

To determine if a graded poset is $r$-thick, it is enough to check that between every $x$ and $y$ with $x<y$ and $\rho(y)-\rho(x)=2$, there are at least $r$ elements. The definition of $r$-thick posets can then be generalized by allowing the lower bound $r$ to vary through the levels of the poset. The results of this section have straightforward analogues in that context.

# Part II: $k$-Eulerian posets

## 4. The $k$-Möbius function

**Definition 4.1.** The *Möbius function* of a graded poset $P$ is defined recursively for any subinterval of $P$ by the formula

$$\mu([x,y])=\begin{cases}1 & \text{if } x=y,\\ -\displaystyle\sum_{x\leqslant z<y}\mu([x,z]) & \text{otherwise.}\end{cases}$$

A graded poset $P$ is *Eulerian* if the Möbius function of every interval $[x,y]$ is given by $\mu([x,y])=(-1)^{\rho(x,y)}$.

(Here, $\rho(x,y)=\rho([x,y])=\rho(y)-\rho(x)$.)

See [9] for a survey of Eulerian posets. The first characterization of all linear equalities holding for the flag vectors of all Eulerian posets was given by Bayer and Billera in [1].

**Theorem 4.2** (Bayer and Billera [1]). *For every Eulerian poset of rank $n+1$, every subset $S\subseteq[1,n]$, and every maximal interval $[i,\ell]$ of $[1,n]\backslash S$,*

$$((-1)^{i-1}+(-1)^{\ell+1})f_S(P)+\sum_{j=i}^{\ell}(-1)^j f_{S\cup\{j\}}(P)=0.$$

*Furthermore, every linear equality holding for the flag vector of all Eulerian posets of rank $n+1$ is a consequence of these equations.*

Next, we present generalizations of the Möbius function and of Eulerian posets.

**Definition 4.3.** The *k-Möbius function* of a graded poset is defined recursively by the formula

$$\mu_k([x,y]) = \begin{cases} 1 & \text{if } x = y, \\ -1 - \frac{1}{k} \sum_{x<z<y} \mu_k([x,z]) & \text{otherwise.} \end{cases}$$

The following proposition gives the $k$-Möbius function of a poset $P$ as a $k$-analogue of the reduced Euler characteristic of the order complex of $P$. It is a generalization of Philip Hall's theorem, and is easy to prove by induction.

**Proposition 4.4.** *If $P$ is a graded poset of rank $n+1$, then*

$$\mu_k(P) = -\sum_{S\subseteq[1,n]} \left(-\frac{1}{k}\right)^{|S|} f_S^{n+1}(P).$$

A graded poset is *$k$-Eulerian* if for every interval $[x,y]\subseteq P$, $\mu_k([x,y]) = (-1)^{\rho(x,y)}$. Note that 1-Eulerian is the same as Eulerian. The following proposition follows easily from the definitions.

**Proposition 4.5.** *If $P$ is a $k$-Eulerian poset of rank $n+1$, then*
(1) *every interval of $P$ is $k$-Eulerian,*
(2) $\sum_{i=1}^{n} (-1)^{i-1} f_i(P) = k(1-(-1)^n)$.

The thickening operation introduced in Section 3 connects the $k$-Möbius function for different values of $k$.

**Proposition 4.6.** *Let $[x,y]$ be an interval of a graded poset $P$ and $\ell$ a positive integer. Consider an interval $[x_i,y_j]\subseteq D^\ell P$ corresponding to $[x,y]\subseteq P$. Then*

$$\mu_k([x,y]) = \mu_{k\ell}([x_i,y_j]).$$

**Proof.** Recall that $f_S(D^\ell P) = \ell^{|S|} f_S(P)$. Since the interval $[x_i,y_j]$ of $D^\ell P$ is isomorphic to $D^\ell[x,y]$, the result is obtained by substitution in Proposition 4.4. □

**Corollary 4.7.** *A poset $P$ is $k$-Eulerian if and only if $D^\ell P$ is $k\ell$-Eulerian.*

In [3] a half-Eulerian poset was defined to be a poset $P$ for which $D^2P$ is Eulerian.

Using Proposition 4.6 we can determine exactly the set of those $k$'s for which $k$-Eulerian posets exist.

**Theorem 4.8.** *For every positive integer $n$, there exists a $k$-Eulerian poset of rank $n+1$ if and only if $k = j/2$ for some positive integer $j$. Moreover, every $k$-Eulerian poset is $2k$-thick.*

**Proof.** The chain $C$ of rank $n+1$ is half-Eulerian. For every positive integer $j$, $D^jC$ is a $j/2$-Eulerian poset. On the other hand, by the definition of the function $\mu_k$, for an interval $[x,y]$ of rank 2 in a $k$-Eulerian poset,

$$(-1)^2 = \mu_k([x,y]) = -1 - 1/k \sum_{x<z<y} \mu_k([x,z]) = -1 - \frac{1}{k}\sum_{x<z<y}(-1).$$

Therefore, $2k$ is the number of elements $z$ strictly between $x$ and $y$. Thus, if $P$ is a $k$-Eulerian poset, then $2k$ is a positive integer, and $P$ is $2k$-thick. □

It is easy to check by induction that a graded poset is half-Eulerian if and only if (1) in every interval $[x,y]$ with $\rho(x,y)$ odd, the number of elements of even rank equals the number of elements of odd rank; and (2) in every interval $[x,y]$ with $\rho(x,y)$ even, the number of elements $z$ with $\rho(x,z)$ even is one more than the number of elements $z$ with $\rho(x,z)$ odd. This characterization can be used to check that the following "vertical doubling" of an arbitrary graded poset produces a half-Eulerian poset. Let $P$ be any graded poset with relation $\prec_P$. Form the set $Q=\{\hat{0},\hat{1}\}\cup\{x_1,x_2\colon\ x\in P\backslash\{\hat{0},\hat{1}\}\}$. Define a relation $\prec_Q$ on $Q$ by $u\prec_Q v$ if and only if one of the following holds:

- $u=\hat{0},\ v\in P\backslash\{\hat{0}\}$,
- $v=\hat{1},\ u\in P\backslash\{\hat{1}\}$,
- $u=x_1$ and $v=x_2$ for some $x\in P\backslash\{\hat{0},\hat{1}\}$,
- $u=x_i$ and $v=y_j$ for some $x,y\in P\backslash\{\hat{0},\hat{1}\}$, with $x\prec_P y$.

If $P$ is a rank $n+1$ graded poset, then the resulting poset $Q$ is a rank $2n+1$ half-Eulerian poset.

For larger $k$, not all $k$-Eulerian posets are obtained by the thickening operation. For an example, consider the poset $P$ of rank $n+1\geqslant 3$ having elements $x_1,x_2,\dots,x_m$ of rank 1, elements $y_1,y_2,\dots,y_m$ of rank 2, with $x_i<y_j$ if and only if $i=j$, and one element of each other rank. It is easy to check that $P$ is half-Eulerian, and so $D^{2k}P$ is $k$-Eulerian. In the Hasse diagram of $D^{2k}P$, the subgraph induced by the elements of ranks 1 and 2 consists of $m$ copies of the complete bipartite graph $K_{2k,2k}$. Replace this subgraph by any other $2k$-regular bipartite graph on these elements. The resulting graph is the Hasse diagram of another $k$-Eulerian poset. (Note that the only relations changed in the poset are those between ranks 1 and 2 elements.)

The definition of $k$-Eulerian, like that of $r$-thick, can be generalized by varying the multiplier $k$ with the rank of the elements. The results of this and the next section can easily be adapted for such posets.

## 5. The flag $L^k$-vector

A certain transformation of the flag $f$-vector was useful in [3,5,8]. It has a natural adaptation to the $k$-Eulerian setting.

**Definition 5.1.** The *flag $L^k$-vector* of a graded partially ordered set $P$ of rank $n+1$ is the vector $(L_S^{k,n+1}(P):\ S\subseteq[1,n])$, where

$$L_S^{k,n+1}(P)=(-1)^{n-|S|}\sum_{T\supseteq[1,n]\setminus S}\left(-\frac{1}{2k}\right)^{|T|} f_T^{n+1}(P).$$

For $k=\frac{1}{2}$ this is the $\ell$-vector of [5]; for $k=1$ this is the "*ce*-index" of [8] and the $L$-vector of [3]. The formula inverts to give

$$f_S^{n+1}(P)=(2k)^{|S|}\sum_{T\subseteq[1,n]\setminus S} L_T^{k,n+1}(P). \tag{4}$$

The $L^k$-vector ignores the effect of the operator $D^\ell$. If $P$ is a graded poset of rank $n+1$, then

$$L_S^{k\ell,n+1}(D^\ell P)=L_S^{k,n+1}(P). \tag{5}$$

A set $S\subseteq[1,n]$ is *even* if $S$ is a disjoint union of intervals of even cardinality. The parameters $L_S^{k,n+1}$ for even sets $S$ play a special role for $k$-Eulerian posets. The $k$-analogue of Theorem 4.2 is the following.

**Theorem 5.2.** *For every $k$-Eulerian poset $P$ of rank $n+1$, every subset $S\subseteq[1,n]$, and every maximal interval $[i,\ell]$ of $[1,n]\setminus S$,*

$$k((-1)^{i-1}+(-1)^{\ell+1})f_S(P)+\sum_{j=i}^{\ell}(-1)^j f_{S\cup\{j\}}(P)=0.$$

*Every linear equality holding for the flag vector of all $k$-Eulerian posets of rank $n+1$ is a consequence of these equations.*

*In $L^k$-vector form, these equations are equivalent to the set of equations, $L_S^{k,n+1}(P)=0$ for all subsets $S\subseteq[1,n]$ that are not even.*

Call these equations the *generalized Dehn–Sommerville equations*, and denote by $DS_{k,n+1}$ the resulting subspace of $\mathbf{R}^{2^n}$.

**Proof.** The fact that the equations (in flag $f$-vector form) hold for all $k$-Eulerian posets follows from Proposition 4.5. Fix a set $S$ with gap $[i,\ell]$. For each $S$-chain identify the rank $i-1$ element $x$ and rank $\ell+1$ element $y$, and apply Eq. (2) to the interval $[x,y]$. Sum the resulting equations for all the $S$-chains.

Convert the flag $f$-vector equations using Eq. (4). Writing $V=[1,n]\setminus S$ and dividing by $2^{|S|}k^{|S|+1}$, the result is

$$((-1)^{i-1}+(-1)^{\ell+1})\sum_{T\subseteq V} L_T^{k,n+1}+2\sum_{j=i}^{\ell}(-1)^j\sum_{T\subseteq V\setminus\{j\}} L_T^{k,n+1}=0. \tag{6}$$

From this we prove by induction that $L_V^{k,n+1}(P)=0$ (abbreviated as $L_V=0$) for all noneven sets $V$. Let $V\subseteq[1,n]$ be any noneven set, and let $[i,\ell]$ be an odd maximal interval of $V$. Eq. (6) gives

$$\sum_{T\subseteq V} L_T+\sum_{j=i}^{\ell}(-1)^{j-i+1}\sum_{T\subseteq V\setminus\{j\}} L_T=0. \tag{7}$$

If $T$ is a noneven proper subset of $V$, then by the induction assumption, $L_T=0$. So consider an even subset $T\subseteq V$. Since, the maximal intervals of $T$ contained in $[i,\ell]$ are even, $[i,\ell]\setminus T=\{j_1,j_2,\ldots,j_t\}$, where $t$ is odd, $j_1-i$ is even, and, for $2\leqslant p\leqslant t$, $j_p-j_{p-1}$ is odd. Thus, for $1\leqslant p\leqslant t$, $j_p-i+1$ has the same parity as $p$. The coefficient of $L_T$ in (7) is $1+\sum_{p=1}^{t}(-1)^{j_p-i+1}=1+\sum_{p=1}^{t}(-1)^p=0$. So Eq. (7) reduces to $L_V=0$.

Conversely, suppose $L_V=0$ for all noneven sets $V\subseteq[1,n]$. We show that the equations in (6) hold. Let $V\subseteq[1,n]$ and $[i,\ell]$ a maximal interval of $V$. For $\ell-i$ even, we need to prove Eq. (7). (The case of $\ell-i$ odd is similar, and is omitted.) It suffices to consider the terms $L_T$ with $T$ an even set. For such $T$, $[i,\ell]\setminus T=\{j_1,j_2,\ldots,j_t\}$ as above, with $t$ odd, and $j_p-i+1\equiv p\,(\mathrm{mod}\,2)$. So the coefficient in (7) of $L_T$ is $1+\sum_{p=1}^{t}(-1)^{j_p-i+1}=1+\sum_{p=1}^{t}(-1)^p=0$. Thus Eq. (7) holds.

To complete the proof, it suffices to show that the linear span of the $L^k$-vectors of $k$-Eulerian posets of rank $n+1$ is the subspace of $\mathbf{R}^{2^n}$ determined by the equations $L_S^{k,n+1}(P)=0$ for all subsets $S\subseteq[1,n]$ that are not even. This can be accomplished by finding a set of linearly independent vectors in the span of the $L^k$-vectors of $k$-Eulerian posets, one vector for each even subset $S\subseteq[1,n]$. In [5] Billera and Hetyei constructed, for each interval system $\mathscr{I}$, a sequence of graded posets $P(n,\mathscr{I},N)$. The construction starts with a rank $n+1$ chain, and replicates intervals of ranks in the poset. For an even set $S$, let $\mathscr{I}[S]$ be the set of maximal intervals in $S$. (For example, for $S=\{1,3,4,7,8,9,10\}$, $\mathscr{I}[S]=\{[1],[3,4],[7,10]\}$.) If $S$ is an even subset of $[1,n]$, then $P(n,\mathscr{I}[S],N)$ is half-Eulerian for all $N$. Furthermore, the sequence of $L^{1/2}$-vectors of these posets satisfies the following. Here $m$ is the number of intervals in $\mathscr{I}[S]$.

$$\lim_{N\to\infty}\frac{1}{N^m}L_T^{1/2,n+1}(P(n,\mathscr{I}[S],N))=\begin{cases}(-1)^j & \text{if } T \text{ is the union}\\ & \text{of } j \text{ intervals of } S,\\ 0 & \text{otherwise.}\end{cases}$$

(See [3] for details.) Using (5), we get for any positive integer $2k$,

$$\lim_{N\to\infty}\frac{1}{N^m}L_T^{k,n+1}(D^{2k}P(n,\mathscr{I}[S],N))=\begin{cases}(-1)^j & \text{if } T \text{ is the union}\\ & \text{of } j \text{ intervals of } S,\\ 0 & \text{otherwise.}\end{cases}$$

For fixed $k$ the limiting $L^k$-vectors for each even interval system $\mathscr{I}[S]$ are linearly independent, since for each even set $S$, the vector formed from the sequence $(P(n,\mathscr{I}[S],N))$ has $T$-entry 0 for all $T$ not containing $S$. $\square$

A flag vector can by chance lie in the subspace $DS_{k,n+1}$ without the poset being $k$-Eulerian. However, $k$-Eulerian posets are characterized by the equations holding locally. The $k=1$ case of this is in [3]. The proof requires the convolution of flag operators, defined by Kalai [7] (see also [6]). It is defined for the flag numbers by $f_S^m * f_T^n = f^{m+n}_{S\cup\{m\}\cup(T+m)}$, and is extended by bilinearity to linear combinations. For $p^{m+1}$ and $q^{n+1}$ linear combinations of chain operators in ranks $m+1$ and $n+1$, respectively, their convolution on a rank $m+n$ poset $P$ satisfies

$$p^{m+1} * q^{n+1}(P) = \sum_{\substack{x\in P\\ \rho(x)=m}} p^{m+1}([\hat{0},x])q^{n+1}([x,\hat{1}]).$$

Convolution behaves nicely on the flag $L^k$-vector. For a rank $m+n+2$ poset $P$,

$$\begin{aligned} L_S^{k,m+1} * L_T^{k,n+1}(P) &= \sum_{\substack{x\in P\\ \rho(x)=m+1}} L_S^{k,m+1}([\hat{0},x])L_T^{k,n+1}([x,\hat{1}]) \\ &= 2kL^{k,m+n+2}_{S\cup(T+m+1)}(P). \end{aligned} \tag{8}$$

**Theorem 5.3.** *A graded partially ordered set $P$ is $k$-Eulerian if and only if for every interval $[x,y]\subseteq P$ of positive even rank $L^{k,\rho(x,y)}_{[1,\rho(x,y)-1]}([x,y])=0$.*

**Proof.** Since every interval of a $k$-Eulerian partially ordered set is $k$-Eulerian, Theorem 5.2 gives that $L^{k,\rho(x,y)}_{[1,\rho(x,y)-1]}([x,y])=0$ for all intervals $[x,y]$ of positive even rank.

Now assume that for every interval $[x,y]$ of positive even rank, $L^{k,\rho(x,y)}_{[1,\rho(x,y)-1]}([x,y])=0$. Then by Eq. (8), for every interval $[x,y]\subseteq P$ and for every $S\subseteq[1,\rho(x,y)-1]$ that is not even, $L_S^{k,\rho([x,y])}([x,y])=0$.

For $P$ of rank $n+1$, by Proposition 4.4,

$$\begin{aligned} \mu_k(P) &= -\sum_{S\subseteq[1,n]}\left(-\frac{1}{k}\right)^{|S|} f_S^{n+1}(P) = -\sum_{S\subseteq[1,n]}\left(-\frac{1}{k}\right)^{|S|}(2k)^{|S|}\sum_{T\subseteq[1,n]\setminus S} L_T^{k,n+1}(P) \\ &= -\sum_{T\subseteq[1,n]} L_T^{k,n+1}(P)\sum_{S\subseteq[1,n]\setminus T}(-2)^{|S|} = -\sum_{T\subseteq[1,n]} L_T^{n+1}(P)(-1)^{n-|T|}. \end{aligned}$$

Since, $L_T^{k,n+1}(P)$ is nonzero only if T is an even set, and then $|T|$ is an even number,

$$\mu_k(P) = (-1)^{n+1}\sum_{T\subseteq[1,n]} L_T^{n+1}(P) = (-1)^{n+1}f_\emptyset^{n+1}(P) = (-1)^{n+1}.$$

The same argument can be repeated for every interval of $P$, showing that it is a $k$-Eulerian poset. □

Using this result, we get the following curious characterization via the Möbius function.

**Theorem 5.4.** *A graded poset $P$ is $k$-Eulerian if and only if the $2k$-Möbius function of every interval $[x,y]\subseteq P$ of even rank is zero.*

**Proof.** Let $P$ be a graded poset. By Corollary 4.7 $P$ is $k$-Eulerian if and only if $D^2P$ is $2k$-Eulerian if and only if for every interval $[x_i,y_j]$ of $D^2P$ with $\rho(x_i,y_j)$ even

$$L^{2k,\rho(x_i,y_j)}_{[1,\rho(x_i,y_j)-1]}([x_i,y_j])=0$$

if and only if for every interval $[x_i,y_j]$ of $D^2P$ with $\rho(x_i,y_j)$ even

$$\sum_{T\subseteq[1,\rho(x_i,y_j)-1]}\left(-\frac{1}{4k}\right)^{|T|} f_T([x_i,y_j])=0$$

if and only if for every interval $[x,y]$ of $P$ with $\rho(x,y)$ even

$$\sum_{T\subseteq[1,\rho(x,y)-1]}\left(-\frac{1}{4k}\right)^{|T|} 2^{|T|} f_T([x,y])=0$$

if and only if for every interval $[x,y]$ of $P$ with $\rho(x,y)$ even

$$\sum_{T\subseteq[1,\rho(x,y)-1]}\left(-\frac{1}{2k}\right)^{|T|} f_T([x,y])=0$$

if and only if for every interval $[x,y]$ of $P$ with $\rho(x,y)$ even $\mu_{2k}([x,y])=0$. □

In particular, a graded poset $P$ is half-Eulerian if and only if the (usual) Möbius function of $[x,y]$ is zero for every $[x,y]\subseteq P$ of even rank.

The $L^1$-vector of a graded poset is the vector of coefficients of the $ce$-index, introduced in [8] as a variation of the $cd$-index of an Eulerian poset. (The $cd$-index of an Eulerian poset, due to Fine (see [4]), is a vector linearly equivalent to the flag vector; it embodies the generalized Dehn–Sommerville equations of Theorem 4.2.) In [8], Stanley observed that the existence of the $cd$-index for a graded poset is equivalent to the vanishing of the coefficients of $ce$-words containing an odd string of $e$'s; in our notation this says $L^{1,n+1}_S(P)=0$ for all subsets $S\subseteq[1,n]$ that are not even. Thus, the last part of Theorem 5.2 (as well as the first part) is already known for $k=1$.

The $L^k$-vector for general $k$ can be presented in the same way. For $P$ any graded poset of rank $n+1$, write a generating function for the flag $f$-vector as follows:

$$\Upsilon(a,b)=\sum_{S\subseteq[1,n]} f_S u_S,$$

where $u_S=u_1u_2\ldots u_n$ with $u_i=a$ if $i\notin S$ and $u_i=b$ if $i\in S$. Then

$$\Upsilon\left(e,\frac{c-e}{2k}\right)=\sum_{T\subseteq[1,n]} L_T v_T,$$

where $v_T = v_1 v_2 \ldots v_n$ with $v_i = c$ if $i \notin T$ and $v_i = e$ if $i \in T$. The equations of Theorem 5.2 for $k$-Eulerian posets can then be rephrased as saying that $\Upsilon(e,(c-e)/(2k))$ is a polynomial in (the noncommuting expressions) $c$ and $ee$.

## 6. The cone of $k$-Eulerian flag vectors

Theorem 3.1, along with the description of the cone of flag vectors of general graded posets [5], can be used to generate all the inequalities valid for all $r$-thick posets. The inequalities for $2k$-thick posets are, in particular, valid for all $k$-Eulerian posets, but they may not be sharp. We would like to know the essential inequalities, that is, the closed cones of flag vectors of Eulerian and of half-Eulerian posets. In [3] these cones are studied and are completely determined up through rank 7. (See also [2] for data on the cone.) In the context of this paper, the results can be stated as follows.

**Theorem 6.1** (Bayer and Hetyei [3]). *For rank $n+1 \leqslant 7$,*
(1) *the closed cone of flag vectors of half-Eulerian posets of rank $n+1$ is the intersection of the cone $\mathscr{C}_{1,n+1}$ of flag vectors of all graded posets of rank $n+1$ with the subspace $DS_{1/2,n+1}$ determined by the half-Eulerian equations of Theorem* 5.2;
(2) *the closed cone of flag vectors of Eulerian posets of rank $n+1$ is the intersection of the cone $\mathscr{C}_{2,n+1}$ of flag vectors of all* 2*-thick graded posets of rank $n+1$ with the generalized Dehn–Sommerville subspace $DS_{1,n+1}$; and*
(3) *the two cones are isomorphic.*

We do not know if this theorem extends to higher ranks. However, for all ranks, part 1 of the theorem implies parts 2 and 3.

**Theorem 6.2.** *Let* $\mathrm{CONE}_{k,n+1}$ *be the statement,*

> *The closed cone of flag vectors of $k$-Eulerian posets of rank $n+1$ is the intersection of the cone $\mathscr{C}_{2k,n+1}$ of flag vectors of all $2k$-thick graded posets of rank $n+1$ with the generalized Dehn–Sommerville space $DS_{k,n+1}$.*

*For all $k \geqslant 1$ (with $2k$ an integer) and all positive integers $n$,*

$$\mathrm{CONE}_{1/2,n+1} \quad \Rightarrow \quad \mathrm{CONE}_{k,n+1}.$$

**Proof.** Recall the map $\alpha_{1,2k}$ of Corollary 3.2; it maps $\mathscr{C}_{1,n+1}$ onto $\mathscr{C}_{2k,n+1}$. Clearly it also maps $DS_{1/2,n+1}$ onto $DS_{k,n+1}$. So $\alpha_{1,2k}(\mathscr{C}_{1,n+1} \cap DS_{1/2,n+1}) = \mathscr{C}_{2k,n+1} \cap DS_{k,n+1}$, which contains the cone of $k$-Eulerian flag vectors. On the other hand, for any half-Eulerian poset $P$, $\alpha_{1,2k}((f_S(P))) = (f_S(D^{2k}P))$, the flag vector of the $k$-Eulerian poset $D^{2k}P$. If $\mathrm{CONE}_{1/2,n+1}$ holds, then $\mathscr{C}_{1,n+1} \cap DS_{1/2,n+1}$ is the cone of half-Eulerian flag vectors, and its image is contained in the cone of $k$-Eulerian flag vectors. Thus, if

$\mathrm{CONE}_{1/2,n+1}$ holds, then $\mathscr{C}_{2k,n+1} \cap DS_{k,n+1}$ is exactly the closed cone of flag vectors of $k$-Eulerian posets. □

Another question raised in [3] on the structure of these cones can be answered. For rank at most 7, all facet inequalities of the half-Eulerian (and with slight modification, Eulerian) cone are generated from two basic types of inequalities.

**Theorem 6.3** (Bayer and Hetyei [3]). *Let $S$ and $T$ be disjoint subsets of $[1,n]$, such that every maximal interval of the complement of $S$ contains at most one element of $T$. Then for every rank $n+1$ half-Eulerian poset $P$,*

$$\sum_{R \subseteq T} (-1)^{|T \setminus R|} f_{S \cup R}(P) \geqslant 0.$$

*Let $1 \leqslant i < j < \ell \leqslant n$. Then for every rank $n+1$ half-Eulerian poset $P$,*

$$f_{i\ell}(P) - f_i(P) - f_\ell(P) + f_j(P) \geqslant 0.$$

Other valid inequalities are obtained by the convolution of inequalities of these types. The question arose whether these generate all inequalities valid for the flag vectors of all half-Eulerian posets. They do not.

**Proposition 6.4.** *For all half-Eulerian posets $P$ of rank* 8,

$$\begin{aligned} &f^8_{1356}(P) - f^8_{135}(P) - f^8_{356}(P) + f^8_{15}(P) - f^8_{16}(P) \\ &\quad + f^8_{35}(P) + f^8_{36}(P) - f^8_{3}(P) \geqslant 0, \end{aligned}$$

*or, in $L^{1/2}$-vector form,*

$$\begin{aligned} &L^{1/2,8}_{45}(P) + L^{1/2,8}_{2345}(P) + L^{1/2,8}_{56}(P) + L^{1/2,8}_{1256}(P) - L^{1/2,8}_{2367}(P) \\ &\quad - L^{1/2,8}_{3467}(P) + L^{1/2,8}_{4567}(P) + L^{1/2,8}_{124567}(P) \leqslant 0. \end{aligned} \tag{9}$$

*This inequality determines a facet of the closed cone of flag vectors of half-Eulerian posets, and does not follow from the inequalities of Proposition* 6.3.

The proposition remains valid if "half-Eulerian" is replaced by $k$-Eulerian, and each $f_S$ is replaced by $(2k)^{n-|S|} f_S$.

**Proof.** We first show the inequality is not a convolution of lower rank inequalities. In $L^k$-vector form the convolution satisfies the rule $L^{k,i+1}_T * L^{k,j+1}_V = 2kL^{k,i+j+2}_{T \cup (V+i+1)}$ (see Eq. (8)). So the convolution of linear expressions for ranks $i+1$ and $j+1$ with $n = i+j+1$ gives a linear combination of $L^{k,n+1}_S$ involving only subsets $S \subseteq [1,n]$ not containing

$i+1$. Since each element of $[1,7]$ occurs in some set $S$ in inequality (9), it is not a convolution of lower rank inequalities.

We now show that the inequality determines a facet of the cone. Billera and Hetyei list the facet inequalities for the general graded cone up through rank 5. The inequality of the proposition comes from applying one of the rank 5 Billera–Hetyei inequalities to the rank-selected subposet $P_{\{1,3,5,6\}}$ of an arbitrary half-Eulerian poset $P$. To check it is a facet of the half-Eulerian cone, we give 20 linearly independent limiting normalized $L^{1/2}$-vectors of half-Eulerian posets, for which the inequality holds with equality. The first 16 posets are Billera–Hetyei limit posets determined by interval systems as in the following table.

| | |
|---|---|
| $P_1$ | $\emptyset$ |
| $P_2$ | $[1,2]$ |
| $P_3$ | $[2,3]$ |
| $P_4$ | $[3,4]$ |
| $P_5$ | $[1,2][3,4]$ |
| $P_6$ | $[2,3][4,5]$ |
| $P_7$ | $[1,2][5,6]$ |
| $P_8$ | $[1,2][3,4][5,6]$ |
| $P_9$ | $[3,6]$ |
| $P_{10}$ | $[6,7]$ |
| $P_{11}$ | $[1,2][6,7]$ |
| $P_{12}$ | $[1,4][6,7]$ |
| $P_{13}$ | $[4,5][6,7]$ |
| $P_{14}$ | $[2,3][4,5][6,7]$ |
| $P_{15}$ | $[1,2][4,7]$ |
| $P_{16}$ | $[2,7]$ |

The next three limit posets are obtained from the rank 7 Extremes 2, 3 and 4 of [3, Theorem 4.8] by inserting a single new element of rank 1, shifting the old elements up one rank.

To describe the last sequence of posets, let us (re)introduce the following generalization of the operator $D^r$. Given a graded poset $P$ of rank $n+1$ denote by $D^r_{[u,v]}(P)$ the poset obtained from $P$ by replacing each $x\in P$ satisfying $\rho(x)\in[u,v]$ with $r$ elements $x_1,x_2,\ldots,x_r$ (keep every $y\in P$ satisfying $\rho(y)\notin[u,v]$ unchanged), and by setting the following order relations. The $([1,n]\backslash[u,v])$-rank-selected subposet of $P$ and of $D^r_{[u,v]}(P)$ are identical. For $x,y\in P$ satisfying $\rho(x)\in[u,v]$ and $\rho(y)\notin[u,v]$ set $x_i<y$ or $x_i>y$ in $D^r_{[u,v]}(P)$ if and only of the same relation holds between $x$ and $y$ in $P$. Finally for $x,y\in P$ satisfying $u\leqslant\rho(x)<\rho(y)\leqslant v$ set $x_i<y_j$ in $D^r_{[u,v]}(P)$ if and only if $i=j$ and $x<y$ in $P$.

For example, Fig. 1 shows $D^2_{[1,2]}(C_4)$ where $C_4$ is a chain of rank 4. Note that for a graded poset $P$ of rank $n+1$ the graded poset $D^r(P)$ is isomorphic to $D^r_{[1,1]}D^r_{[2,2]}\cdots D^r_{[n,n]}(P)$. The same notation is used in [3].

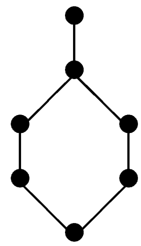


Fig. 1. $D^2_{[1,2]}(C_4)$.

Let $N$ be an arbitrary positive integer, and $C_8$ be a chain of rank 8. Consider now the following four graded posets.

$$P^{\mathrm{I}}(N) = D^{N+1}_{[1,2]} D^{N+1}_{[2,3]} D^{N+1}_{[4,5]} D^{N}_{[1,7]}(C_8),$$

$$P^{\mathrm{II}}(N) = D^{N^2}_{[1,3]} D^{N+1}_{[1,5]} D^{N}_{[1,7]}(C_8),$$

$$P^{\mathrm{III}}(N) = D^{N^2-N+2}_{[1,4]} D^{N+2}_{[4,5]} D^{N}_{[6,7]}(C_8),$$

$$P^{\mathrm{IV}}(N) = D^{N+2}_{[1,2]} D^{N^3-N^2+2}_{[2,7]}(C_8).$$

The $\{4,5,6,7\}$-rank-selected subposets of $P^{\mathrm{I}}(N)$ and $P^{\mathrm{II}}(N)$ are both isomorphic to $D^{N+1}_{[1,2]} D^{N}_{[1,4]}(C_5)$, where $C_5$ is a chain of rank 5; the $\{6,7\}$-rank-selected subposets of $P^{\mathrm{I}}(N)$, $P^{\mathrm{II}}(N)$, and $P^{\mathrm{III}}(N)$ are all isomorphic to $D^{N}_{[1,2]}(C_3)$ where $C_3$ is a chain of rank 3. Let $P(N)$ be the graded poset of rank 8 obtained from $P^{\mathrm{I}}(N)$, $P^{\mathrm{II}}(N)$, $P^{\mathrm{III}}(N)$, and $P^{\mathrm{IV}}(N)$ by performing the following identifications:

- identify the bottom element $\hat{0}$ of all four posets,
- identify the top element $\hat{1}$ of all four posets,
- identify $P^{\mathrm{I}}(N)_{\{4,5,6,7\}}$ with $P^{\mathrm{II}}(N)_{\{4,5,6,7\}}$,
- identify $P^{\mathrm{I}}(N)_{\{6,7\}}$ with $P^{\mathrm{III}}(N)_{\{6,7\}}$.

Fig. 2 indicates how the four posets are identified, in a schematic way.

Straightforward calculation shows that $P(N)$ is a half-Eulerian poset, for each positive $N$. Furthermore, the normalized $L^{1/2}$-vectors, $(L^{1/2,8}_S(P(N))/N^4)$, converge.

The rows of the matrix below are the normalized $L^{1/2}$-vectors of the 20 limit posets. In the columns are the values of $L^{1/2,8}_S$ (divided by the appropriate power of $N$), with the sets $S$ in the order

$\emptyset$, $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{1,2,3,4\}$, $\{4,5\}$, $\{1,2,4,5\}$, $\{2,3,4,5\}$, $\{5,6\}$, $\{1,2,5,6\}$, $\{2,3,5,6\}$, $\{3,4,5,6\}$, $\{1,2,3,4,5,6\}$, $\{6,7\}$, $\{1,2,6,7\}$, $\{2,3,6,7\}$, $\{3,4,6,7\}$, $\{1,2,3,4,6,7\}$, $\{4,5,6,7\}$, $\{1,2,4,5,6,7\}$, $\{2,3,4,5,6,7\}$.

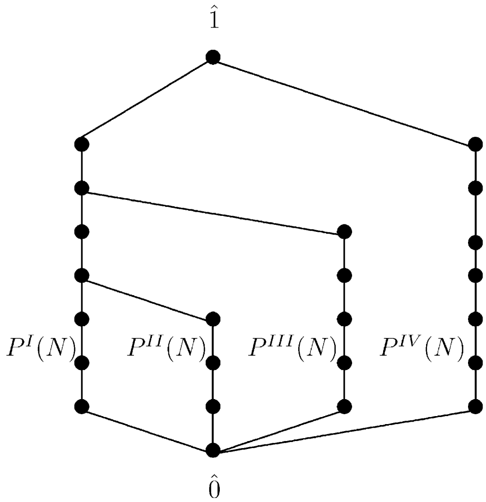


Fig. 2. $P(N)$.

It is easy to check the rows are linearly independent.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
3 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
3 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
3 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\\
1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
4 & -2 & -1 & 0 & -1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1
\end{bmatrix}
\qquad \square
$$

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