

Available online at www.sciencedirect.com

Journal of Functional Analysis 260 (2011) 2222–2246

**JOURNAL OF
Functional
Analysis**www.elsevier.com/locate/jfa

Direct sums and the Szlenk index [☆]

Philip A.H. Brooker ¹*Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia*

Received 1 April 2010; accepted 16 December 2010

Available online 20 January 2011

Communicated by Gilles Godefroy

Abstract

For α an ordinal and $1 < p < \infty$, we determine a necessary and sufficient condition for an ℓ_p -direct sum of operators to have Szlenk index not exceeding ω^α . It follows from our results that the Szlenk index of an ℓ_p -direct sum of operators is determined in a natural way by the behaviour of the ε -Szlenk indices of its summands. Our methods give similar results for c_0 -direct sums.

© 2010 Elsevier Inc. All rights reserved.

Keywords: Szlenk index; Asplund operators; Direct sums; Banach spaces

0. Introduction

The Szlenk index was introduced by W. Szlenk in his influential paper [22], where an ordinal index was used to show that the class of all separable, reflexive Banach spaces contains no universal element. Since then, the Szlenk index and its variants have taken on an increasingly important role in the study of Banach spaces and their operators. We refer the reader to the surveys [14] and [19] for details on some of the main applications of the Szlenk index.

A class of closed operator ideals naturally related to the Szlenk index has been introduced and systematically studied by the present author in [2]. These operator ideals are denoted \mathcal{SL}_α , where α is an ordinal, and elements of \mathcal{SL}_α are known as α -Szlenk operators. The operator

[☆] Research supported by an ANU PhD Scholarship.

E-mail address: philip.a.h.brooker@gmail.com.

¹ This work forms part of the author's doctoral dissertation, written at the Australian National University under the supervision of Dr. Richard J. Loy.

ideals \mathcal{L}_α are studied in [2] with regard to their operator ideal properties and their relationship to other closed operator ideals, in particular the class of Asplund operators.

The purpose of the present paper is to present a detailed analysis of the behaviour of the Szlenk index under the process of taking c_0 and ℓ_p -direct sums of operators. In particular, we give a precise formulation of the Szlenk index of a direct sum of operators in terms of the behaviour of the ε -Szlenk indices of the summands. Our motivation for this is as follows. Firstly, forming direct sums is a fundamental construction in Banach space theory, often being used to construct examples with a particular property, and so we feel it essential to understand precisely how the Szlenk index behaves under this procedure. Secondly, we are motivated by the following basic question of operator ideal theory:

Question 0.1. Let \mathcal{I} be a given operator ideal. Does \mathcal{I} have the factorisation property? That is, does every element of \mathcal{I} factor continuously and linearly through a Banach space whose identity operator belongs to \mathcal{I} ?

In [2], results and techniques developed in the current paper are applied to obtain both positive and negative answers to Question 0.1 for the case $\mathcal{I} = \mathcal{L}_\alpha$, with the answer depending upon ordinal properties of α .

We now outline the structure of the current paper. In Section 1 we detail necessary notation and background results regarding the Szlenk index, including several relevant results from [2]. Our main results are presented in Section 2. Firstly, we consider the Szlenk index of ℓ_1 and ℓ_∞ -direct sums; this case is rather straightforward, but worth noting explicitly for the sake of completeness. We then move on to our main concern, providing a formulation of the Szlenk index of c_0 and ℓ_p -direct sums of operators, where $1 < p < \infty$ (see, in particular, Theorem 2.10). This case is far more subtle than the case of ℓ_1 and ℓ_∞ -direct sums and, as such, requires substantially more effort to accomplish the desired formulation of the Szlenk index of the direct sum. Section 2 concludes with some applications of the earlier operator theoretic results to the Szlenk index of Banach spaces. The final section, Section 3, constitutes almost half of the paper and is devoted to proving the main technical lemma used in Section 2, namely Lemma 2.5.

1. Preliminaries

Banach spaces are typically denoted by the letters E and F . For a Banach space E and nonempty bounded $S \subseteq E$, we define $|S| := \sup\{\|x\| \mid x \in S\}$. By B_E we denote the closed unit ball of E , and by I_E the identity operator of E . The class of all bounded linear operators between arbitrary Banach spaces is denoted by \mathcal{B} , and the class of all compact operators by \mathcal{K} . We write ORD for the class of all ordinals, whose elements are typically denoted by the lower-case Greek letters α , β and γ . For Λ a set, $\Lambda^{<\infty}$ denotes the set of all nonempty finite subsets of Λ . When Λ denotes the index set over which we take a direct sum or direct product, it is always assumed that Λ is nonempty.

Let $p \in \{0\} \cup (1, \infty)$ and $q \in [1, \infty)$. We say that q is dual to p , or equivalently, p is predual to q , if $(p, q) \in \{(0, 1)\} \cup \{(r, r(r-1)^{-1}) \mid r \in (1, \infty)\}$.

For $1 \leq p \leq \infty$, a set Λ and Banach spaces E_λ , $\lambda \in \Lambda$, the ℓ_p -direct sum of $\{E_\lambda \mid \lambda \in \Lambda\}$ is denoted $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$, and the c_0 -direct sum of $\{E_\lambda \mid \lambda \in \Lambda\}$ is denoted $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_0$. If there is a Banach space E such that $E_\lambda = E$ for all $\lambda \in \Lambda$, then we may also write the ℓ_p -direct sum and the c_0 -direct sum as $\ell_p(\Lambda, E)$ and $c_0(\Lambda, E)$, respectively. Throughout, for $1 < p, q < \infty$ satisfying $p + q = pq$, we implicitly identify $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$ with $(\bigoplus_{\lambda \in \Lambda} E_\lambda^*)_q$, so that the dual of a

direct sum is the dual direct sum of the duals of the spaces E_λ . Making this identification allows us to consider direct products of the form $\prod_{\lambda \in \Lambda} K_\lambda$, where $K_\lambda \subseteq E_\lambda^*$ and $(\{K_\lambda\}_{\lambda \in \Lambda}) \in \ell_q(\Lambda)$, as subsets of $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$. Similarly, $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_0^*$ is naturally identified with $(\bigoplus_{\lambda \in \Lambda} E_\lambda^*)_1$ throughout.

Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ a family of Banach spaces indexed by Λ and $p = 0$ or $1 < p < \infty$. For $\mathcal{R} \subseteq \Lambda$, we denote by $U_{\mathcal{R}}$ the canonical injection of $(\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p$ into $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$, and by $P_{\mathcal{R}}$ the canonical surjection of $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$ onto $(\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p$.

For a set Λ , a family of Banach spaces $\{E_\lambda \mid \lambda \in \Lambda\}$ and nonempty, bounded subsets $S_\lambda \subseteq E_\lambda$, $\lambda \in \Lambda$, we say that $\{S_\lambda \subseteq E_\lambda \mid \lambda \in \Lambda\}$ is *uniformly bounded* if $\sup\{\|S_\lambda\| \mid \lambda \in \Lambda\} < \infty$. If $\{F_\lambda \mid \lambda \in \Lambda\}$ is also a family of Banach spaces indexed by Λ , a set of operators $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ is said to be *uniformly bounded* if $\sup\{\|T_\lambda\| \mid \lambda \in \Lambda\} < \infty$. Given $1 \leq p \leq \infty$ and a uniformly bounded family of operators $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$, the ℓ_p -direct sum of $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$, denoted $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$, is the continuous linear map that sends $(x_\lambda)_{\lambda \in \Lambda} \in (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$ to $(T_\lambda x_\lambda)_{\lambda \in \Lambda} \in (\bigoplus_{\lambda \in \Lambda} F_\lambda)_p$. Each of the operators T_λ , $\lambda \in \Lambda$, is a *summand* of the direct sum $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$.

A Banach space E over the field \mathbb{R} of real scalars is said to be *Asplund* if every real-valued convex continuous function defined on a convex open subset U of E is Fréchet differentiable on a dense G_δ subset of U . Our arguments hold for Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} ; note that the notion of Asplund space may be extended (somewhat artificially) to complex Banach spaces by declaring a complex Banach space to be Asplund precisely when its underlying real Banach space structure is an Asplund space in the real scalar sense. By extending the notion of Asplund space to complex Banach spaces in this way, many of the well-known characterisations of Asplund spaces – for instance, a Banach space is Asplund if and only if each of its separable subspaces has separable dual [3, Theorem 5.7] – then hold also for complex Asplund spaces.

For Banach spaces E and F , an operator $T : E \rightarrow F$ is *Asplund* if for any finite positive measure space (Ω, Σ, μ) , any $S \in \mathcal{B}(F, L_\infty(\Omega, \Sigma, \mu))$ and any $\varepsilon > 0$, there exists $B \in \Sigma$ such that $\mu(B) > \mu(\Omega) - \varepsilon$ and $\{f \chi_B \mid f \in ST(B_E)\}$ is relatively compact in $L_\infty(\Omega, \Sigma, \mu)$ (here χ_B denotes the characteristic function of B on Ω). We note that some authors, for example in [17] and [10], refer to Asplund operators as *decomposing operators*. Standard references for Asplund operators are [17] and [21], where it is shown that the Asplund operators form a closed operator ideal and that a Banach space is an Asplund space if and only if its identity operator is an Asplund operator. A further impressive result is that every Asplund operator factors through an Asplund space; this is due independently to O. Reřnov [18], S. Heinrich [10] and C. Stegall [21].

We now define the Szlenk index, noting that our definition varies from that given by W. Szlenk in [22]. However, the two definitions give the same index for operators acting on separable Banach spaces containing no isomorphic copy of ℓ_1 (see the proof of [12, Proposition 3.3] for details).

Let E be a Banach space, $K \subseteq E^*$ a w^* -compact set and $\varepsilon > 0$. Define

$$s_\varepsilon(K) := \{x \in K \mid \text{diam}(K \cap V) > \varepsilon \text{ for every } w^*\text{-open } V \ni x\}.$$

We iterate s_ε transfinitely as follows: let $s_\varepsilon^0(K) = K$, $s_\varepsilon^{\alpha+1}(K) = s_\varepsilon(s_\varepsilon^\alpha(K))$ for each ordinal α and, if α is a limit ordinal, $s_\varepsilon^\alpha(K) = \bigcap_{\beta < \alpha} s_\varepsilon^\beta(K)$.

The ε -Szlenk index of K , denoted $\text{Sz}_\varepsilon(K)$, is the class of all ordinals α such that $s_\varepsilon^\alpha(K) \neq \emptyset$. The Szlenk index of K is the class $\bigcup_{\varepsilon > 0} \text{Sz}_\varepsilon(K)$. Note that $\text{Sz}_\varepsilon(K)$ (resp., $\text{Sz}(K)$) is either an ordinal or the class ORD of all ordinals. If $\text{Sz}_\varepsilon(K)$ (resp., $\text{Sz}(K)$) is an ordinal, then we write $\text{Sz}_\varepsilon(K) < \infty$ (resp., $\text{Sz}(K) < \infty$), and otherwise we write $\text{Sz}_\varepsilon(K) = \infty$ (resp., $\text{Sz}(K) = \infty$).

For a Banach space E , the ε -Szlenk index of E is $Sz_\varepsilon(E) = Sz_\varepsilon(B_{E^*})$, and the Szlenk index of E is $Sz(E) = Sz(B_{E^*})$. If $T : E \rightarrow F$ is an operator, the ε -Szlenk index of T is $Sz_\varepsilon(T) = Sz_\varepsilon(T^*B_{F^*})$, whilst the Szlenk index of T is $Sz(T) = Sz(T^*B_{F^*})$.

It is clear that the Szlenk index of a nonempty w^* -compact set cannot be 0. We note also that, by w^* -compactness, the ε -Szlenk index of a nonempty w^* -compact set K is never a limit ordinal.

The following proposition states some known facts about the Szlenk index.

Proposition 1.1. *Let E and F be Banach spaces, $T : E \rightarrow F$ an operator and $K \subseteq E^*$ a nonempty w^* -compact set.*

- (i) *If E is isomorphic to a quotient or subspace of F , then $Sz(E) \leq Sz(F)$. In particular, the Szlenk index is an isomorphic invariant of a Banach space.*
- (ii) *$Sz(E) < \infty$ if and only if E is an Asplund space. Similarly, $Sz(T) < \infty$ if and only if T is an Asplund operator.*
- (iii) *If K is absolutely convex and $Sz(K) < \infty$, then there exists an ordinal α such that $Sz(K) = \omega^\alpha$. In particular, the Szlenk index of an Asplund space or Asplund operator is of the form ω^α for some (unique) ordinal α .*
- (iv) *$Sz(K) = 1$ if and only if K is norm-compact. In particular, $Sz(E) = 1$ if and only if $\dim(E) < \infty$, and $Sz(T) = 1$ if and only if T is compact.*
- (v) *$Sz(E \oplus F) = \max\{Sz(E), Sz(F)\}$.*

Part (i) of Proposition 1.1 is discussed in [8]. Part (ii) is discussed in [8] in the case of spaces, and the more general case of operators is established in [2, Proposition 2.10]. Part (iii) was proved for $K = B_{E^*}$ in [13]; see also p. 64 of [9]. As the proof of the case $K = B_{E^*}$ relies only upon the fact that B_{E^*} is convex and symmetric (that is, absolutely convex), the proof applies also to arbitrary absolutely convex K . Part (iv) is a consequence of the fact that a w^* -compact set is norm-compact if and only if its relative w^* and norm topologies coincide (see, e.g., [4, Corollary 3.1.14]), with the final assertion regarding operators requiring the use of Schauder’s theorem. Part (v) is essentially Proposition 2.4 of [7] (see also [15, Proposition 14] for the separable case), and will be improved upon in Theorem 2.11 below.

Definition 1.2. For each ordinal α , define $\mathcal{SL}_\alpha := \{T \in \mathcal{B} \mid Sz(T) \leq \omega^\alpha\}$.

As noted in the introduction, elements of \mathcal{SL}_α are known as α -Szlenk operators. We have the following:

Theorem 1.3. (See [2, Theorem 2.2].) *Let α be an ordinal. Then \mathcal{SL}_α is a closed operator ideal.*

2. Main results

It is obvious that a direct sum of operators factors any of its summands. Thus, since $\{T \in \mathcal{B} \mid Sz(T) < \infty\}$ is the operator ideal of Asplund operators (see Proposition 1.1(ii)), it is only interesting to consider the Szlenk index of a direct sum of operators in the case that all of the summands are Asplund. With this in mind, we henceforth consider direct sums of Asplund operators only.

2.1. ℓ_1 -Direct sums and ℓ_∞ -direct sums

The task of determining the Szlenk index of ℓ_1 -direct sums and ℓ_∞ -direct sums of operators is made considerably easier by the fact that the Banach spaces ℓ_1 and ℓ_∞ fail to be Asplund, for this ensures that the norms of the summand operators must exhibit c_0 -like behaviour in order for the direct sum operator to be Asplund. More precisely, we have the following result.

Proposition 2.1. *Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ a uniformly bounded family of Asplund operators and $p = 1$ or $p = \infty$. The following are equivalent:*

- (i) $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) < \infty$ (that is, $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ is Asplund).
- (ii) $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \sup\{Sz(T_\lambda) \mid \lambda \in \Lambda\}$.
- (iii) $(\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.

Proof. We prove (iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii).

Suppose (iii) holds; we will show $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \sup\{Sz(T_\lambda) \mid \lambda \in \Lambda\}$. By Proposition 1.1(iii) there exist ordinals α_λ , $\lambda \in \Lambda$, with $Sz(T_\lambda) = \omega^{\alpha_\lambda}$ for each λ . Let $\alpha_\Lambda = \sup\{\alpha_\lambda \mid \lambda \in \Lambda\}$, so that $\sup\{Sz(T_\lambda) \mid \lambda \in \Lambda\} = \omega^{\alpha_\Lambda}$. To see that $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p \in \mathcal{SZ}_{\alpha_\Lambda}$, for $n \in \mathbb{N}$ and $\lambda \in \Lambda$ let

$$T_{\lambda,n} = \begin{cases} T_\lambda & \text{if } \|T_\lambda\| > 1/n, \\ 0 & \text{otherwise} \end{cases}$$

and $V_n = (\bigoplus_{\lambda \in \Lambda} T_{\lambda,n})_p$. Note that $\{T_{\lambda,n} \mid \lambda \in \Lambda, n \in \mathbb{N}\} \subseteq \mathcal{SZ}_{\alpha_\Lambda}$, hence $V_n \in \mathcal{SZ}_{\alpha_\Lambda}$ also since each V_n can be written as a (finite) sum of operators that factor some element of $\{T_{\lambda,n} \mid \lambda \in \Lambda, n \in \mathbb{N}\}$. It follows from the definitions that $\|V_n - (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p\| \leq 1/n$ for each $n \in \mathbb{N}$, hence $V_n \rightarrow (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ as $n \rightarrow \infty$. Since $V_n \in \mathcal{SZ}_{\alpha_\Lambda}$ for all n and $\mathcal{SZ}_{\alpha_\Lambda}$ is closed (Theorem 1.3), we have $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p \in \mathcal{SZ}_{\alpha_\Lambda}$. In particular, $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) \leq \omega^{\alpha_\Lambda} = \sup\{Sz(T_\lambda) \mid \lambda \in \Lambda\}$. The reverse inequality follows by Theorem 1.3 and the fact that $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ factors each of the operators T_λ , $\lambda \in \Lambda$. We have now shown (iii) \Rightarrow (ii).

It is trivial that (ii) \Rightarrow (i), so remains only to show that (i) \Rightarrow (iii). To this end, suppose that (iii) does not hold. Then there exists $\delta > 0$ and an infinite set $\Lambda' \subseteq \Lambda$ such that $\|T_\lambda\| > \delta$ for all $\lambda \in \Lambda'$, and so $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ factors an isomorphic embedding of the non-Asplund space ℓ_p . By Proposition 1.1(ii), $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \infty$. \square

2.2. c_0 -Direct sums and ℓ_p -direct sums ($1 < p < \infty$)

In this section we consider the Szlenk index of a direct sum operator $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$, where $p = 0$ or $1 < p < \infty$. As in the cases $p = 1$ and $p = \infty$, if $(\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$ then $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = \sup\{Sz(T_\lambda) \mid \lambda \in \Lambda\}$. However, the situation is not so clear if $(\|T_\lambda\|)_{\lambda \in \Lambda} \notin c_0(\Lambda)$, and we demonstrate this by way of an example. For an ordinal γ , we may equip the ordinal $\gamma + 1$ with its order topology, thereby making it a compact Hausdorff space. C. Samuel has shown that for each $\alpha < \omega_1$, $Sz(C(\omega^{\omega^\alpha} + 1)) = \omega^{\alpha+1}$ (Samuel’s calculation is found in [20], however a more direct approach has been discovered by P. Hájek and G. Lancien [7]). By the Bessaga–Pełczyński linear isomorphic classification of $C(K)$ spaces with K countable [1, Theorem 1], $C(\omega^n + 1)$ is linearly isomorphic to $C(\omega + 1)$ for all $0 < n < \omega$. Thus, in particular,

$Sz(C(\omega^n + 1)) = Sz(C(\omega + 1)) = \omega$ for all $0 < n < \omega$. For each $0 < n < \omega$, let T_n denote the identity operator on $C(\omega^n + 1)$. As $(\bigoplus_{0 < n < \omega} C(\omega^n + 1))_0$ is linearly isomorphic to $C(\omega^\omega + 1)$, by Samuel’s result we have

$$Sz\left(\left(\bigoplus_{0 < n < \omega} T_n\right)_0\right) = Sz(C(\omega^\omega + 1)) = \omega^2 > \omega = \sup\{Sz(T_n) \mid 0 < n < \omega\}.$$

Thus the situation under consideration in this section is more subtle than the cases of ℓ_1 -direct sums and ℓ_∞ -direct sums. Our goal is to determine precisely the Szlenk index of a c_0 -direct sum or ℓ_p -direct sum ($1 < p < \infty$) of operators in terms of the overall behaviour of the ε -Szlenk indices of the summand operators. To this end, we now introduce some notation.

Given a set Λ , a family of Banach spaces $\{E_\lambda \mid \lambda \in \Lambda\}$, a corresponding uniformly bounded family $\{K_\lambda \subseteq E_\lambda^* \mid \lambda \in \Lambda\}$ of absolutely convex, w^* -compact sets and $1 \leq p < \infty$, we define

$$B_q(K_\lambda \mid \lambda \in \Lambda) := \bigcup_{(a_\lambda)_{\lambda \in \Lambda} \in B_{\ell_q(\Lambda)}} \prod_{\lambda \in \Lambda} a_\lambda K_\lambda,$$

and always consider $B_q(K_\lambda \mid \lambda \in \Lambda)$ as a subset of $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$, where p is predual to q (recall from Section 1 that $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$ is naturally identified with $(\bigoplus_{\lambda \in \Lambda} E_\lambda^*)_q$). Such a set $B_q(K_\lambda \mid \lambda \in \Lambda)$ so defined is clearly bounded, and it is not difficult to see that it is also w^* -compact. Indeed, for each $\lambda \in \Lambda$ define $T_\lambda : E_\lambda \rightarrow C(K_\lambda)$ to be the map that sends $x \in E_\lambda$ to the continuous function $k \mapsto \langle k, x \rangle$ ($k \in K_\lambda$). Then the Kreĭn–Mil’man theorem, along with other classical results regarding extreme points (see, for example, [5, Lemma 3.42] and [6, Exercise 2.4]), implies that $T_\lambda^* B_{C(K_\lambda)^*} = K_\lambda$ for each $\lambda \in \Lambda$. Hence $B_q(K_\lambda \mid \lambda \in \Lambda) = (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p^* B_{(\bigoplus_{\lambda \in \Lambda} C(K_\lambda))_p^*}$, ensuring the w^* -compactness of $B_q(K_\lambda \mid \lambda \in \Lambda)$.

We first deal explicitly with the case where the Szlenk index of a direct sum of operators has Szlenk index $\omega^0 = 1$. The following result describes the situation for this case.

Proposition 2.2. *Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ a uniformly bounded family of operators and $p \in \{0\} \cup [1, \infty]$. The following are equivalent:*

- (i) $Sz((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) = 1$.
- (ii) $Sz(T_\lambda) = 1$ for every $\lambda \in \Lambda$ and $(\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.

Proposition 2.2 follows immediately from Proposition 1.1(iv) and the following proposition.

Proposition 2.3. *Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda \in \mathcal{B}(E_\lambda, F_\lambda) \mid \lambda \in \Lambda\}$ a uniformly bounded family of operators and $p \in \{0\} \cup [1, \infty]$. The following are equivalent:*

- (i) $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ is compact.
- (ii) T_λ is compact for every $\lambda \in \Lambda$ and $(\|T_\lambda\|)_{\lambda \in \Lambda} \in c_0(\Lambda)$.

We omit the straightforward proof of Proposition 2.3, but note that it is similar to the proof of Proposition 2.1 presented earlier.

The general case for c_0 -direct sums and ℓ_p -direct sums of operators, where $1 < p < \infty$, will be deduced from the following key result.

Proposition 2.4. *Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ a family of Banach spaces, $\{K_\lambda \subseteq E_\lambda^* \mid \lambda \in \Lambda, K_\lambda \neq \emptyset\}$ a uniformly bounded family of nonempty absolutely convex w^* -compact sets, $\alpha > 0$ an ordinal and $1 \leq q < \infty$. The following are equivalent:*

- (i) $Sz(B_q(K_\lambda \mid \lambda \in \Lambda)) \leq \omega^\alpha$.
- (ii) $\sup\{Sz_\varepsilon(K_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha$ for every $\varepsilon > 0$.
- (iii) $\sup\{Sz_\varepsilon(B_q(K_\lambda \mid \lambda \in \mathcal{F})) \mid \mathcal{F} \in \Lambda^{<\infty}\} < \omega^\alpha$ for every $\varepsilon > 0$.

To establish Proposition 2.4, we prove (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). In proving the implication (ii) \Rightarrow (iii), we shall call upon the following technical result:

Lemma 2.5. *Let E_1, \dots, E_n be Banach spaces, $K_1 \subseteq E_1^*, \dots, K_n \subseteq E_n^*$ nonempty, absolutely convex, w^* -compact sets, $1 \leq q < \infty$ and $\varepsilon > 0$. Let $d = \max\{\text{diam}(K_i) \mid 1 \leq i \leq n\}$ and let m and M be natural numbers such that $M \geq m \geq 2$ and $(2^q - 1)\varepsilon^q M \geq 8^q d^q (m - 1)$. Suppose α is an ordinal such that $s_\varepsilon^{\omega^\alpha \cdot M}(B_q(K_i \mid 1 \leq i \leq n)) \neq \emptyset$. Then for every $\delta \in (0, \varepsilon/16)$ there is $i \leq n$ such that $s_\delta^{\omega^\alpha \cdot m}(K_i) \neq \emptyset$.*

The proof of Lemma 2.5 is delayed until Section 3. To show (iii) \Rightarrow (i) we require the following discrete variant of [7, Lemma 3.3]:

Lemma 2.6. *Let Λ be a set, $(E_\lambda)_{\lambda \in \Lambda}$ a family of Banach spaces, $1 \leq q < \infty$, p predual to q and $K \subseteq (\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*$ nonempty and w^* -compact. Let α be an ordinal, $\mathcal{R} \subseteq \Lambda$ and $\varepsilon > \delta > 0$. If $x \in s_\varepsilon^\alpha(K)$ and $\|U_{\mathcal{R}}^* x\|^q > |K|^q - (\frac{\varepsilon - \delta}{2})^q$, then $U_{\mathcal{R}}^* x \in s_\delta^\alpha(U_{\mathcal{R}}^* K)$.*

Proof. We fix ε, δ and \mathcal{R} and proceed by induction on α . The conclusion of the lemma is trivially true for $\alpha = 0$. So suppose that β is an ordinal such that the conclusion of the lemma holds with $\alpha = \beta$; we show that it holds then also for $\alpha = \beta + 1$. To this end, let $x \in K$ be such that $\|U_{\mathcal{R}}^* x\|^q > |K|^q - (\frac{\varepsilon - \delta}{2})^q$ and $U_{\mathcal{R}}^* x \notin s_\delta^{\beta+1}(U_{\mathcal{R}}^* K)$. Our goal is to show that $x \notin s_\varepsilon^{\beta+1}(K)$, so we may assume that $x \in s_\varepsilon^\beta(K)$, hence $U_{\mathcal{R}}^* x \in s_\delta^\beta(U_{\mathcal{R}}^* K)$ by the inductive hypothesis. It follows that there is w^* -open $V \subseteq (\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p^*$ such that $U_{\mathcal{R}}^* x \in V$ and $d := \text{diam}(V \cap s_\delta^\beta(U_{\mathcal{R}}^* K)) \leq \delta$. As $U_{\mathcal{R}}^* x$ does not belong to the w^* -closed set $(|K|^q - (\frac{\varepsilon - \delta}{2})^q)^{1/q} B_{(\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p^*}$, we may assume

$$V \cap \left(|K|^q - \left(\frac{\varepsilon - \delta}{2}\right)^q \right)^{1/q} B_{(\bigoplus_{\lambda \in \mathcal{R}} E_\lambda)_p^*} = \emptyset.$$

Let $W = (U_{\mathcal{R}}^*)^{-1}(V)$ and let $u \in W \cap s_\varepsilon^\beta(K)$. Then $\|U_{\mathcal{R}}^* u\|^q > |K|^q - (\frac{\varepsilon - \delta}{2})^q$ and $u \in s_\varepsilon^\beta(K)$, hence by the induction hypothesis $U_{\mathcal{R}}^* u \in V \cap s_\delta^\beta(U_{\mathcal{R}}^* K)$. So for $u_1, u_2 \in W \cap s_\varepsilon^\beta(K)$ we have $\|U_{\mathcal{R}}^* u_1 - U_{\mathcal{R}}^* u_2\|^q \leq d^q \leq \delta^q$. Moreover, since $\|U_{\mathcal{R}}^* u_1\|^q > |K|^q - (\frac{\varepsilon - \delta}{2})^q$ it follows that

$$\|u_1 - P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_1\| \leq (|K|^q - \|P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_1\|^q)^{1/q} = (|K|^q - \|U_{\mathcal{R}}^* u_1\|^q)^{1/q} < \frac{\varepsilon - \delta}{2}.$$

Similarly, $\|u_2 - P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_2\| < \frac{\varepsilon - \delta}{2}$. We now deduce that

$$\begin{aligned} \|u_1 - u_2\|^q &= \|P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_1 - P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_2\|^q + \|(u_1 - P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_1) - (u_2 - P_{\mathcal{R}}^* U_{\mathcal{R}}^* u_2)\|^q \\ &\leq \|U_{\mathcal{R}}^* u_1 - U_{\mathcal{R}}^* u_2\|^q + \left(2 \cdot \frac{\varepsilon - \delta}{2}\right)^q \\ &\leq \delta^q + (\varepsilon - \delta)^q \\ &\leq \varepsilon^q. \end{aligned}$$

In particular, $\text{diam}(W \cap s_{\varepsilon}^{\beta}(K)) \leq \varepsilon$. It follows that $x \notin s_{\varepsilon}^{\beta+1}(K)$, as desired.

The lemma passes easily to limit ordinals, so we are done. \square

In order to state the third (and final) lemma required in the proof of Proposition 2.4, we give the following definition.

Definition 2.7. For real numbers $a \geq 0, b > c > 0$ and $1 \leq d < \infty$, define

$$\sigma(a, b, c, d) := \inf \left\{ n \in \mathbb{N} \mid n \geq \left(\frac{2a}{b-c}\right)^d - \left(\frac{b}{b-c}\right)^d + 1 \right\}.$$

With regards to Definition 2.7, note that $\sigma(a, b, c, d) = 1$ whenever $2a \leq b$.

Lemma 2.8. Let Λ be a set, $\{E_{\lambda} \mid \lambda \in \Lambda\}$ a family of Banach spaces, $1 \leq q < \infty, p$ predual to $q, K \subseteq (\bigoplus_{\lambda \in \Lambda} E_{\lambda})_p^*$ a nonempty, w^* -compact set and $\varepsilon > \delta > 0$. Suppose η_{δ} is a nonzero ordinal such that $s_{\delta}^{\eta_{\delta}}(U_{\mathcal{F}}^* K) = \emptyset$ for every $\mathcal{F} \in \Lambda^{<\infty}$. Then $s_{\varepsilon}^{\eta_{\delta} \cdot \sigma(|K|, \varepsilon, \delta, q)}(K) = \emptyset$.

Proof. We claim that for each $n < \omega$, either $s_{\varepsilon}^{\eta_{\delta} \cdot n}(K)$ is empty or

$$|s_{\varepsilon}^{\eta_{\delta} \cdot n}(K)|^q \leq |K|^q - n \left(\frac{\varepsilon - \delta}{2}\right)^q. \tag{2.1}$$

To prove the claim, we proceed by induction on n . (2.1) holds trivially for $n = 0$. Suppose the claim holds for $n = m$; we will show that it holds for $n = m + 1$. For every $\mathcal{F} \in \Lambda^{<\infty}$ we have

$$s_{\delta}^{\eta_{\delta}}(U_{\mathcal{F}}^* s_{\varepsilon}^{\eta_{\delta} \cdot (m+1)}(K)) \subseteq s_{\delta}^{\eta_{\delta}}(U_{\mathcal{F}}^* K) = \emptyset. \tag{2.2}$$

If $s_{\varepsilon}^{\eta_{\delta} \cdot m}(K) = \emptyset$, we are done. Otherwise, by the induction hypothesis,

$$|s_{\varepsilon}^{\eta_{\delta} \cdot (m+1)}(K)|^q \leq |K|^q - m \left(\frac{\varepsilon - \delta}{2}\right)^q. \tag{2.3}$$

If $s_{\varepsilon}^{\eta_{\delta} \cdot (m+1)}(K) \neq \emptyset$, then applying (2.2), (2.3) and Lemma 2.6 implies that for every $x \in s_{\varepsilon}^{\eta_{\delta} \cdot (m+1)}(K)$ and $\mathcal{F} \in \Lambda^{<\infty}$, we have

$$\|U_{\mathcal{F}}^* x\|^q \leq |K|^q - m \left(\frac{\varepsilon - \delta}{2}\right)^q - \left(\frac{\varepsilon - \delta}{2}\right)^q = |K|^q - (m + 1) \left(\frac{\varepsilon - \delta}{2}\right)^q.$$

Thus $x \in s_\varepsilon^{\eta_\delta \cdot (m+1)}(K)$ implies

$$\|x\|^q = \sup\{\|U_{\mathcal{F}}^*x\|^q \mid \mathcal{F} \in \Lambda^{<\infty}\} \leq |K|^q - (m + 1)\left(\frac{\varepsilon - \delta}{2}\right)^q,$$

and so (2.1) holds for $n = m + 1$. The inductive proof of the claim is complete.

By definition (precisely, Definition 2.7), we have

$$|K|^q - (\sigma(|K|, \varepsilon, \delta, q) - 1)\left(\frac{\varepsilon - \delta}{2}\right)^q \leq \left(\frac{\varepsilon}{2}\right)^q. \tag{2.4}$$

Thus, by (2.4) and the claim proved above we have

$$\text{diam}(s_\varepsilon^{\eta_\delta \cdot (\sigma(|K|, \varepsilon, \delta, q) - 1)}(K)) \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon,$$

and we thus deduce that

$$s_\varepsilon^{\eta_\delta \cdot \sigma(|K|, \varepsilon, \delta, q)}(K) \subseteq s_\varepsilon^{\eta_\delta \cdot (\sigma(|K|, \varepsilon, \delta, q) - 1) + 1}(K) = s_\varepsilon(s_\varepsilon^{\eta_\delta \cdot (\sigma(|K|, \varepsilon, \delta, q) - 1)}(K)) = \emptyset. \quad \square$$

We now give the proof of Proposition 2.4, assuming Lemma 2.5.

Proof of Proposition 2.4. We prove (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). Throughout, p shall denote the real number preducal to q .

To show (i) \Rightarrow (ii), suppose by way of a contraposition that there is $\varepsilon > 0$ such that $\sup\{\text{Sz}_\varepsilon(K_\lambda) \mid \lambda \in \Lambda\} \geq \omega^\alpha$. For each $\lambda' \in \Lambda$, the restriction $P_{[\lambda']}^*|_{K_{\lambda'}}$ is a norm-isometric, w^* -homeomorphic embedding of $K_{\lambda'}$ into $B_q(K_\lambda \mid \lambda \in \Lambda)$, hence $\text{Sz}_\delta(B_q(K_\lambda \mid \lambda \in \Lambda)) \geq \text{Sz}_\delta(K_{\lambda'})$ for all $\delta > 0$ and $\lambda' \in \Lambda$. Thus

$$\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda)) \geq \sup\{\text{Sz}_\varepsilon(K_\lambda) \mid \lambda \in \Lambda\} \geq \omega^\alpha. \tag{2.5}$$

As $\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda))$ cannot be a limit ordinal, we deduce from (2.5) that

$$\text{Sz}(B_q(K_\lambda \mid \lambda \in \Lambda)) \geq \text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda)) > \omega^\alpha.$$

This proves (i) \Rightarrow (ii).

Suppose (ii) holds. For each $\varepsilon > 0$ let $1 < m_\varepsilon < \omega$ and $\beta_\varepsilon < \alpha$ be such that $\sup\{\text{Sz}_{\varepsilon/32}(K_\lambda) \mid \lambda \in \Lambda\} < \omega^{\beta_\varepsilon \cdot m_\varepsilon}$. Set $d = \sup\{\text{diam}(K_\lambda) \mid \lambda \in \Lambda\}$ and for each $\varepsilon \in (0, 1)$ let $M_\varepsilon \in \mathbb{N}$ be such that $(2^q - 1)\varepsilon^q M_\varepsilon \geq 8^q d^q (m_\varepsilon - 1)$. By Lemma 2.5, for $\mathcal{F} \in \Lambda^{<\infty}$ we have $\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \mathcal{F})) < \omega^{\beta_\varepsilon} \cdot M_\varepsilon$, hence

$$\sup\{\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda)) \mid \mathcal{F} \in \Lambda^{<\infty}\} \leq \omega^{\beta_\varepsilon} \cdot M_\varepsilon < \omega^\alpha.$$

Thus, (ii) \Rightarrow (iii).

Suppose that (iii) holds. As $U_{\mathcal{F}}^*B_q(K_\lambda \mid \lambda \in \Lambda) = B_q(K_\lambda \mid \lambda \in \mathcal{F})$ for each $\mathcal{F} \in \Lambda^{<\infty}$, applying Lemma 2.8 with $K = B_q(K_\lambda \mid \lambda \in \Lambda)$, $\delta = \delta(\varepsilon) = \varepsilon/2$ and $\eta_{\delta(\varepsilon)} = \sup\{\text{Sz}_{\varepsilon/2}(U_{\mathcal{F}}^*B_q(K_\lambda \mid \lambda \in \Lambda)) \mid \mathcal{F} \in \Lambda^{<\infty}\} (< \omega^\alpha)$ yields

$$\begin{aligned} \text{Sz}(B_q(K_\lambda \mid \lambda \in \Lambda)) &= \sup\{\text{Sz}_\varepsilon(B_q(K_\lambda \mid \lambda \in \Lambda)) \mid \varepsilon > 0\} \\ &\leq \sup\{\eta_{\delta(\varepsilon)} \cdot \sigma(\sup\{|K_\lambda| \mid \lambda \in \Lambda\}, \varepsilon, \varepsilon/2, q) \mid \varepsilon > 0\} \\ &\leq \omega^\alpha, \end{aligned}$$

hence (iii) \Rightarrow (i). \square

Remark 2.9. The idea that an iterated implementation of Lemma 2.6 (c.f. Lemma 2.8 and its proof) might be used to prove the implication (iii) \Rightarrow (i) in the proof of Proposition 2.4 was essentially suggested to the author by Gilles Lancien; previous versions of the main results of this chapter used a slightly different argument (also using Lemma 2.6, but just a single direct application) and required the additional hypothesis that $K_\lambda = B_{E_\lambda^*}$ for all λ (see Theorem 2.11).

The following result, along with Proposition 2.2, determines precisely the Szlenk index of a c_0 -direct sum or ℓ_p -direct sum of operators ($1 < p < \infty$) in terms of properties of the ε -Szlenk indices of the summands.

Theorem 2.10. *Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ and $\{F_\lambda \mid \lambda \in \Lambda\}$ families of Banach spaces, $\{T_\lambda : E_\lambda \rightarrow F_\lambda \mid \lambda \in \Lambda\}$ a uniformly bounded family of Asplund operators, $\alpha > 0$ an ordinal and $p = 0$ or $1 < p < \infty$. The following are equivalent:*

- (i) $\text{Sz}((\bigoplus_{\lambda \in \Lambda} T_\lambda)_p) \leq \omega^\alpha$.
- (ii) $\sup\{\text{Sz}_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha$ for all $\varepsilon > 0$.

It follows that if $(\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$ is noncompact, then

$$\text{Sz}\left(\left(\bigoplus_{\lambda \in \Lambda} T_\lambda\right)_p\right) = \inf\{\omega^\alpha \mid \sup\{\text{Sz}_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for all } \varepsilon > 0\}.$$

Proof. For convenience we set $T = (\bigoplus_{\lambda \in \Lambda} T_\lambda)_p$. The equivalence of (i) and (ii) is achieved by applying Proposition 2.4 with $K_\lambda = T_\lambda^* B_{F_\lambda^*}$ for all $\lambda \in \Lambda$, for in this case $T^* B_{(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p^*} = B_q(T_\lambda^* B_{F_\lambda^*} \mid \lambda \in \Lambda)$, where $q \in [1, \infty)$ is dual to p .

For each $\lambda \in \Lambda$ let α_λ denote the unique ordinal satisfying $\text{Sz}(T_\lambda) = \omega^{\alpha_\lambda}$. Let $\alpha_\Lambda = \sup\{\alpha_\lambda \mid \lambda \in \Lambda\}$. The set

$$\{\omega^\alpha \mid \sup\{\text{Sz}_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for all } \varepsilon > 0\} \ni \omega^{\alpha_\Lambda+1}$$

is nonempty, hence

$$\text{Sz}(T) \leq \inf\{\omega^\alpha \mid \sup\{\text{Sz}_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for all } \varepsilon > 0\}$$

by the implication (ii) \Rightarrow (i) above.

To complete the proof, we now suppose that T is noncompact. As $\text{Sz}(T)$ is a power of ω , it is enough to show that $\text{Sz}(T) > \omega^\beta$ holds for β satisfying $\omega^\beta < \inf\{\omega^\alpha \mid \sup\{\text{Sz}_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for all } \varepsilon > 0\}$. Take such β . If $\beta = 0$, then $\text{Sz}(T) > \omega^\beta$ by noncompactness of T . On the other hand, if $\beta > 0$ then there is $\varepsilon > 0$ so small that $\text{Sz}_\varepsilon(T) \geq \sup\{\text{Sz}_\varepsilon(T_\lambda) \mid \lambda \in \Lambda\} \geq \omega^\beta$. As $\text{Sz}_\varepsilon(T)$ cannot be a limit ordinal, we conclude that $\text{Sz}(T) \geq \text{Sz}_\varepsilon(T) > \omega^\beta$. \square

2.3. Applications

Our first result here is the following Banach space analogue of Theorem 2.10 which determines precisely the Szlenk index of a c_0 -direct sum or ℓ_p -direct sum of Banach spaces in terms of the behaviour of the ε -Szlenk indices of the summand spaces.

Theorem 2.11. *Let Λ be a set, $\{E_\lambda \mid \lambda \in \Lambda\}$ a family of Asplund spaces, $\alpha > 0$ an ordinal and $p = 0$ or $1 < p < \infty$. The following are equivalent:*

- (i) $Sz((\bigoplus_{\lambda \in \Lambda} E_\lambda)_p) \leq \omega^\alpha$.
- (ii) $\sup\{Sz_\varepsilon(E_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha$ for all $\varepsilon > 0$.

It follows that if $(\bigoplus_{\lambda \in \Lambda} E_\lambda)_p$ is infinite dimensional, then

$$Sz\left(\left(\bigoplus_{\lambda \in \Lambda} E_\lambda\right)_p\right) = \inf\{\omega^\alpha \mid \sup\{Sz_\varepsilon(E_\lambda) \mid \lambda \in \Lambda\} < \omega^\alpha \text{ for all } \varepsilon > 0\}.$$

Proof. The conclusions of the theorem follow by taking T_λ to be the identity operator of E_λ for each $\lambda \in \Lambda$ in the statement of Theorem 2.10. \square

Theorem 2.12. *Let Λ be a set, E an infinite dimensional Banach space and $1 < p < \infty$. Then*

$$Sz(E) = Sz(c_0(\Lambda, E)) = Sz(\ell_p(\Lambda, E)).$$

Proof. Apply Theorem 2.11 with $E_\lambda = E$ for all $\lambda \in \Lambda$. \square

The previous theorem, Theorem 2.12, allows us to add to the class of ordinals γ for which the Szlenk index of $C(\gamma + 1)$ is known (here, $\gamma + 1$ is equipped with its order topology). The computation of the Szlenk index of $C(\omega_1 + 1)$, in particular $Sz(C(\omega_1 + 1)) = \omega_1 \cdot \omega$, is due to Hájek and Lancien [7]. Essentially using the fact that $Sz(C(\xi + 1)) = Sz(C(\zeta + 1))$ for ordinals ξ and ζ satisfying $\xi \leq \zeta < \xi \cdot \omega$ (an easy consequence of Proposition 1.1(v)), Hájek and Lancien deduce that $Sz(C(\gamma + 1)) = \omega_1 \cdot \omega$ whenever $\omega_1 \leq \gamma < \omega_1 \cdot \omega$. We claim that $Sz(C(\gamma + 1)) = \omega_1 \cdot \omega$ whenever $\omega_1 \leq \gamma < \omega_1 \cdot \omega^\omega$, a fact that will follow once we have shown that $Sz(C(\xi + 1)) = Sz(C(\zeta + 1))$ whenever ξ and ζ are ordinals satisfying $\omega \leq \xi \leq \zeta < \xi \cdot \omega^\omega$. If ξ and ζ are ordinals satisfying $\omega \leq \xi \leq \zeta < \xi \cdot \omega^\omega$, then there exists $n < \omega$ such that $C(\zeta + 1)$ is isomorphic to a subspace of $C(\xi \cdot \omega^n + 1)$. Thus, by Proposition 1.1(i), it suffices to show that $Sz(C(\xi + 1)) = Sz(C(\xi \cdot \omega^n + 1))$ for all $n < \omega$. This is obviously true for $n = 0$, and if true for some n then, since $C(\xi \cdot \omega^{n+1} + 1)$ is isomorphic to $c_0(\omega, C(\xi \cdot \omega^n + 1))$, Theorem 2.12 yields

$$\begin{aligned} Sz(C(\xi \cdot \omega^{n+1} + 1)) &= Sz(c_0(\omega, C(\xi \cdot \omega^n + 1))) \\ &= Sz(C(\xi \cdot \omega^n + 1)) \\ &= Sz(C(\xi + 1)), \end{aligned}$$

which completes the proof.

The following proposition asserts that the set of all countable values of the Szlenk index of Banach spaces is attained by the class of Banach spaces with a shrinking basis. A further consequence of this result is that if for $\alpha < \omega_1$ there exists a Banach space of Szlenk index ω^α , then Pełczyński’s complementably universal basis space (see [16]) has a complemented subspace of Szlenk index ω^α .

Proposition 2.13. *Let $0 < \alpha < \omega_1$. The following are equivalent:*

- (i) *There exists a Banach space E with $Sz(E) = \omega^\alpha$.*
- (ii) *There exists a Banach space E with a shrinking basis and $Sz(E) = \omega^\alpha$.*

To prove Proposition 2.13, we shall call on the following result regarding subspaces and quotients, due to G. Lancien [13] and [11, Theorem III.1]:

Proposition 2.14. *Let $\beta < \omega_1$ and let E be a Banach space such that $Sz(E) > \beta$.*

- (i) *There is a separable closed subspace F of E such that $Sz(F) > \beta$.*
- (ii) *If E^* is norm separable, then for every $\delta > 0$ there is a closed subspace F of E such that $Sz(E/F) > \beta$ and E/F has a shrinking basis with basis constant not exceeding $1 + \delta$.*

With the exception of the basis constant assertion of part (ii), Proposition 2.14 is proved in [13]. Lancien’s proof follows closely the proof of [11, Theorem III.1], and the extra assertion above regarding the basis constant is easily added to Lancien’s result using the observations regarding basis constants in the proof of [11, Theorem III.1].

Proposition 2.13 is an immediate consequence of the following:

Proposition 2.15. *Let $\alpha > 0$ be a countable ordinal and E a Banach space with $Sz(E) = \omega^\alpha$. Then there exist closed subspaces $F \subseteq E$ and $G \subseteq \ell_2(F)$ such that $\ell_2(F)/G$ has a shrinking basis and $Sz(\ell_2(F)/G) = \omega^\alpha$.*

Proof. For each $n \in \mathbb{N}$, Proposition 2.14(i) yields a separable closed subspace D_n of E such that $Sz(D_n) > Sz_{1/n}(E)$. Let $F = \overline{\text{span}}(\bigcup_{n \in \mathbb{N}} D_n)$. Then

$$\omega^\alpha = Sz(E) = \sup_n Sz_{1/n}(E) \leq \sup_n Sz(D_n) \leq Sz(F) \leq Sz(E) = \omega^\alpha,$$

hence equality holds throughout. In particular, $Sz(F) = \omega^\alpha$ and, as F is a separable Asplund space (indeed, $Sz(F) < \infty$), F^* is norm separable. For each $n \in \mathbb{N}$ let $F_n = F$. Then, by Proposition 2.14(ii), for each $n \in \mathbb{N}$ there is a closed subspace G_n of F_n such that $Sz(F_n/G_n) > Sz_{1/n}(E)$ and F_n/G_n has a shrinking basis with basis constant not exceeding 2. Let G denote the image of $(\bigoplus_{n \in \mathbb{N}} G_n)_2$ under its natural embedding into $(\bigoplus_{n \in \mathbb{N}} F_n)_2$. Then $(\bigoplus_{n \in \mathbb{N}} F_n)_2/G$ is naturally isometrically isomorphic to $(\bigoplus_{n \in \mathbb{N}} F_n/G_n)_2$. Note that $(\bigoplus_{n \in \mathbb{N}} F_n/G_n)_2$ has a shrinking basis since it is the ℓ_2 -direct sum of a countable family of Banach spaces with shrinking bases that have uniformly bounded basis constants. On the one hand, by Theorem 2.12 we have

$$Sz\left(\left(\bigoplus_{n \in \mathbb{N}} F_n\right)_2 / G\right) \leq Sz\left(\left(\bigoplus_{n \in \mathbb{N}} F_n\right)_2\right) = Sz(F) = \omega^\alpha.$$

On the other hand,

$$\text{Sz}\left(\left(\bigoplus_{n \in \mathbb{N}} F_n\right)_2 / G\right) = \text{Sz}\left(\left(\bigoplus_{n \in \mathbb{N}} F_n / G_n\right)_2\right) \geq \sup_n \text{Sz}_{1/n}(E_n) = \text{Sz}(E) = \omega^\alpha.$$

Thus $(\bigoplus_{n \in \mathbb{N}} F_n)_2 / G$ has a shrinking basis and Szlenk index ω^α . \square

Proposition 2.16. *Let α be an ordinal. Then there exists a Banach space of Szlenk index $\omega^{\alpha+1}$.*

Proof. Our proof is based on the construction of Szlenk in [22], by which we construct Banach spaces E_β indexed by the class of ordinals β . Let $E_0 = \{0\}$, $E_{\beta+1} = E_\beta \oplus_1 \ell_2$ and, if β is a limit ordinal, $E_\beta = (\bigoplus_{\gamma < \beta} E_\gamma)_2$. It is shown in [14, Theorem 4] that for this construction we have $\text{Sz}_1(E_\beta) > \beta$ for all ordinals β . As the assertion of the proposition is known to be true for $\alpha = 0$ (for example, $\text{Sz}(\ell_2) = \omega$), we assume that $\alpha > 0$ and let β' denote the least ordinal such that $\text{Sz}(E_{\beta'}) > \omega^\alpha$. Then, by Proposition 1.1(iii), $\text{Sz}(E_{\beta'}) \geq \omega^{\alpha+1}$. By Proposition 1.1(v) and the definition of β' , it must be that β' is a limit ordinal, hence $E_{\beta'} = (\bigoplus_{\beta'' < \beta'} E_{\beta''})_2$. It follows that $\text{Sz}(E_{\beta'}) = \text{Sz}((\bigoplus_{\beta'' < \beta'} E_{\beta''})_2) \leq \omega^{\alpha+1}$, where the final inequality here follows from Theorem 2.11 and the fact that, for all $\varepsilon > 0$,

$$\sup\{\text{Sz}_\varepsilon(E_{\beta''}) \mid \beta'' < \beta'\} \leq \sup\{\text{Sz}(E_{\beta''}) \mid \beta'' < \beta'\} \leq \omega^\alpha < \omega^{\alpha+1}.$$

It is now clear that $\text{Sz}(E_{\beta'}) = \omega^{\alpha+1}$, so we are done. \square

Implicit in the proof of Proposition 2.16 is the following fact: for a set Λ , Banach spaces $\{E_\lambda \mid \lambda \in \Lambda\}$, $p = 0$ or $1 < p < \infty$ and α an ordinal satisfying $\sup\{\text{Sz}(E_\lambda) \mid \lambda \in \Lambda\} \leq \omega^\alpha$, we have $\text{Sz}((\bigoplus_{\lambda \in \Lambda} E_\lambda)_p) \leq \omega^{\alpha+1}$. This follows easily from Theorem 2.11, but seems to have been known for some time. For example, the separable case was established in [15, Proposition 15], and the result is also implicit in the proof of [14, Proposition 5].

Propositions 2.16 and 2.13 concern themselves with the existence of Banach spaces having a particular Szlenk index. The author is not aware of a complete classification of the possible values of the Szlenk index of a Banach space. Proposition 1.1(iii) asserts that the Szlenk index of a Banach space is a power of ω . On the other hand, as the Szlenk index of a Banach space E is the supremum of the countable set $\{\text{Sz}_{1/n}(E) \mid n \in \mathbb{N}\}$, it follows that the Szlenk index of a Banach space is of countable cofinality. In particular, if α is an ordinal of uncountable cofinality, then α is a limit ordinal and ω^α cannot be the Szlenk index a Banach space since $cf(\omega^\alpha) = cf(\alpha) \geq \omega_1$. In view of this fact and Proposition 2.16, a complete classification of values of the Szlenk index of Banach spaces will be achieved if one establishes an affirmative answer to the following question, which we believe to be open:

Question 2.17. Let α be an ordinal with $cf(\alpha) = \omega$. Does there exist a Banach space with Szlenk index equal to ω^α ?

A partial answer to Question 2.17 is found in [15] where it is shown that if $\mathcal{T}_{\omega^\alpha}$ denotes the ω^α th Tsirel'son space, where $\alpha < \omega_1$, then $\text{Sz}(\mathcal{T}_{\omega^\alpha}) = \omega^{\omega^{\alpha+1}}$. The values taken by the Szlenk index on the class of all operators between Banach spaces will be determined in Proposition 2.18 below.

To conclude the current section, we now apply Proposition 2.16 to obtain, amongst other things, a characterisation of those limit ordinals α for which the operator ideal $\bigcup_{\beta < \alpha} \mathcal{S}\mathcal{L}_\alpha$ is closed.

Proposition 2.18. *Let $\alpha > 0$ be an ordinal. The following are equivalent:*

- (i) $cf(\alpha) \geq \omega_1$.
- (ii) ω^α is not the Szlenk index of any operator between Banach spaces.
- (iii) $\mathcal{S}\mathcal{L}_\alpha = \bigcup_{\beta < \alpha} \mathcal{S}\mathcal{L}_\beta$.
- (iv) α is a limit ordinal and $\bigcup_{\beta < \alpha} \mathcal{S}\mathcal{L}_\beta$ is closed.

Proof. We will show that (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).

To see that (i) \Rightarrow (ii), suppose that there exists an operator T such that $\omega^\alpha = Sz(T) = \sup\{Sz_{1/n}(T) \mid n \in \mathbb{N}\}$. Then $cf(\alpha) \leq cf(\omega^\alpha) = \omega < \omega_1$.

The implication (ii) \Rightarrow (iii) is immediate from Proposition 1.1(iii).

Now suppose that (iii) holds. Then $\bigcup_{\beta < \alpha} \mathcal{S}\mathcal{L}_\beta$ is closed by Theorem 1.3. Moreover, α is a limit ordinal. Indeed, otherwise we may write $\alpha = \zeta + 1$, where ζ is an ordinal, and by Proposition 2.16 there exists a Banach space E such that $I_E \in \mathcal{S}\mathcal{L}_{\zeta+1} \setminus \mathcal{S}\mathcal{L}_\zeta = \mathcal{S}\mathcal{L}_\alpha \setminus \bigcup_{\beta < \alpha} \mathcal{S}\mathcal{L}_\beta = \emptyset$, which is absurd.

Finally, we show that (iv) \Rightarrow (i). Suppose by way of a contraposition that $cf(\alpha) = \omega$ and let $\{\alpha_n \mid n < \omega\} \subseteq \alpha$ be cofinal in α . Then $\{\alpha_n + 1 \mid n < \omega\}$ is also cofinal in α , and $\bigcup_{n < \omega} \mathcal{S}\mathcal{L}_{\alpha_n+1} = \bigcup_{\beta < \alpha} \mathcal{S}\mathcal{L}_\beta$. So to complete the proof, it suffices to construct an operator $T \in \overline{\bigcup_{n < \omega} \mathcal{S}\mathcal{L}_{\alpha_n+1}} \setminus \bigcup_{n < \omega} \mathcal{S}\mathcal{L}_{\alpha_n+1}$. To this end, for each $n < \omega$ let E_n be a Banach space whose Szlenk index is ω^{α_n+1} (cf. Proposition 2.16), and set $E = (\bigoplus_{n < \omega} E_n)_2$. Define $T \in \mathcal{B}(E)$ by setting $T(x_n)_{n < \omega} = ((n + 1)^{-1}x_n)_{n < \omega}$ for each $(x_n)_{n < \omega} \in E$. Since T factors I_{E_n} for each $n < \omega$, we have

$$Sz(T) \geq \sup\{Sz(E_n) \mid n < \omega\} = \sup\{\omega^{\alpha_n+1} \mid n < \omega\} = \omega^\alpha,$$

hence $T \notin \bigcup_{n < \omega} \mathcal{S}\mathcal{L}_{\alpha_n+1}$. On the other hand, with A_m ($m < \omega$) denoting the operator on E that sends $(x_n)_{n < \omega} \in E$ to the element $(y_n)_{n < \omega}$ of E that satisfies $y_n = x_n$ if $n \leq m$, and $y_n = 0$ otherwise, we have that $I_{E_1 \oplus \dots \oplus E_m}$ factors $A_m T$ for all $m < \omega$, hence

$$Sz(A_m T) \leq Sz(E_1 \oplus \dots \oplus E_m) = \max\{\omega^{\alpha_i+1} \mid 1 \leq i \leq m\}.$$

In particular, $A_m T \in \bigcup_{n < \omega} \mathcal{S}\mathcal{L}_{\alpha_n+1}$ for $m < \omega$. As $\lim_{m \rightarrow \omega} \|A_m T - T\| = 0$, it follows that $T \in \overline{\bigcup_{n < \omega} \mathcal{S}\mathcal{L}_{\alpha_n+1}(E)}$. \square

Remark 2.19. The existence of an operator of Szlenk index ω^α whenever $cf(\alpha) \leq \omega$ (Proposition 2.18(ii) \Rightarrow (i)) is used in the proof of [2, Theorem 5.1], where it is shown that if β is an ordinal with $cf(\beta) \leq \omega$, then $\mathcal{S}\mathcal{L}_{\omega^\beta}$ does not have the factorisation property.

3. Proof of Lemma 2.5

Our goal in this section is to prove Lemma 2.5. We proceed via a sequence of lemmas, whose general theme is to establish upper bounds (in terms of set containment) on various derived

sets $s_\varepsilon^\alpha(K)$, where K is w^* -compact, α is an ordinal and $\varepsilon > 0$. The sets K that we shall consider are typically direct products, for it will be seen later that the set $B_q(K_i \mid 1 \leq i \leq n)$ in the statement of Lemma 2.5 can be ‘approximated’ from above (with respect to set containment) in a convenient way by a finite union of direct products of w^* -compact sets. Indeed, this so-called approximation of $B_q(K_i \mid 1 \leq i \leq n)$ plays a key role in our proof.

We mention another important aspect of our results in this section. As noted earlier, Lemma 2.5 is used to establish the implication (ii) \Rightarrow (iii) of Proposition 2.4. Note that in the statement of Proposition 2.4(iii), there is no (finite) upper bound on the cardinality of the finite sets $\mathcal{F} \in \Lambda^{<\omega}$. It is thus important for us in this section, when aiming for estimates of ε -Szlenk indices of direct products, to obtain estimates that are independent of the (finite) number of factors in a given direct product. Our efforts in this regard are reflected in the fact that the numbers M and n in the statement of Lemma 2.5 are independent of one another.

We first establish the following general result regarding the behaviour of s_ε^α derivatives of finite unions of w^* -compact sets.

Lemma 3.1. *Let E be a Banach space, $K_1, \dots, K_n \subseteq E^*$ w^* -compact sets and $\varepsilon > 0$. Let α be an ordinal and $m < \omega$. Then:*

- (i) $s_\varepsilon^\alpha(\bigcup_{i=1}^n K_i) \subseteq \bigcup_{i=1}^n s_{\varepsilon/2}^\alpha(K_i)$.
- (ii) $s_\varepsilon^{mn}(\bigcup_{i=1}^n K_i) \subseteq \bigcup_{i=1}^n s_\varepsilon^m(K_i)$.
- (iii) *If α is a limit ordinal, then $s_\varepsilon^\alpha(\bigcup_{i=1}^n K_i) \subseteq \bigcup_{i=1}^n s_\varepsilon^\alpha(K_i)$.*

Proof. (i) holds trivially for $\alpha = 0$. Suppose that β is an ordinal such that (i) holds for all $\alpha \leq \beta$ and let $x \in E^* \setminus \bigcup_{i=1}^n s_{\varepsilon/2}^{\beta+1}(K_i)$. Then for $1 \leq i \leq n$ there is w^* -open $U_i \ni x$ such that $\text{diam}(U_i \cap s_{\varepsilon/2}^\beta(K_i)) \leq \varepsilon/2$. It follows that for $x_1, x_2 \in (\bigcap_{i=1}^n U_i) \cap (s_\varepsilon^\beta(\bigcup_{i=1}^n K_i))$ we have

$$\|x_1 - x_2\| \leq \|x_1 - x\| + \|x - x_2\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $\text{diam}((\bigcap_{i=1}^n U_i) \cap (s_\varepsilon^\beta(\bigcup_{i=1}^n K_i))) \leq \varepsilon$. In particular, $x \notin s_\varepsilon^{\beta+1}(\bigcup_{i=1}^n K_i)$, and so (i) passes to successor ordinals.

Suppose that β is a limit ordinal such that (i) holds for all $\alpha < \beta$. Then

$$s_\varepsilon^\beta\left(\bigcup_{i=1}^n K_i\right) = \bigcap_{\alpha < \beta} s_\varepsilon^\alpha\left(\bigcup_{i=1}^n K_i\right) \subseteq \bigcap_{\alpha < \beta} \bigcup_{i=1}^n s_{\varepsilon/2}^\alpha(K_i). \tag{3.1}$$

Let $x \in s_\varepsilon^\beta(\bigcup_{i=1}^n K_i)$. Then for each $\alpha < \beta$ we may choose $i_\alpha \in \{1, \dots, n\}$ such that $x \in s_{\varepsilon/2}^\alpha(K_{i_\alpha})$, and for some $i' \in \{1, \dots, n\}$ the set $\{\alpha < \beta \mid i_\alpha = i'\}$ is cofinal in β . Hence

$$x \in \bigcap_{i_\alpha=i'} s_{\varepsilon/2}^\alpha(K_{i'}) = \bigcap_{\alpha < \beta} s_{\varepsilon/2}^\alpha(K_{i'}) = s_{\varepsilon/2}^\beta(K_{i'}) \subseteq \bigcup_{i=1}^n s_{\varepsilon/2}^\beta(K_i). \tag{3.2}$$

Since $x \in s_\varepsilon^\beta(\bigcup_{i=1}^n K_i)$ was arbitrary, (i) passes to limit ordinals, and thus holds for all ordinals α .

Statement (ii) is trivial for $m = 0$. To see that it is true for $m = 1$, we first let $\mathbb{P}_k = \{\mathcal{F} \subseteq \{1, \dots, n\} \mid |\mathcal{F}| = k\}$, $k \in \mathbb{N}$. It suffices to show that for all $l < \omega$,

$$s_\varepsilon^l \left(\bigcup_{i=1}^n K_i \right) \subseteq \left(\bigcup_{i=1}^n s_\varepsilon(K_i) \right) \cup \left(\bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i \right). \tag{3.3}$$

Indeed, taking $l = n$ in (3.3) gives (ii) with $m = 1$ (since $\bigcup_{\mathcal{F} \in \mathbb{P}_{l+1}} \bigcap_{i \in \mathcal{F}} K_i = \emptyset$ when $l = n$). It is clear that (3.3) holds for $l = 0$. Suppose now $l' < \omega$ is such that (3.3) holds for $l = l'$; we show that it holds also for $l = l' + 1$. Let

$$x \in E^* \setminus \left(\left(\bigcup_{i=1}^n s_\varepsilon(K_i) \right) \cup \left(\bigcup_{\mathcal{G} \in \mathbb{P}_{l'+2}} \bigcap_{j \in \mathcal{G}} K_j \right) \right).$$

We want to show that $x \notin s_\varepsilon^{l'+1}(\bigcup_{i=1}^n K_i)$, so by the induction hypothesis it suffices to assume that

$$x \in s_\varepsilon^{l'} \left(\bigcup_{i=1}^n K_i \right) \subseteq \left(\bigcup_{i=1}^n s_\varepsilon(K_i) \right) \cup \left(\bigcup_{\mathcal{F} \in \mathbb{P}_{l'+1}} \bigcap_{i \in \mathcal{F}} K_i \right),$$

hence

$$x \in \left(\bigcup_{\mathcal{F} \in \mathbb{P}_{l'+1}} \bigcap_{i \in \mathcal{F}} K_i \right) \setminus \left(\left(\bigcup_{i=1}^n s_\varepsilon(K_i) \right) \cup \left(\bigcup_{\mathcal{G} \in \mathbb{P}_{l'+2}} \bigcap_{j \in \mathcal{G}} K_j \right) \right). \tag{3.4}$$

By (3.4) there is (a unique) $\mathcal{F}_x \in \mathbb{P}_{l'+1}$ such that $x \in (\bigcap_{i \in \mathcal{F}_x} K_i) \setminus (\bigcup_{i' \notin \mathcal{F}_x} K_{i'})$. For each $i \in \mathcal{F}_x$ let $U_i \ni x$ be w^* -open and such that $\text{diam}(U_i \cap K_i) \leq \varepsilon$ and $U_i \cap \bigcup_{i' \notin \mathcal{F}_x} K_{i'} = \emptyset$. Then $U = \bigcap_{i \in \mathcal{F}_x} U_i$ is a w^* -neighbourhood of x and

$$U \cap \left(\left(\bigcup_{i=1}^n s_\varepsilon(K_i) \right) \cup \left(\bigcup_{\mathcal{F} \in \mathbb{P}_{l'+1}} \bigcap_{i \in \mathcal{F}} K_i \right) \right) = U \cap \bigcap_{i \in \mathcal{F}_x} K_i = \bigcap_{i \in \mathcal{F}_x} U_i \cap K_i$$

has norm diameter not exceeding ε (because $\text{diam}(U_i \cap K_i) \leq \varepsilon$ for $i \in \mathcal{F}_x$). It follows then by (3.3) and the induction hypothesis on $l = l'$ that

$$x \notin s_\varepsilon \left(\left(\bigcup_{i=1}^n s_\varepsilon(K_i) \right) \cup \left(\bigcup_{\mathcal{F} \in \mathbb{P}_{l'+1}} \bigcap_{i \in \mathcal{F}} K_i \right) \right) \supseteq s_\varepsilon^{l'+1} \left(\bigcup_{i=1}^n K_i \right),$$

as required. In particular, (3.3) holds for all $l < \omega$ and (ii) holds for $m = 1$.

Suppose $h < \omega$ is such that (ii) holds for all $m \leq h$. Then

$$s_\varepsilon^{(h+1)n} \left(\bigcup_{i=1}^n K_i \right) \subseteq s_\varepsilon^n \left(\bigcup_{i=1}^n s_\varepsilon^h(K_i) \right) \subseteq \bigcup_{i=1}^n s_\varepsilon^{h+1}(K_i),$$

so that (ii) holds for $m = h + 1$, and thus for all m by induction.

For (iii), we prove the case $n = 2$, with the general case then following from this case and a straightforward induction on n . So we want to show that if α is a nonzero limit ordinal, then

$$s_\varepsilon^\alpha(K_1 \cup K_2) \subseteq s_\varepsilon^\alpha(K_1) \cup s_\varepsilon^\alpha(K_2). \tag{3.5}$$

To this end, it suffices to consider the case $\alpha = \omega^\beta$, $\beta > 0$, since the general case follows from finitely many iterations of this case. Indeed, every limit ordinal α is the sum of finitely many ordinals of the form ω^β , $\beta > 0$. We proceed by induction on β . For $\beta = 1$ we note that, by (ii),

$$s_\varepsilon^\omega(K_1 \cup K_2) = \bigcap_{m < \omega} s_\varepsilon^{2^m}(K_1 \cup K_2) \subseteq \bigcap_{m < \omega} (s_\varepsilon^m(K_1) \cup s_\varepsilon^m(K_2)), \tag{3.6}$$

and then a similar argument to that used to obtain (3.2) from (3.1) yields (iii) for $\alpha = \omega$. Suppose now that (3.5) holds for $\alpha = \omega^\beta$, some $\beta > 0$. Then a straightforward induction on $l < \omega$ shows that for all such l we have

$$s_\varepsilon^{\omega^\beta \cdot l}(K_1 \cup K_2) \subseteq s_\varepsilon^{\omega^\beta \cdot l}(K_1) \cup s_\varepsilon^{\omega^\beta \cdot l}(K_2). \tag{3.7}$$

(3.7) and an argument similar to that used to obtain (3.2) from (3.1) yields

$$s_\varepsilon^{\omega^{\beta+1}}(K_1 \cup K_2) \subseteq s_\varepsilon^{\omega^{\beta+1}}(K_1) \cup s_\varepsilon^{\omega^{\beta+1}}(K_2);$$

in particular, (iii) passes to successor ordinals. The straightforward proof that (iii) passes to limit ordinals uses, once again, a similar cofinality argument to that used to obtain (3.2) from (3.1) above. \square

The next three lemmas are specifically concerned with s_ε^α derivatives of direct products of w^* -compact sets, considered as w^* -compact subsets of dual spaces of direct sums of Banach spaces.

We require more notation. Given Banach spaces E_1, \dots, E_n , nonempty w^* -compact sets $K_1^* \subseteq E_1^*, \dots, K_n^* \subseteq E_n^*$, $1 \leq q < \infty$ and $a_1, \dots, a_n \geq 0$ real numbers such that $\sum_{i=1}^n a_i^q \leq 1$, for each $\varepsilon > 0$ we define

$$A_\varepsilon := \left\{ (\varepsilon_i)_{i=1}^n \in \mathbb{R}^n \mid \sum_{i=1}^n a_i^q \varepsilon_i^q \geq \varepsilon^q \text{ and } 0 \leq \varepsilon_i \leq \text{diam}(K_i), 1 \leq i \leq n \right\}.$$

In all places where we use the notation A_ε , the w^* -compact sets K_1, \dots, K_n , real numbers a_1, \dots, a_n and $1 \leq q < \infty$ will be fixed, so no ambiguity should arise from this notation. It is elementary to see that $A_\varepsilon = \emptyset$ if and only if $\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$.

We adopt the notational convention that $s_0^\alpha(K) = K$ for every ordinal α and w^* -compact K .

Lemma 3.2. *Let E_1, \dots, E_n be Banach spaces and $K_1 \subseteq E_1^*, \dots, K_n \subseteq E_n^*$ w^* -compact sets. Let $1 \leq q < \infty$, $\varepsilon > 0$ and let $a_1, \dots, a_n \geq 0$ be real numbers such that $\sum_{i=1}^n a_i^q \leq 1$. Let p be predual to q and consider $\prod_{i=1}^n a_i K_i$ as a subset of $(\bigoplus_{i=1}^n E_i)_p^*$. Then, for every $\delta \in (0, \varepsilon)$,*

$$s_\varepsilon \left(\prod_{i=1}^n a_i K_i \right) \subseteq \bigcup_{(\varepsilon_i) \in A_\delta} \prod_{i=1}^n a_i s_{\varepsilon_i}(K_i).$$

Proof. We first suppose that $\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$. Then $s_\varepsilon(\prod_{i=1}^n a_i K_i)$ is empty since $\text{diam}(\prod_{i=1}^n a_i K_i) < \varepsilon$. The assertion of the lemma follows.

Suppose now that $\varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$, so that $A_{\varepsilon'} \neq \emptyset$ for $0 < \varepsilon' \leq \varepsilon$. Let $\delta \in (0, \varepsilon)$, $(a_i x_i)_{i=1}^n \in s_\varepsilon(\prod_{i=1}^n a_i K_i)$ and, for $1 \leq i \leq n$, define

$$\delta_i := \inf\{\text{diam}(K_i \cap U_i) \mid U_i \text{ a } w^*\text{-neighbourhood of } x_i\}.$$

Then $\sum_{i=1}^n a_i^q \delta_i^q \geq \varepsilon^q > \delta^q$. Let $f : \{1, \dots, n\} \rightarrow \mathbb{R}$ be a map such that $\sum_{i=1}^n a_i^q f(i)^q \geq \delta^q$ and $f(i) \in \{0\} \cup (0, \delta_i)$ for all i (note that $[0, \delta_i)$ is empty whenever $\delta_i = 0$). We claim that with f so defined, $x_i \in s_{f(i)}(K_i)$ for $1 \leq i \leq n$. Indeed, if $\delta_i = 0$ then $f(i) = 0$, hence $x_i \in K_i = s_{f(i)}(K_i)$ by convention. On the other hand, if $\delta_i > 0$, then for all w^* -open $U_i \ni x_i$ we have $\text{diam}(K_i \cap U_i) \geq \delta_i > f(i)$, hence $x_i \in s_{f(i)}(K_i)$ in this case too. Note that $(f(i))_{i=1}^n \in A_\delta$ since $f(i) \leq \delta_i \leq \text{diam}(K_i)$ for all i and $\sum_{i=1}^n a_i^q f(i)^q \geq \delta^q$, hence

$$(a_i x_i)_{i=1}^n \in \prod_{i=1}^n a_i s_{f(i)}(K_i) \subseteq \bigcup_{(\varepsilon_i) \in A_\delta} \prod_{i=1}^n a_i s_{\varepsilon_i}(K_i). \quad \square$$

Lemma 3.3. Let E_1, \dots, E_n be Banach spaces and $K_1 \subseteq E_1^*, \dots, K_n \subseteq E_n^*$ w^* -compact sets. Let $1 \leq q < \infty$, $\varepsilon > 0$ and let $a_1, \dots, a_n \geq 0$ be real numbers such that $\sum_{i=1}^n a_i^q \leq 1$. Let p be predual to q and consider $\prod_{i=1}^n a_i K_i$ as a subset of $(\bigoplus_{i=1}^n E_i)_p^*$. Then, for every $\delta \in (0, \varepsilon)$, $0 < m < \omega$ and ordinal α ,

$$s_\varepsilon^{\omega^\alpha \cdot m} \left(\prod_{i=1}^n a_i K_i \right) \subseteq \bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,m}) \in A_{\delta/2}} \prod_{i=1}^n a_i s_{\varepsilon_{i,m}}^{\omega^\alpha} (s_{\varepsilon_{i,m-1}}^{\omega^\alpha} (\dots s_{\varepsilon_{i,1}}^{\omega^\alpha} (K_i) \dots)). \quad (3.8)$$

Proof. If $\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$, then $s_\varepsilon^{\omega^\alpha \cdot m}(\prod_{i=1}^n a_i K_i)$ is empty since $\text{diam}(\prod_{i=1}^n a_i K_i) < \varepsilon$ and $\omega^\alpha \cdot m \geq 1$. The assertion of the lemma follows.

Suppose now that $\varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$, so that $A_{\varepsilon'} \neq \emptyset$ whenever $0 < \varepsilon' \leq \varepsilon$. For $\alpha = 0$ and $m = 1$, (3.8) is a consequence of Lemma 3.2. Suppose that α is an ordinal such that (3.8) holds for $m = 1, 2, \dots, k$, for some $0 < k < \omega$. We will show that (3.8) holds for α and $m = k + 1$. Fix $\delta \in (0, \varepsilon)$ and note that $A_{(\varepsilon+\delta)/4} \subseteq A_{\delta/2}$ since $\delta/2 < (\varepsilon + \delta)/4$. We now detail a method that assigns to each $(\varepsilon_i)_{i=1}^n \in A_{(\varepsilon+\delta)/4}$ an element $(\bar{\varepsilon}_i)_{i=1}^n$ of a certain finite subset of $A_{\delta/2}$. For $(\varepsilon_i)_{i=1}^n \in A_{(\varepsilon+\delta)/4}$ and $1 \leq i \leq n$, define

$$j_i := \max\{j \in \mathbb{N} \cup \{0\} \mid j(\varepsilon - \delta) \leq 4\varepsilon_i\}$$

and set $\bar{\varepsilon}_i = j_i(\varepsilon - \delta)/4$. Note that $\bar{\varepsilon}_i \leq \varepsilon_i \leq \text{diam}(K_i)$ and

$$\begin{aligned} \left(\sum_{i=1}^n a_i^q \bar{\varepsilon}_i^q \right)^{1/q} &\geq \left(\sum_{i=1}^n a_i^q \varepsilon_i^q \right)^{1/q} - \left(\sum_{i=1}^n a_i^q (\varepsilon_i - \bar{\varepsilon}_i)^q \right)^{1/q} \\ &\geq \frac{\varepsilon + \delta}{4} - \frac{\varepsilon - \delta}{4} \\ &= \frac{\delta}{2}, \end{aligned}$$

hence $(\bar{\varepsilon}_i)_{i=1}^n \in A_{\delta/2}$. Moreover, for $(\varepsilon_{i,1})_{i=1}^n, \dots, (\varepsilon_{i,m})_{i=1}^n \in A_{(\varepsilon+\delta)/4}$ we have

$$s_{\varepsilon_i, m}^{\omega^\alpha} \left(s_{\varepsilon_i, m-1}^{\omega^\alpha} \left(\dots s_{\varepsilon_i, 1}^{\omega^\alpha} (K_i) \dots \right) \right) \subseteq s_{\bar{\varepsilon}_i, m}^{\omega^\alpha} \left(s_{\bar{\varepsilon}_i, m-1}^{\omega^\alpha} \left(\dots s_{\bar{\varepsilon}_i, 1}^{\omega^\alpha} (K_i) \dots \right) \right) \tag{3.9}$$

for all $1 \leq i \leq n$. Let $A = \{(\bar{\varepsilon}_i)_{i=1}^n \mid (\varepsilon_{i,1})_{i=1}^n \in A_{(\varepsilon+\delta)/4}\} \subseteq A_{\delta/2}$. Then A is finite, with

$$|A| \leq \left\lceil \frac{4 \cdot \max_{1 \leq i \leq n} \text{diam}(K_i)}{\varepsilon - \delta} + 1 \right\rceil^n.$$

The finiteness of A will allow us to invoke Lemma 3.1 in the next step of our proof. To complete our demonstration that (3.8) holds for $m = k + 1$, we henceforth treat the cases $\alpha = 0$ and $\alpha > 0$ separately.

If $\alpha = 0$, then for $\delta \in (0, \varepsilon)$ we have, by the induction hypothesis, (3.9), Lemma 3.1(i) and Lemma 3.2,

$$\begin{aligned} s_\varepsilon^{k+1} \left(\prod_{i=1}^n a_i K_i \right) &\subseteq s_\varepsilon \left(\overline{\bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^n a_i s_{\varepsilon_{i,k}} (s_{\varepsilon_{i,k-1}} (\dots s_{\varepsilon_{i,1}} (K_i) \dots))} \right)^{w^*} \\ &\subseteq s_\varepsilon \left(\bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_{i,k}} (s_{\bar{\varepsilon}_{i,k-1}} (\dots s_{\bar{\varepsilon}_{i,1}} (K_i) \dots)) \right) \\ &\subseteq \bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}) \in A_{(\varepsilon+\delta)/4}} s_{\varepsilon/2} \left(\prod_{i=1}^n a_i s_{\bar{\varepsilon}_{i,k}} (s_{\bar{\varepsilon}_{i,k-1}} (\dots s_{\bar{\varepsilon}_{i,1}} (K_i) \dots)) \right) \\ &\subseteq \bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}), (\varepsilon_{i,k+1}) \in A_{\delta/2}} \prod_{i=1}^n a_i s_{\varepsilon_{i,k+1}} (s_{\varepsilon_{i,k}} (\dots s_{\varepsilon_{i,1}} (K_i) \dots)), \end{aligned}$$

as required.

On the other hand, if $\alpha > 0$ then it follows from the induction hypothesis, (3.9) and Lemma 3.1(iii) that

$$\begin{aligned} s_\varepsilon^{\omega^\alpha \cdot (k+1)} \left(\prod_{i=1}^n a_i K_i \right) &\subseteq s_\varepsilon^{\omega^\alpha} \left(\overline{\bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^n a_i s_{\varepsilon_{i,k}}^{\omega^\alpha} (s_{\varepsilon_{i,k-1}}^{\omega^\alpha} (\dots s_{\varepsilon_{i,1}}^{\omega^\alpha} (K_i) \dots))} \right)^{w^*} \\ &\subseteq s_\varepsilon^{\omega^\alpha} \left(\bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_{i,k}}^{\omega^\alpha} (s_{\bar{\varepsilon}_{i,k-1}}^{\omega^\alpha} (\dots s_{\bar{\varepsilon}_{i,1}}^{\omega^\alpha} (K_i) \dots)) \right) \\ &\subseteq \bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}) \in A_{(\varepsilon+\delta)/4}} s_\varepsilon^{\omega^\alpha} \left(\prod_{i=1}^n a_i s_{\bar{\varepsilon}_{i,k}}^{\omega^\alpha} (s_{\bar{\varepsilon}_{i,k-1}}^{\omega^\alpha} (\dots s_{\bar{\varepsilon}_{i,1}}^{\omega^\alpha} (K_i) \dots)) \right) \\ &\subseteq \bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,k}), (\varepsilon_{i,k+1}) \in A_{\delta/2}} \prod_{i=1}^n a_i s_{\varepsilon_{i,k+1}}^{\omega^\alpha} (s_{\varepsilon_{i,k}}^{\omega^\alpha} (\dots s_{\varepsilon_{i,1}}^{\omega^\alpha} (K_i) \dots)), \end{aligned}$$

as we would like.

Finally, suppose that β is a nonzero ordinal (either limit or successor) such that (3.8) holds for all $m < \omega$ and $\alpha < \beta$; we show that (3.8) then holds for $m = 1$ and $\alpha = \beta$. Fix $\delta \in (0, \varepsilon)$ and let A be defined as above. Then, since $A \subseteq A_{\delta/2}$, to complete the induction it suffices to show that

$$s_\varepsilon^{\omega\beta} \left(\prod_{i=1}^n a_i K_i \right) \subseteq \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega\beta} (K_i). \tag{3.10}$$

To prove (3.10), we shall establish the following two inclusions:

$$s_\varepsilon^{\omega\beta} \left(\prod_{i=1}^n a_i K_i \right) \subseteq \bigcap_{(l,\alpha) \in (0,\omega) \times \beta} \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega^\alpha \cdot l} (K_i) \tag{3.11}$$

and

$$\bigcap_{(l,\alpha) \in (0,\omega) \times \beta} \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega^\alpha \cdot l} (K_i) \subseteq \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega\beta} (K_i). \tag{3.12}$$

We first deal with (3.11). To this end, let

$$x \in s_\varepsilon^{\omega\beta} \left(\prod_{i=1}^n a_i K_i \right) = \bigcap_{(m,\alpha) \in (0,\omega) \times \beta} s_\varepsilon^{\omega^\alpha \cdot m} \left(\prod_{i=1}^n a_i K_i \right).$$

Then, since $\frac{\varepsilon+\delta}{2} < \varepsilon$, it follows from the induction hypothesis and (3.9) that

$$\begin{aligned} x &\in \bigcap_{(m,\alpha) \in (0,\omega) \times \beta} \bigcup_{(\varepsilon_{i,1}), \dots, (\varepsilon_{i,m}) \in A_{(\varepsilon+\delta)/4}} \prod_{i=1}^n a_i s_{\varepsilon_{i,m}}^{\omega^\alpha} (s_{\varepsilon_{i,m-1}}^{\omega^\alpha} (\dots s_{\varepsilon_{i,1}}^{\omega^\alpha} (K_i) \dots)) \\ &\subseteq \bigcap_{(m,\alpha) \in (0,\omega) \times \beta} \bigcup_{(\bar{\varepsilon}_{i,1}), \dots, (\bar{\varepsilon}_{i,m}) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_{i,m}}^{\omega^\alpha} (s_{\bar{\varepsilon}_{i,m-1}}^{\omega^\alpha} (\dots s_{\bar{\varepsilon}_{i,1}}^{\omega^\alpha} (K_i) \dots)). \end{aligned}$$

So for each $(m, \alpha) \in (0, \omega) \times \beta$ there are $(\bar{\varepsilon}_{i,1,m,\alpha})_{i=1}^n, \dots, (\bar{\varepsilon}_{i,m,m,\alpha})_{i=1}^n \in A$ such that

$$x \in \prod_{i=1}^n a_i s_{\bar{\varepsilon}_{i,m,m,\alpha}}^{\omega^\alpha} (s_{\bar{\varepsilon}_{i,m-1,m,\alpha}}^{\omega^\alpha} (\dots s_{\bar{\varepsilon}_{i,1,m,\alpha}}^{\omega^\alpha} (K_i) \dots)). \tag{3.13}$$

Suppose $l \in (0, \omega)$ and $\alpha < \beta$ and set $m_l = |A| \cdot l$. Then there is a subset $J_{l,\alpha} \subseteq \{1, 2, \dots, m_l\}$ with $|J_{l,\alpha}| = l$ and $|\{(\bar{\varepsilon}_{i,j,m_l,\alpha})_{i=1}^n \mid j \in J_{l,\alpha}\}| = 1$. Let $(\bar{\varepsilon}_{i,l,\alpha})_{i=1}^n$ denote the unique element of $\{(\bar{\varepsilon}_{i,j,m_l,\alpha})_{i=1}^n \mid j \in J_{l,\alpha}\} (\subseteq A)$. We may write $J_{l,\alpha} = \{j_1 < j_2 < \dots < j_l\}$, and then by (3.13) we have, in particular,

$$\begin{aligned}
 x &\in \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i, m_i, \alpha}^{\omega^\alpha} \left(s_{\bar{\varepsilon}_i, m_i - 1, \alpha}^{\omega^\alpha} \left(\dots s_{\bar{\varepsilon}_i, 1, \alpha}^{\omega^\alpha} (K_i) \dots \right) \right) \\
 &\subseteq \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i, j_i, m_i, \alpha}^{\omega^\alpha} \left(s_{\bar{\varepsilon}_i, j_i - 1, m_i, \alpha}^{\omega^\alpha} \left(\dots s_{\bar{\varepsilon}_i, 1, m_i, \alpha}^{\omega^\alpha} (K_i) \dots \right) \right) \\
 &= \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i, \alpha}^{\omega^\alpha \cdot l} (K_i) \\
 &\subseteq \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega^\alpha \cdot l} (K_i).
 \end{aligned}$$

As $l \in (0, \omega)$ and $\alpha < \beta$ were arbitrary, (3.11) follows.

We now prove (3.12). Let

$$y \in \bigcap_{(l, \alpha) \in (0, \omega) \times \beta} \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega^\alpha \cdot l} (K_i),$$

and for each $l \in (0, \omega)$ and $\alpha < \beta$ let $(\bar{\varepsilon}_i)_{i=1}^n \in A$ be such that

$$y \in \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i, (l, \alpha)}^{\omega^\alpha \cdot l} (K_i).$$

For each $(\bar{\varepsilon}_i)_{i=1}^n \in A$, let

$$\mathcal{A}[(\bar{\varepsilon}_i)_{i=1}^n] = \{ \omega^\alpha \cdot l \mid 0 < l < \omega, \alpha < \beta, (\bar{\varepsilon}_i, (l, \alpha))_{i=1}^n = (\bar{\varepsilon}_i)_{i=1}^n \}.$$

Since $\{ \omega^\alpha \cdot l \mid 0 < l < \omega, \alpha < \beta \}$ is cofinal in ω^β and $\{ \mathcal{A}[(\bar{\varepsilon}_i)_{i=1}^n] \mid (\bar{\varepsilon}_i)_{i=1}^n \in A \}$ is a finite partition of $\{ \omega^\alpha \cdot l \mid 0 < l < \omega, \alpha < \beta \}$, there exists $(\bar{\rho}_i)_{i=1}^n \in A$ such that $\mathcal{A}[(\bar{\rho}_i)_{i=1}^n]$ is cofinal in ω^β . It follows that

$$\begin{aligned}
 y &\in \bigcap_{\xi \in \mathcal{A}[(\bar{\rho}_i)_{i=1}^n]} \prod_{i=1}^n a_i s_{\bar{\rho}_i}^\xi (K_i) \subseteq \prod_{i=1}^n a_i \left(\bigcap_{\xi \in \mathcal{A}[(\bar{\rho}_i)_{i=1}^n]} s_{\bar{\rho}_i}^\xi (K_i) \right) \\
 &= \prod_{i=1}^n a_i \left(\bigcap_{\xi < \omega^\beta} s_{\bar{\rho}_i}^\xi (K_i) \right) \\
 &= \prod_{i=1}^n a_i s_{\bar{\rho}_i}^{\omega^\beta} (K_i) \\
 &\subseteq \bigcup_{(\bar{\varepsilon}_i) \in A} \prod_{i=1}^n a_i s_{\bar{\varepsilon}_i}^{\omega^\beta} (K_i).
 \end{aligned}$$

At last, the proof of Lemma 3.3 is complete. \square

Lemma 3.4. *Let E_1, \dots, E_n be Banach spaces and $K_1 \subseteq E_1^*, \dots, K_n \subseteq E_n^*$ nonempty w^* -compact sets. Let $1 \leq q < \infty$, $\varepsilon > 0$ and let $a_1, \dots, a_n \geq 0$ be real numbers such that $\sum_{i=1}^n a_i^q \leq 1$. Let $d = \max\{\text{diam}(K_i) \mid 1 \leq i \leq n\}$ and let $m, M \in \mathbb{N}$ be such that $M \geq m \geq 2$ and $(2^q - 1)\varepsilon^q M \geq 8^q d^q (m - 1)$. Let p be predual to q and consider $\prod_{i=1}^n a_i K_i$ as a subset of $(\bigoplus_{i=1}^n E_i)_p^*$. If α is an ordinal such that $s_{\varepsilon/8}^{\omega_\alpha \cdot m}(K_i) = \emptyset$ for all $1 \leq i \leq n$, then $s_\varepsilon^{\omega_\alpha \cdot M}(\prod_{i=1}^n a_i K_i) = \emptyset$.*

Proof. If $\varepsilon^q > \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$, then $s_\varepsilon^{\omega_\alpha \cdot M}(\prod_{i=1}^n a_i K_i)$ is empty since $\text{diam}(\prod_{i=1}^n a_i K_i) < \varepsilon$ and $\omega_\alpha \cdot M \geq 1$. The assertion of the lemma follows.

So suppose now that $\varepsilon^q \leq \sum_{i=1}^n [a_i \cdot \text{diam}(K_i)]^q$. Then $A_{\varepsilon'} \neq \emptyset$ whenever $0 < \varepsilon' \leq \varepsilon$. Applying Lemma 3.3 with $\delta = \varepsilon/2$, we see that $s_\varepsilon^{\omega_\alpha \cdot M}(\prod_{i=1}^n a_i K_i)$ is contained in a union of sets of the form

$$\prod_{i=1}^n a_i s_{\varepsilon_i, M}^{\omega_\alpha} (s_{\varepsilon_i, M-1}^{\omega_\alpha} (\dots s_{\varepsilon_i, 1}^{\omega_\alpha} (K_i) \dots)), \tag{3.14}$$

where $(\varepsilon_{i,1})_{i=1}^n, (\varepsilon_{i,2})_{i=1}^n, \dots, (\varepsilon_{i,M})_{i=1}^n \in A_{\varepsilon/4}$. For each such product (3.14),

$$a_1^q \left(\sum_{j=1}^M \varepsilon_{1,j}^q \right) + a_2^q \left(\sum_{j=1}^M \varepsilon_{2,j}^q \right) + \dots + a_n^q \left(\sum_{j=1}^M \varepsilon_{n,j}^q \right) \geq \frac{M\varepsilon^q}{4^q}.$$

Since $\sum_{i=1}^n a_i^q \leq 1$, there is $h \in \{1, \dots, n\}$ such that $\sum_{j=1}^M \varepsilon_{h,j}^q \geq M\varepsilon^q/4^q$. We claim that at least one of the following two conditions holds for such h :

- (a) There exists a subset $\{j_1 < j_2 < \dots < j_m\} \subseteq \{1, 2, \dots, M\}$ such that $\min\{\varepsilon_{h,j_1}, \dots, \varepsilon_{h,j_m}\} \geq \varepsilon/8$.
- (b) There exists $j \leq M$ such that $\varepsilon_{h,j} > d$.

Indeed, suppose that (a) does not hold. Then there are distinct j_1, \dots, j_{m-1} in $\{1, \dots, M\}$ such that $\varepsilon_{h,j} < \varepsilon/8$ whenever $j \in \{1, \dots, M\} \setminus \{j_1, \dots, j_{m-1}\}$. It follows then that

$$\begin{aligned} \sum_{k=1}^{m-1} \varepsilon_{h,j_k}^q &> \frac{M\varepsilon^q}{4^q} - (M - m + 1) \left(\frac{\varepsilon}{8}\right)^q \\ &= M \left(\left(\frac{\varepsilon}{4}\right)^q - \left(\frac{\varepsilon}{8}\right)^q \right) + (m - 1) \left(\frac{\varepsilon}{8}\right)^q \\ &> M \left(\left(\frac{\varepsilon}{4}\right)^q - \left(\frac{\varepsilon}{8}\right)^q \right) \\ &\geq d^q (m - 1). \end{aligned}$$

Thus $\varepsilon_{h,j_k}^q > d^q$ for some $k \leq m - 1$, hence $\varepsilon_{h,j_k} > d$ for some $k \leq m - 1$. In particular, (b) holds whenever (a) does not.

If (b) holds, then the factor $a_h s_{\varepsilon_h, M}^{\omega^\alpha} (s_{\varepsilon_h, M-1}^{\omega^\alpha} (\dots s_{\varepsilon_h, 1}^{\omega^\alpha} (K_h) \dots))$ is empty since $\text{diam}(K_h) \leq d < \varepsilon_{h, j}$ for j satisfying (b). It follows then that the product $\prod_{i=1}^n a_i s_{\varepsilon_i, M}^{\omega^\alpha} (s_{\varepsilon_i, M-1}^{\omega^\alpha} (\dots s_{\varepsilon_i, 1}^{\omega^\alpha} (K_i) \dots))$ is empty also, giving the desired result. On the other hand, if (a) holds then

$$s_{\varepsilon_h, M}^{\omega^\alpha} (s_{\varepsilon_h, M-1}^{\omega^\alpha} (\dots s_{\varepsilon_h, 1}^{\omega^\alpha} (K_h) \dots)) \subseteq s_{\varepsilon_h, jm}^{\omega^\alpha} (s_{\varepsilon_h, jm-1}^{\omega^\alpha} (\dots s_{\varepsilon_h, j1}^{\omega^\alpha} (K_h) \dots)) \subseteq s_{\varepsilon/8}^{\omega^\alpha \cdot m} (K_h).$$

We conclude that $s_\varepsilon^{\omega^\alpha \cdot M} (\prod_{i=1}^n a_i K_i)$ is contained in a union of direct products of the form (3.14), with each such direct product having a factor contained in a scalar multiple of one of the sets $s_{\varepsilon/8}^{\omega^\alpha \cdot m} (K_i)$, $1 \leq i \leq n$. From this it is clear that if $s_{\varepsilon/8}^{\omega^\alpha \cdot m} (K_i) = \emptyset$ for all $1 \leq i \leq n$, then $s_\varepsilon^{\omega^\alpha \cdot M} (\prod_{i=1}^n a_i K_i) \subseteq \emptyset$. \square

The next and final lemma required for our proof of Lemma 2.5 shows how we can put a set $B_q(K_i \mid 1 \leq i \leq n)$ inside a finite union of direct products of w^* -compact sets in a way that will be useful for us.

Lemma 3.5. *Let E_1, \dots, E_n be Banach spaces, $K_1 \subseteq E_1^*, \dots, K_n \subseteq E_n^*$ nonempty, absolutely convex, w^* -compact sets, $1 \leq q < \infty$ and $l \in \mathbb{N}$. Let $L = \mathbb{N}^n \cap (l + n^{1/q})B_{\ell_q^n}$. Then*

$$B_q(K_i \mid 1 \leq i \leq n) \subseteq \bigcup_{(k_i)_{i=1}^n \in L} \prod_{i=1}^n \frac{k_i}{l} K_i.$$

Proof. Let $(a_i)_{i=1}^n \in B_{\ell_q^n}$ and set $j_i = \inf\{j \in \mathbb{N} \mid l|a_i| < j\}$, $1 \leq i \leq n$. Then $j_i - 1 \leq l|a_i|$ for all i , hence $\|(j_i)_{i=1}^n\|_{\ell_q^n} \leq \|(l a_i)_{i=1}^n\|_{\ell_q^n} + n^{1/q} \leq l + n^{1/q}$. In particular, $(j_i)_{i=1}^n \in L$. As the sets K_i , $1 \leq i \leq n$, are absolutely convex, we have $a_i K_i \subseteq \frac{j_i}{l} K_i$ for all i , hence $\prod_{i=1}^n a_i K_i \subseteq \prod_{i=1}^n \frac{j_i}{l} K_i$. It follows that

$$B_q(K_i \mid 1 \leq i \leq n) = \bigcup_{(a_i) \in B_{\ell_q^n}} \prod_{i=1}^n a_i K_i \subseteq \bigcup_{(k_i)_{i=1}^n \in L} \prod_{i=1}^n \frac{k_i}{l} K_i. \quad \square$$

We note a few points of interest regarding the sets $\bigcup_{(k_i) \in L} \prod_{i=1}^n \frac{k_i}{l} K_i$ from Lemma 3.5. For each $l \in \mathbb{N}$, let $L_l = \mathbb{N}^n \cap (l + n^{1/q})B_{\ell_q^n}$. Then the intersection of the collection $\{\bigcup_{(k_i) \in L_l} \prod_{i=1}^n \frac{k_i}{l} K_i \mid l \in \mathbb{N}\}$ is precisely $B_q(K_i \mid 1 \leq i \leq n)$; this follows from the observation that for $l \in \mathbb{N}$, each point of $\bigcup_{(k_i) \in L_l} \prod_{i=1}^n \frac{k_i}{l} K_i$ is no greater than $n^{1/q} \cdot l^{-1} \cdot \max\{\text{diam}(K_i) \mid 1 \leq i \leq n\}$ in norm distance from $B_q(K_i \mid 1 \leq i \leq n)$. We may thus think of $\{\bigcup_{(k_i) \in L_l} \prod_{i=1}^n \frac{k_i}{l} K_i \mid l \in \mathbb{N}\}$ as a sequence of increasingly closer approximations to the set $B_q(K_i \mid 1 \leq i \leq n)$, and our need to closely approximate $B_q(K_i \mid 1 \leq i \leq n)$ is reflected by our choice of l in the following proof of Lemma 2.5.

Proof of Lemma 2.5. Fix $\delta \in (0, \varepsilon/16)$. Let $l \geq 16\delta n^{1/q}(\varepsilon - 16\delta)^{-1}$ be an integer and let $L = \mathbb{N}^n \cap (l + n^{1/q})B_{\ell_q^n}$. By Lemma 3.5 and the hypothesis of Lemma 2.5,

$$s_\varepsilon^{\omega^\alpha \cdot M} \left(\bigcup_{(k_i) \in L} \prod_{i=1}^n \frac{k_i}{l} K_i \right) \supseteq s_\varepsilon^{\omega^\alpha \cdot M} (B_q(K_i \mid 1 \leq i \leq n)) \supsetneq \emptyset.$$

Thus, since L is finite, by Lemma 3.1(i) there exists $(h_i)_{i=1}^n \in L$ such that

$$s_{\varepsilon/2}^{\omega^\alpha \cdot M} \left(\prod_{i=1}^n \frac{h_i}{l} K_i \right) \neq \emptyset. \tag{3.15}$$

Let $\rho = (1 + \frac{n^{1/q}}{l})^{-1}$. By (3.15) and the homogeneity of the derivations $s_{\varepsilon'}^\gamma$ (where γ is an ordinal and $\varepsilon' > 0$), we have

$$s_{\rho\varepsilon/2}^{\omega^\alpha \cdot M} \left(\prod_{i=1}^n \frac{\rho h_i}{l} K_i \right) = \rho s_{\varepsilon/2}^{\omega^\alpha \cdot M} \left(\prod_{i=1}^n \frac{h_i}{l} K_i \right) \neq \emptyset. \tag{3.16}$$

Thus, since $\|(\frac{\rho h_i}{l})_{i=1}^n\|_{\ell_q^n} \leq 1$, it follows from (3.16) and Lemma 3.4 that there is $i \leq n$ such that $s_{\rho\varepsilon/16}^{\omega^\alpha \cdot m}(K_i) \neq \emptyset$. As $\rho\varepsilon/16 \geq \delta$, we conclude that $s_\delta^{\omega^\alpha \cdot m}(K_i) \supseteq s_{\rho\varepsilon/16}^{\omega^\alpha \cdot m}(K_i) \supsetneq \emptyset$. This completes the proof. \square

Remark 3.6. Lemma 2.5 is similar to [2, Lemma 5.9]. Though many of the arguments and preliminary results used here in the proof of Lemma 2.5 have been employed similarly in the proof of [2, Lemma 5.9], neither of these technical lemmas are strong enough to be used in place of the other in the proofs of the respective theorems for which they have been developed.

Acknowledgments

The author thanks Dr. Rick Loy for his invaluable support and the anonymous referee for many helpful suggestions that have improved the presentation of the results.

Part of the research presented here was completed during a visit of the author to the Département de Mathématiques at the Université de Franche-Comté, and the author thanks the department for their kind hospitality. The author is especially grateful to Prof. Gilles Lancien for many stimulating discussions during his stay.

References

- [1] C. Bessaga, A. Pełczyński, Spaces of continuous functions, IV. On isomorphical classification of spaces of continuous functions, *Studia Math.* 19 (1960) 53–62.
- [2] P.A.H. Brooker, Asplund operators and the Szlenk index, *J. Operator Theory*, in press, a preprint is online at arXiv: 1003.5710v3 [math.FA].
- [3] R. Deville, G. Godefroy, V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monogr. Surv. Pure Appl. Math., vol. 64, Longman Scientific & Technical, Harlow, 1993.
- [4] R. Engelking, General Topology, second edition, Sigma Ser. Pure Math., vol. 6, Heldermann-Verlag, Berlin, 1989.
- [5] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, V. Zizler, Functional Analysis and Infinite-Dimensional Geometry, CMS Books Math./Ouvrages Math. SMC, vol. 8, Springer-Verlag, New York, 2001.

- [6] J. Giles, *Convex Analysis with Application in the Differentiation of Convex Functions*, Res. Notes Math., vol. 58, Pitman (Advanced Publishing Program), Boston, MA, 1982.
- [7] P. Hájek, G. Lancien, Various slicing indices on Banach spaces, *Mediterr. J. Math.* 4 (2) (2007) 179–190.
- [8] P. Hájek, G. Lancien, V. Montesinos, Universality of Asplund spaces, *Proc. Amer. Math. Soc.* 135 (7) (2007) 2031–2035 (electronic).
- [9] P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, V. Zizler, *Biorthogonal Systems in Banach Spaces*, CMS Books Math./Ouvrages Math. SMC, vol. 26, Springer-Verlag, New York, 2008.
- [10] S. Heinrich, Closed operator ideals and interpolation, *J. Funct. Anal.* 35 (3) (1980) 397–411.
- [11] W.B. Johnson, H.P. Rosenthal, On ω^* -basic sequences and their applications to the study of Banach spaces, *Studia Math.* 43 (1972) 77–92.
- [12] G. Lancien, Dentability indices and locally uniformly convex renormings, *Rocky Mountain J. Math.* 23 (2) (1993) 635–647.
- [13] G. Lancien, On the Szlenk index and the weak*-dentability index, *Quart. J. Math. Oxford Ser. (2)* 47 (185) (1996) 59–71.
- [14] G. Lancien, A survey on the Szlenk index and some of its applications, *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 100 (1–2) (2006) 209–235.
- [15] E. Odell, Th. Schlumprecht, A. Zsák, Banach spaces of bounded Szlenk index, *Studia Math.* 183 (1) (2007) 63–97.
- [16] A. Pelczyński, Universal bases, *Studia Math.* 32 (1969) 247–268.
- [17] A. Pietsch, *Operator Ideals*, North-Holland Math. Library, vol. 20, North-Holland, Amsterdam, 1980.
- [18] O. Reĭnov, RN-sets in Banach spaces, *Funktsional. Anal. i Prilozhen.* 12 (1) (1978) 80–81, 96.
- [19] H. Rosenthal, The Banach spaces $C(K)$, in: *Handbook of the Geometry of Banach Spaces*, vol. 2, North-Holland, Amsterdam, 2003, pp. 1547–1602.
- [20] C. Samuel, Indice de Szlenk des $C(K)$ (K espace topologique compact dénombrable), in: *Seminar on the Geometry of Banach Spaces*, vols. I, II, Paris, 1983, in: *Publ. Math. Univ. Paris VII*, vol. 18, Univ. Paris VII, Paris, 1984, pp. 81–91.
- [21] C. Stegall, The Radon–Nikodým property in conjugate Banach spaces, II, *Trans. Amer. Math. Soc.* 264 (2) (1981) 507–519.
- [22] W. Szlenk, The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces, *Studia Math.* 30 (1968) 53–61.