The Based Ring of the Lowest Two-Sided Cell of an Affine Weyl Group

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1. INTRODUCTION

1.1. The purpose of this paper is to give a clear description for the based ring of the lowest two-sided cell of an affine Weyl group. Our results support the conjecture proposed in [4, IV].

1.2. Let $G$ be a simply connected, almost simple complex algebraic group and $T$ a maximal torus of $G$. Let $R \subset X = \text{Hom}(T, C^*)$ be the root system, $P \subset X$ the root lattice. Let $\Delta$ be the set of simple roots in $R$ and $R^+$ be the set of positive roots in $R$. The Weyl group $W_0 = N_G(T)/T$ of $G$ acts on $X$ in a natural way and this action is stable on $P$ and $R$. Thus we can form the affine Weyl group $W' = W_0 \ltimes P$ which is a normal subgroup of the extended affine Weyl group $W = W_0 \ltimes X$. There exists a finite abelian subgroup $\Omega$ of $W$ such that $W = \Omega \ltimes W'$. Let $S$ be the set of simple reflections of $W'$. Then we have a standard length function $l$ on $W'$ which can be extended to $W$ by defining $l(rw) = l(w)$ for any $r \in \Omega$, $w \in W'$. We keep the same notation for the extension of $l$.

1.3. For any $u = \omega_1 w_1$, $w = \omega_2 w_1$, $\omega_1, \omega_2 \in \Omega$, $u_1, w_1 \in W'$, we define $P_{u, w}$ to be $P_{u_1, w_1}$ as in [1] if $\omega_1 = \omega_2$ and define $P_{u, w}$ to be 0 if $\omega_1 \neq \omega_2$. We say that $u \leq_{LR} w$ or $u \leq_{L} w$ or $u \leq_{R} w$ if $u_1 \leq_{LR} w_1$ or $u_1 \leq_{L} w_1$ or $u_1 \leq_{R} w_1$ in the sense of [1]. These relations generate equivalence relations $\sim_{LR}$, $\sim_{L}$, $\sim_{R}$ in $W$, respectively, and the corresponding equivalence classes are called two-sided cells, left cells, right cells of $W$, respectively. The relation $\leq_{LR}$ (resp. $\leq_{L}$, $\leq_{R}$) in $W$ then induces a partial order $\leq_{LR}$ (resp. $\leq_{L}$, $\leq_{R}$) in the set of two-sided (resp. left, right) cells of $W$. We extend the Bruhat order $\leq$ in $W'$ to $W$ by defining $u \leq w$ if and only if $\omega_1 = \omega_2$ and $u_1 \leq w_1$. 

356
1.4. Let \( \mathcal{A} = \mathbb{C}[q, q^{-1}] \) be the ring of all Laurent polynomials in an indeterminate \( q \) with coefficients in \( \mathbb{C} \). The Hecke algebra \( \mathcal{H} \) of \( W \) over \( \mathcal{A} \) is a free \( \mathcal{A} \)-module with a basis \( T_w, w \in W \), and the multiplication law is given by the formulae

\[
T_r^i = q^i + (q^2 - 1) T_r \quad \text{if} \quad r \in S,
\]

\[
T_w T_{w'} = T_{ww'} \quad \text{if} \quad w, w' \in W \text{ and } l(ww') = l(w) + l(w').
\]

We have another basis of \( \mathcal{H} \), \( C_w = q^{-l(w)} \sum_{u \leq w} P_{u,w}(q^2) T_u, w \in W \). For any \( z \in W \) let \( a(z) = \max \{ \deg h_{w,u,z} | w, u \in W \} \), where \( h_{w,u,z} \in \mathcal{A} \) is determined by the expression \( C_w C_u = \sum_x h_{w,u,x} C_x \) for any \( w, u \in W \). Then \( a(z) \leq v = |R^+| \) for all \( z \in W \) (see [4, I]) and \( e_0 = \{ w \in W | a(w) = v \} \) is a two-sided cell of \( W \) (see [6, I]) which is the lowest one for the partial order \( \leq_{LR} \).

1.5. A ring with 1 is called a based ring if its additive group is a free abelian group with a basis \( \Theta \) such that the following two conditions are satisfied:

(a) If \( \theta, \theta' \in \Theta \), \( \theta \theta' = \sum_{\rho \leq \theta} n_{\rho, \theta'} \), then \( n_{\theta, \theta'} \geq 0 \).

(b) There exists an involution \( \tau \) of this ring as a group such that if \( \theta \in \Theta \) then \( \theta \theta' = \tau(\theta \theta') \), \( \theta' \in \Theta \), and

\[
\tau(\theta \theta') = \begin{cases} 1 & \text{if } \theta' = \theta \\ 0 & \text{if } \theta' \neq \theta \end{cases}
\]

where \( \tau \) is a group homomorphism from the ring to \( \mathbb{Z} \) defined by \( \tau(\sum_{\theta} n_{\theta, \theta}) = \sum_{\theta} n_{\theta} \).

From (a) one knows immediately that there exists a finite subset \( \Theta_0 \) of \( \Theta \) such that \( 1 = \sum_{\theta \in \Theta_0} \theta \) and \( \theta^2 = \theta \) if \( \theta \in \Theta_0 \), \( \theta \theta' = 0 \) if \( \theta, \theta' \in \Theta_0 \) are different.

1.6. Let \( h_{w,u,z} = h_{w,u,z}^\theta + \text{lower power terms}, w, u, z \in W \), then \( h_{w,u,z} \in \mathcal{A} \). Let \( \mathcal{J} \) be the free \( \mathbb{Z} \)-module with basis \( \{ t_w | w \in W \} \). \( \mathcal{J} \) becomes a based ring if we set \( t_w t_u = \sum_x h_{w,u,z} x_z \) and define \( \tilde{t}_w = t_{w^{-1}} \). In this case we have \( \Theta = \{ t_w | w \in W \} \) and \( \Theta_0 = \{ t_d | d \in W', a(d) = l(d) - 2 \deg P_{e,d} \} \) (\( e \) is the unit of \( W \)) (see [4, II]).

Let \( \mathcal{D} = \{ d \in \mathcal{J} | a(d) - 2 \deg P_{e,d} \} \). The elements in \( \mathcal{D} \) all are involutions and are called distinguished involutions of \( W \). Let \( d \in \mathcal{D} \) be a distinguished involution then \( \gamma_{w,u,d} \neq 0 \) implies that \( w = u^{-1} \) and \( \gamma_{w,u,d} = 1 \) (see [4, II]).

Let \( w, u, z \in W \). If \( \gamma_{w,u,z} \neq 0 \), then \( w \sim_L u^{-1}, u \sim_L z, w \sim_R z \). Conversely if \( w \sim_L u^{-1} \) then \( \gamma_{w,u,z} \neq 0 \) for some \( z \in W \). Thus for any two-sided cell \( c \) of \( W \)
the \(\mathbb{Z}\)-submodule \(J_c\) of \(J\) generated by \(t_w\) \((w \in c)\) is an ideal of \(J\) and \(J_c\) is also a based ring with \(\Theta = \{t_w|w \in c\}\) and \(\Theta_0 = \{t_d|d \in \mathcal{D} \cap c\}\). In particular we know that \(J_{c_0} = J_0\) is a based ring with \(\Theta = \{t_w|w \in c_0\}\) and \(\Theta_0 = \{t_d|d \in \mathcal{D}_0\}\), where \(\mathcal{D}_0 = \mathcal{D} \cap c_0\). \(J_0\) is the main object which we shall discuss in this paper. We have a decomposition \(J = \bigoplus_c J_c\), where the sum is over all two-sided cells \(c\) of \(W\) (see \([4, \II]\)). It is easy to see that for any left cell \(I\) of \(W\), \(J_{I \cap I^{-1}}\) is a based ring, the definition of \(J_{I \cap I^{-1}}\) is similar to that of \(J_c\).

1.7. Let \(J_{\mathcal{A}} = J \otimes \mathcal{A}\). The map \(C_w \mapsto \sum_{d \in \mathcal{A}, d \in \lambda} h_{w, d, z} t_z\) defines an algebra homomorphism \(\phi'\) from \(\mathcal{H}\) to \(J_{\mathcal{A}}\) which is injective (loc. cit.). Therefore map \(C_w \mapsto \sum_{d \in \mathcal{A}_0} h_{w, d, z} t_z\) gives an algebra homomorphism \(\phi\) from \(\mathcal{H}\) to \(J_0 \otimes \mathcal{A}\).

1.8. Similarly for \((W', S')\) we can define its Hecke algebra \(\mathcal{H}'\) over \(\mathcal{A}\) and define its based ring \(J'\), which can be regarded as subalgebras of \(\mathcal{A}\) and \(J\) in a natural way, respectively. We have \(\phi'(\mathcal{H}') \subset J' \otimes \mathcal{A} = J_{\mathcal{A}}\).

1.9. Let \(G'\) be a reductive complex algebraic group which acts algebraically on a finite set \(Y\). A \(G'\)-equivariant complex vector bundle \((= G'-\text{v.b.})\) \(V\) on \(Y\) then is just a collection of finite dimensional complex vector spaces \(V_y\) \((y \in Y)\) with a given rational representation of \(G'\) on \(\bigoplus_{y \in Y} V_y\) and \(gV_y = V_{gy}\) for all \(g \in G', y \in Y\). The direct sum of \(G'\)-v.b. can be defined naturally and the Grothendieck group \(K_{G'}(Y)\) of the category of \(G'\)-v.b. on \(Y\) then is well defined. The set \(\mathcal{F}\) of isomorphism classes of irreducible \(G'\)-v.b. on \(Y\) is a basis of \(K_{G'}(Y)\). For any \(y \in Y\) let \(G'_y\) be the stabilizer of \(y\) in \(G'\). It is easy to see that there exists a bijection between \(\mathcal{F}\) and the set of pairs \((y, \rho)\) where \(y \in Y, \rho \in \text{Irr} G'_y\) (the set of isomorphism classes of irreducible rational representations of \(G'_y\)), modulo the obvious action of \(G'_y\).

Let \(Y\) be a \(G'\)-set (i.e., \(G'\) has an algebraic action on \(Y\)). Then \(Y \times Y\) is also a \(G'\)-set. \(K_{G'}(Y \times Y)\) is a based ring if we define the multiplication \(*\) in \(K_{G'}(Y \times Y)\) by setting \((V_1 * V_2)_{(y_1, y_2)} = \bigoplus_{y \in Y} V_1(y_1, y) \otimes V_2(y_2, y)\) for any \(V_1, V_2 \in K_{G'}(Y \times Y)\) and define \(\tilde{\mathcal{F}} (y_1, y_2) = V_1^{*}(y_1, y_2)\) (the complex dual of \(V_1(y_1, y_2)\)). In this case we have \(\Theta = \text{the set of isomorphism classes of irreducible } G'\text{-v.b. on } Y \times Y\) and \(\Theta_0 = \{V \in \Theta | V(y, y) = \mathbb{C} \text{ for some } y \in Y\}\).

When \(G'\) is connected then any algebraic action of \(G'\) on a finite set has to be trivial. In this case \(\mathcal{F}\) has a bijection with the set of pairs \((y, \rho)\) where \(y \in Y, \rho \in \text{Irr} G'\).

Now we can state our main result of this paper.

**Theorem 1.10.** Let \(G\) be as in 1.2, \(Y\) a \(G\)-set, and \(|Y| = |W_0|\). Then \(J_0\) and \(K_G(Y \times Y)\) are isomorphic as based rings; i.e., there exists a bijection \(\psi: \{t_w|w \in c_0\} \rightarrow \{\text{isomorphism classes of irreducible } G\text{-v.b. on } Y \times Y\}\) such
that \( \psi \) gives rise to an isomorphism of rings \( \psi: J_0 \rightarrow K_0(Y \times Y) \) and 
\( \psi(t_{w^{-1}}) = \overline{\psi(t_w)} \) for any \( w \in c_0 \).

**Corollary 1.11.** For any \( y \in Y \) let
\[
\Gamma_y = \{ w \in c_0 | \psi(t_w)(y', y) \neq 0 \text{ for some } y' \in Y \}
\]
\[
\Gamma'_y = \{ w \in c_0 | \overline{\psi(t_w)(y, y')} \neq 0 \text{ for some } y' \in Y \}.
\]

Then \( y \rightarrow \Gamma_y \) (resp. \( y \rightarrow \Gamma'_y \)) gives a bijection between \( Y \) and the set of left (resp. right) cells in \( c_0 \) and \( J_{r_1 \cap r_1^{-1}} \cong R_G \), the ring of rational representations of \( G \).

**Proof.** These statements follow from \( t_w t_u \neq 0 \) if and only if \( w \sim_L u^{-1} \) and \( K_0(\{ y \} \times \{ y' \}) \) is isomorphic to \( R_G \).

2. THE GEOMETRIC REALIZATION OF \((W', S)\)

In this section we shall give a geometric realization of \((W', S)\) following [2] and prove some results which are crucial for our proof of Theorem 1.10.

2.1. Consider the real space \( E = X \otimes \mathbb{R} \). Let \( \alpha^\vee \in \text{Hom}(X, \mathbb{Z}) \) be the coroot corresponding to \( \alpha \in \mathfrak{g} \). For any \( \alpha \in \mathfrak{h}^+ \), \( n \in \mathbb{Z} \), let \( H_{\alpha, n} \) be the hyperplane \( \{ e \in E | \langle e, \alpha^\vee \rangle = n \} \) and let \( E_{\alpha, n} \) be the half-space \( \{ e \in E | \langle e, \alpha^\vee \rangle > n \} \), \( E_{\alpha, n}^- \) the half-space \( \{ e \in E | \langle e, \alpha^\vee \rangle < n \} \), respectively. The hyperplane \( H_{\alpha, n} \) determines a reflection \( \sigma_{\alpha, n} \) of \( E \) by \( \sigma_{\alpha, n}(e) = e - (\langle e, \alpha^\vee \rangle - n) \alpha \). All such reflections generate an affine motions group \( A \) of \( E \) and \( A \) acts simply transitively on the set \( M \) of the connected components of \( E - \bigcup_{\alpha \in \mathfrak{h}^+, \ n \in \mathbb{Z}} H_{\alpha, n} \). These connected components are called alcoves. We shall regard \( A \) acting on the right on \( M \) or \( E \).

Each alcove has \( k + 1 \) faces (facets with codimension 1), where \( k = |A| \). Then the set \( S_1 \) of \( A \)-orbits of faces consists of \( k + 1 \) elements. For each element \( r_1 \in S_1 \), we define a permutation of \( M \) and denote it yet by \( r_1 \): for any alcove \( A \in M \) we set \( r_1 A \) to be the unique alcove which is not equal to \( A \) and has a common face of type \( r_1 \) with \( A \). \( r_1 \) is an involution. All such involutions generate a permutation group \((W_1, S_1)\) of \( M \) which is an affine Weyl group and is isomorphic to \((W', S)\). We shall equal \((W_1, S_1)\) and \((W', S)\).

A point \( v \in E \) is called a special point if there are \( v \) hyperplanes in \( \{ H_{\alpha, n} | \alpha \in \mathfrak{h}^+, \ n \in \mathbb{Z} \} \) passing \( v \). The original point 0 is a special point. Let \( \mathcal{C}_0 = \{ e \in E | \langle e, \alpha^\vee \rangle > 0 \text{ for any } \alpha \in \mathfrak{h}^+ \} \). It is a quarter with vertex 0. For any special point \( v \) let \( \mathcal{C}_v \) be the unique quarter such that \( \mathcal{C}_0 \) is a translate
of \( C_0 \) and \( v \) is the vertex of \( C_v \). Let \( A_v \) be the unique alcove contained in \( C_v \) with closure containing \( v \). The connected components of \( E - \bigcup_{x \in A, n \in \mathbb{Z}} H_{a,n} \) are called boxes. For each special point \( v \) there exists a unique box \( \Pi_v \) such that \( A_v \subset \Pi_v \). Let \( W_v \) be the subgroup of \( W' \) stabilizing the set of alcoves with closure containing \( v \). Then for any \( w \in W' \) one has \( wA_v \subset C_v \) if and only if \( l(ww_v) = l(w) + l(w_v) \), where \( w_v \) is the longest element of \( W_v \).

2.2. We introduce a partial order \( \leq \) on \( M \). Let \( A, B \in M \) be alcoves. For any hyperplane \( H_{a,n} (a \in \mathbb{R}^+, n \in \mathbb{Z}) \) separating \( A \) and \( B \) we count 1 if \( A \) is in \( E_{a,n}^- \) and count \(-1 \) if \( A \) is in \( E_{a,n}^+ \). The sum of these \( \pm 1 \) over all hyperplanes \( H_{a,n} (a \in \mathbb{R}, n \in \mathbb{Z}) \) separating \( A \) and \( B \) is denoted by \( d(A, B) \). We say that \( A \leq B \) if there exists a sequence of alcoves \( A = A_0, A_1, \ldots, A_m = B \) such that for any \( i \) \( (1 \leq i \leq m) \) we have \( d(A_{i-1}, A_i) = 1 \) and \( A_i = A_{i-1} \sigma_{H_i} \) for some \( H_i \in \{ H_{a,n} | a \in \mathbb{R}, n \in \mathbb{Z} \} \).

2.3. Let \( A \in M \), we set

\[ L(A) = \{ r \in S | A \subset C_v \) and \( rA \notin C_v \) for some special point \( v \} \). \]

Let \( \mathcal{M} \) be the free \( \mathcal{A} \)-module with basis \( M \). \( \mathcal{M} \) has an unique \( \mathcal{H}' \)-module structure such that

\[ T_r A = rA \quad \text{if} \quad r \notin L(A) \]
\[ T_r A = q^2 rA + (q^2 - 1)A \quad \text{if} \quad r \in L(A). \]

Let \( w \in W' \), \( A \in M \), \( T_w A = \sum_{B \in M} \pi_{w,A,B} B \). It is easy to see that if \( \pi_{w,A,B} \neq 0 \) then \( \pi_{w,A,B} \) is a polynomial in \( q \) (in fact in \( q^2 \)) with positive leading coefficient (the coefficient of the highest power of \( q \) in \( \pi_{w,A,B} \)).

We shall need some results due to Lusztig.

**Proposition 2.4** (see [2, 4.2]). Let \( A \in M \) be such that \( \overline{A} \ni v \) a special point of \( E \). If \( wA_v \subset C_v \), then \( \deg \pi_{w,A,B} \leq d(B, wA_v) \), and if \( \deg \pi_{w,A,B} = d(B, wA_v) \) then \( B = wA_v \lambda \) for some translation \( \lambda \) in \( \Lambda \).

2.5. Let \( v \in E \) be a special point and let \( w \in W' \) be such that \( wA_v \subset C_v \).

We set

\[ E_w = q^{-l(w)} \sum_{u \leq _{ww_v} \in \mathbb{Z}} P_{ww_v} T_u. \]

It is easy to check that \( E_w C_{w_v} = C_{ww_v} \).

Let \( A_v^- = w_v A_v \). If \( wA_v \subset C_v \), then

\[ D_C = q^{l(wv)} C_{wv} A_v^- = \sum_{A \in M} Q_A \cdot A \]
has the following properties (see [2, 5.2]):

(a) \( Q_{A,C} \neq 0 \) implies that \( A \leq C \).

(b) If \( A \leq C \) then \( Q_{A,C} \) is a polynomial of degree \( \leq d(A,C) - 1 \) if \( A \neq C \), and \( Q_{C,C} = 1 \).

Note that \( D_C = q^{(w_0 E_w D_A, w \in W', wA_c \subset \Pi_v, C = wA_v}.

Let \( h \) be the anti-isomorphism of \( A \) defined by \( h(T) = T^{-1} \) \( u \in W \) and let \( F_w = h(E_w) \) if \( w \in W' \), \( wA_c \subset \mathcal{C}_w \). We then have \( C_{w_f} F_w = h(E_w C_{w_f}) = h(C_{w_f}) = C_{w_f}^{-1} \).

**Proposition 2.6** (see [2, 5.4]). Let \( v, u \in E \) be a special point and \( w \in W' \) be such that \( uA_v = C \in \mathcal{C}_w \). Then,

\[
q^{(w_0 v) A_v^-} = \sum_B n_B q^{d(B, C)} D_B,
\]

where \( n_B = \sum A \in B \) are determined by \( \pi_{A,B} = n_B q^{d(B, C)} + \) lower power terms,

\[
T_u \sum_{A \in B} A = \sum_B \pi_{A,B} B, \pi_{A,B} \in A.
\]

The following lemma is a key to Theorem 1.10.

**Lemma 2.7.** Let \( v, u \) be as in 2.6. Suppose that \( uA_v = A_v' \), i.e., \( uA_v \) is the translate of \( A_v \). Let \( w \in W' \) be such that \( wA_v \subset \mathcal{C}_w \). Then, \( C_{w_f} A_{w_f} = C_{w_f} \).

**Proof.** Let \( T_u \sum_{A \in v} A' = \sum_B \pi_{u,B} B \) and \( \pi_{u,B} = n_B q^{d(B, C)} + \) lower power terms, where \( C = uA_v = A_v' \). By 2.6 then we have

\[
q^{(w_0 v) A_v^-} = \sum_B n_B q^{d(B, C)} D_B.
\]

Using 2.4 we see that if \( n_B \neq 0 \) then \( B = C A \cdot A = A \cdot A \) for some translation \( A \in A \). Hence we have \( wB \subset \Pi_v \) since \( wA_v \subset \Pi_v \). Applying 2.5 we obtain

\[
q^{(w_0 v) E_w C_{w_f} A_{w_f}^{-}} = \sum_B n_B q^{d(B, C)} D_{w_B}.
\]

Let \( D_{w_B} = \sum A^{w_B} Q_{A,w_B} A \), then \( \deg Q_{A,w_B} \leq d(A, wB) - 1 \) if \( A \neq wB \) and \( Q_{wB,w_B} = 1 \). Let \( C' = wC = wuA_v \), then \( d(B, C) = d(wB, wC) = d(wB, C') \) if \( n_B \neq 0 \). Thus in the expression \( q^{(w_0 w_0 v) E_w C_{w_f} A_{w_f}^{-}} = \sum A \pi_A A \) we have \( \deg \pi_A \leq d(A, C') \) and \( \deg \pi_A = d(A, C') \) if and only if \( A \neq wB \) for some \( B \) with \( n_B \neq 0 \). In this case we have \( \pi_A = n_B q^{d(A, C')} + \) lower power terms. Therefore we can describe \( n_B \) as the coefficient of \( q^{d(A, C')} \) in \( \pi_A \) for some \( A \in M \).

On the other hand, we have \( T_{w_0} \sum_{A \in v} A' = T_{w} \sum_B \pi_{u,B} B = \sum_A \pi_A A \). If
deg $\pi_{u,B} = d(B, C)$ then $B = A;\lambda$ for some translation $\lambda \in A$, hence one knows that $T_u, B = wB$. Note that $\pi_{w', B', B'}$ has a positive leading coefficient for any $w' \in W'$, $B', B'' \in M (2.3)$ and that the Kazhdan–Lusztig polynomial $P_{w', w''}$ has non-negative coefficients for any $w', w'' \in W$. According to the above description of $n_B$ we see that $\deg \pi_A \leq d(A, C')$ and $\deg \pi_A = d(A, C')$ if and only if $A = wB$ for some $B$ with $n_B \neq 0$, and $\pi_A$ has leading coefficient $n_B$ in this case. By 2.6 we have the equality

$$E_w C_{uw_v} A_v^- = C_{uw_v} A_v^-.$$

The lemma therefore can be deduced from the following result.

**Lemma 2.8.** Let $v \in E$ be a special point and let $U_v = \{ u \in W' | l(uw_v) = l(u) + l(w_v) \}$, then $\sum_{u \in U_v} a_u C_{uw_v} A_v^- = \sum_{u \in U_v} b_u C_{uw_v} A_v^- \quad \text{(finite sums)}$ implies that $\sum_{u \in U_v} a_u C_{uw_v} = \sum_{u \in U_v} b_u C_{uw_v}$.

**Proof:** Let $u_0$ be an element in $\{ u \in U_v | a_u \neq 0 \text{ or } b_u \neq 0 \}$ with maximal length, then

$$\sum_{u \in U_v} a_u C_{uw_v} A_v^- = a_{u_0} u_0 A_v + \sum_{u \in A_v \neq B} a_B B = b_{u_0} u_0 A_v + \sum_{u \in A_v \neq B} b_B B.$$

Hence $a_{u_0} = b_{u_0}$. Using induction on $l(u_0)$ we see that the lemma holds.

**Theorem 2.9.** Let $w, u, v$ be as in 2.7. Let $w' \in W'$ be such that $w'A_v \in c_v$, then $E_w C_{uw_v} F_{w'} = C_{uw_v} F_{w'}$.

**Proof:** (1) For any $x \in W$, set $\tilde{T}_x = q^{-l(x)} T_x$. Let $x, y \in W$, write that

$$\tilde{T}_x \tilde{T}_y = \sum_{z \in W} f_{x,y,z} \tilde{T}_z, \quad f_{x,y,z} \in A.$$

Then we have

(a) $f_{x,y,z}$ is a polynomial in $q - q^{-1} = \xi$ with non-negative integer coefficients and $\deg \xi f_{x,y,z} \leq v$ (see [4, 1, Theorem 7.2]).

Let $f_{x,y,z} = \gamma_{x,y,z} \xi^r + \text{lower degree terms}$, then we have

(b) If $x, y, z \in c$, then $\gamma_{x,y,z} = \gamma_{x', y', z'}$ (see [4, I, 5.2 and 7.10]). Let $x, y \in c_0, z \in W$ be such that $l(xz^{-1}) = l(x) + l(z)$ and $l(zy) = l(z) + l(y)$, then

$$\tilde{T}_{xz^{-1}} \tilde{T}_{zy} = \tilde{T}_x (\tilde{T}_{z^{-1}} \tilde{T}_z) \tilde{T}_y$$

$$= \tilde{T}_x \left( \sum_{z' \in \Xi_{x^{-1}, z, z'}} \tilde{T}_{z'} \right) \tilde{T}_y.$$
Note that $f_{z^{-1},z,e} = 1$, by (a) and (b) we see that
\[(c) \quad \gamma_{z^{-1},z,e} = 1, \quad \gamma_{z^{-1},z,e} = 1, \quad \gamma_{z^{-1},z,e} = 1, \quad \gamma_{z^{-1},z,e} = 1.
\]
It is easy to see that
\[(d) \quad \gamma_{w_z, w_e, w_e} = 1.
\]
(2) We have
\[(d') \quad E_w C_{uw_1} C_{w_2} F_w = C_{uw_2} C_{w_1 w'_1} \quad (2.7, 2.5) = \sum_{z} h_{wwzw, w'w'_1w^{-1}z} C_z \quad (1.4).
\]
Because $w'w_2w'_1$ is a distinguished involution of $W$ (see [6, II, Theorem 6.1]), we know that
\[(e) \quad \gamma_{wwzw, w'_1w'_1w^{-1}z} \neq 0 \quad \text{if and only if} \quad z = wuw_z w'_1w^{-1}, \quad \text{and in this case} \quad \gamma_{wwzw, w'_1w'_1w^{-1}z} = 1 \quad \text{(see [4, II, 1.4 and 1.8]).}
\]
By (b), (d), and (e) we see that
\[(f) \quad \gamma_{wwzw, w'_1w'_1w^{-1}z} \neq 0 \quad \text{if and only if} \quad z = wuw_z w'_1w^{-1}.
\]
That is equivalent to saying that
\[(g) \quad \deg h_{wwzw, w'_1w'_1w^{-1}z} = v \quad \text{if and only if} \quad z = wuw_z w'_1w^{-1}.
\]
(3) We have
\[(h) \quad E_w C_{uw_2} C_{w_3} F_w = q^{-v}(\sum_{w 
 This is a proof of the theorem.

2.10. Let $u \in E$ be a special point and let $X_u^+ = \{x \in X | l(w,x) = l(w) + l(x)\}$. For any $x \in X_u^+$ there exist some $\omega \in \Omega$, $u \in U_v$ such that $w_x w_u = \omega u$. It is obvious that $uA_v \subset C_v$ is a translate $A_v'$ of $A_v$. Let $w, w' \in W'$ be such that $w_A \subset \Pi_v, w' A_v \subset \Pi_v$, then we have the following

**Corollary 2.11.** In the setup of 2.10, we have $E_w C_{w_2} F_w = C_{w_1 w'w^{-1}}$. 

**Proof.** Note that $w_x w_u = \omega u$ and one knows that $(\omega^{-1} w \omega) A_v \subset \Pi_v$. 


Hence

\[ E_w C_{w,x} F_{w'} = E_w T_\omega C_{w,w_x} F_{w'} = T_\omega E_{\omega^{-1} w_0} C_{w,w_x} F_{w'} = T_\omega C_{\omega^{-1} w_0 w_x,w'}^{-1} \]  

(2.9)

\[ = C_{w_0 w_x w^{-1}} = C_{w_0 x w'}^{-1}. \]

The corollary is proved.

3. THE CENTER OF \( J_0 \)

In this section we describe the center of \( J_0 \) explicitly.

3.1. For any \( x \in X \) we choose \( x', x'' \in X^+ = X^+_0 \) such that \( x = x' x'' \) and then define \( \bar{\gamma}_x = q^{l(x')} T_x (q^{l(x')} T_x)^{-1} \). \( \bar{\gamma}_x \) is independent of the choices of \( x' \) and \( x'' \). We denote the conjugacy class of \( x \in X \) in \( \mathcal{W} \) by \( \mathcal{O}_x \), and let \( z_x = \sum_{x' \in \mathcal{O}_x} \bar{\gamma}_{x'} \). Then the additive subgroup \( Z \) of \( \mathcal{H} \) generated by \( \{ z_x | x \in X \} \) is a free abelian group and \( Z \otimes \mathcal{A} \) is just the center of \( \mathcal{H} \). Note that \( z_x \), \( x \in X^+ \) is a basis of \( Z \). For any \( x \in X^+ \) let \( V(x) \) be the unique (up to isomorphism) rational irreducible representation of \( G \) with the highest weight \( x \). Denote \( d(x',x) \) as the dimension of the \( x' \)-weight space \( V(x)_{x'} \) of \( V(x) \). Then \( S_x = \sum_{x' \in X^+} d(x',x) z_{x'} \), \( x \in X^+ \) is another basis of \( Z \).

3.2. Let \( \sigma \in \mathcal{W} \) be the set \( \{ \omega w | \omega \in \Omega, w \in \mathcal{W}' \text{ and } w A_0 \subset \Pi_0 \} \). Then

\[ c_0 = \{ \omega w_0 w^{-1} | \omega \in \Omega, u \in U_0, w \in \sigma \} \quad \text{(see [6, 1])}, \]

where \( w_0 \) is the longest element in \( W_0 \). For any \( \omega \in \Omega, u \in U_0 \), there exists unique \( w' \in \sigma, x \in X^+ \) such that \( \omega u = w' w_0 x w_0 \). Hence

\[ c_0 = \{ w' w_0 x w^{-1} | w', w' \in \sigma, x \in X^+ \}. \]

Each element in \( c_0 \) has a unique expression of the form \( w' w_0 x w^{-1} \).

Let \( z_1 = w_1' w_0 x w_1^{-1}, z_2 = w_2' w_0 x w_2^{-1} \), \( w_i, w_i \in \sigma \) \((i = 1, 2)\), \( x, x' \in X^+ \). Then \( z_1 \sim L z_2 \) if and only if \( w_1 = w_2 \) (see [6, II]). As a result one has \( z_1 \sim L z_i^{-1} \) if and only if \( w_1 = w_i' \).

For any \( w = \omega w_1, \omega \in \Omega, w_1 \in \mathcal{W}', \) and \( w_1 A_0 \subset \Pi_0 \) we set \( E_w = T_\omega E_{w_1} \) and \( F_w = h(E_w) = F_{w_1} T_\omega^{-1} \). Then Corollary 2.11 can be re-expressed as follows.

**Lemma 3.3.** Let \( w, w' \in \sigma, x \in X^+ \), then \( E_w C_{w_0,x} F_{w'} = C_{w_0 x w^{-1}} \).

**Lemma 3.4.** Let \( x \in X^+ \), \( w, w' \in \sigma \), then \( S_x C_{w_0,x} = C_{w_0 x w^{-1}}. \)
Proof. By 3.3 we see that $S_x C_{w_0 w^{-1}} = E_w S_x C_{w_0 F_w}$. But $S_x C_{w_0} = C_{w_0 x}$ (see [3, 8.6]). Hence $S_x C_{w_0 w^{-1}} = E_w C_{w_0 x} F_w = C_{w_0 x w^{-1}}$ (3.3). The lemma is proved.

3.5. Let $v \in E$ be a special point and let $w, w' \in W'$ be such that $w A_v \subset \Pi_v$, $w' A_v \subset \Pi_v$. Let $x \in X^+$. Note that there exists $\omega \in \Omega$ such that $w_v = \omega^{-1} w_0 \omega$ and $\omega x w^{-1} \in X^+$ one has $S_{\omega x w^{-1}} C_{w w_0 w^{-1}} = C_{w w_0 x w^{-1}}$ by 3.4.

Theorem 3.6. $\phi(Z)$ is just the center of $J_0$. Let $\Gamma$ be a left cell in $c_0$ and $d \in \Gamma$ be a distinguished involution, then $\phi(Z) d = J_{\Gamma \cap \Gamma^{-1}}$.

Proof. First we know that $\phi(Z)$ is in the center of $J_0 \otimes \mathcal{A}$ (see [4, III]). Note that $\mathcal{O}_0 = \{w w_0 w^{-1} | w \in \sigma\}$ (see [6, II, Theorem 6.1]). By the definition of $\phi$ and 3.4 we have

$$\phi(S_x) = \sum_{w \in \sigma} t_{w w_0 w^{-1}} \in J_{\Gamma}, \quad x \in X^+.$$

Now let $b = \sum_{u} a_u t_u$ be in the center of $J_0$. We need to prove that $b$ is a linear combination of some $\phi(S_x), x \in X^+$. Suppose that $a_u \neq 0, u \in c_0$. Choose $d \in \mathcal{O}_0$ such that $d \sim_L u$, then $t_d$ appears in $t_{u^{-1} b}$ with coefficient $a_u \neq 0$. But $t_{u^{-1} b} = b t_{u^{-1}}$, hence $d \sim_L u^{-1}$. Thus $u \sim_L u^{-1}$ and there exist some $w \in \sigma, x \in X^+$ such that $u = w w_0 x w^{-1}$. Let $u' = w' w_0 x w'^{-1}, w' \in \sigma$, we assert that $a_{u'} = a_u$. In fact let $u_1 = w w_0 x w_{1}^{-1}$, then $t_{w w_0 x w_{1}^{-1}}$ appears in $b t_{u_1}$ with coefficient $a_u$ and appears in $t_{u_1} b$ with coefficient $a_{u'}$. Therefore $b$ is a linear combination of some $\phi(S_{x'}), x' \in X^+$. The first assertion is proved. The second one follows from the explicit expression of $\phi(S_x)$ and 3.2.

4. The Proof of Theorem 1.10

4.1. Let $x, x', x'' \in X^+$, we denote $m(x, x', x'')$ as the multiplicity of $V(x'')$ appearing in the tensor product $V(x) \otimes V(x')$. Then $S_x S_{x'} = \sum_{x'' \in X^+} m(x, x', x'') S_{x''}$. 

Lemma 4.2. Let $w_1 = w w_0 x w^{-1}, w_2 = w w_0 x' w^{-1}, w \in \sigma, x, x' \in X^+$. Then $\gamma_{w_1, w_2, w_3} \neq 0, w_3 \in c_0$, implies that $w_3 = w w_0 x'' w^{-1}$ for some $x'' \in X^+$ and in this case we have $\gamma_{w_1, w_2, w_3} = m(x, x', x'')$.

Proof. If $\gamma_{w_1, w_2, w_3} \neq 0$ then $w_3 \sim_L w_2, w_3^{-1} \sim_L w_1^{-1} \sim_L w_2$. Hence $w_3 \sim_L w_1^{-1}$ and $w_3 = w w_0 x'' w^{-1}$ for some $x'' \in X^+$ (3.2). $\gamma_{w_1, w_2, w_3} = m(x, x', x'')$ follows from 3.4, 4.1, and $\phi$ is an algebra homomorphism from $\mathcal{A}$ to $J_0 \otimes \mathcal{A}$. 
Corollary 4.3. Let \( w_i \in \sigma, \; i = 1, 2, 3, 4, \; x, x' \in X^+ \) and let \( u_1 = w_1 w_3 w_2^{-1}, \; u_2 = w_3 w_0 x' w_4^{-1} \). Then

(a) \( \gamma_{u_1, u_2, u_3} = 0 \) for any \( u_3 \in c_0 \) if \( w_2 \neq w_3 \).

(b) If \( w_2 = w_3 \) and \( \gamma_{u_1, u_2, u_3} \neq 0 \) then \( u_3 = w_1 w_0 x'' w_4^{-1} \) for some \( x'' \in X \) and \( \gamma_{u_1, u_2, u_3} = m(x, x', x'') \).

Proof. (a) follows from \( u_1 \not\sim u_2^{-1} \).

Now suppose that \( w_2 = w_3 \). Let \( u_1' = w_0 x w_2^{-1}, \; u_2' = w_2 w_0 x', \) then \( h_{u_1', u_2', u_3'} \neq 0 \) implies that \( u_3' = w_0 x'' \) for some \( x'' \in X^+ \) since \( u_3' \sim u_2' \) and \( u_3' \sim_R u_1' \). Using 4.2 we see that \( \gamma_{u_1', u_2', u_3'} = m(x, x', x'') \). By 3.3 we know that \( h_{u_1, u_2, u_1 w_0 x'' w_4^{-1}} = h_{u_1, u_2, u_3'} \). The corollary is proved.

4.4. Proof of Theorem 1.10. Let \( Y = \sigma, \) then \( |Y| = |W_0| \). Let \( G \) act on \( Y \) trivially. For any \( u \in c_0, \; u = w w_0 x w' w^{-1}, \) \( w, w' \in \sigma, \; x \in X^+ \), let \( V(u) \) be the unique irreducible \( G \)-v.b. on \( Y \times Y \) such that \( V(u)(w, w') = V(x) \). The map \( t_u \rightarrow V(u) \) defines a bijection between the set \( \{ t_u | u \in c_0 \} \) and the set of isomorphism classes of irreducible \( G \)-v.b. on \( Y \times Y \). By 4.3 we know that the bijection gives rise to an isomorphism \( \psi \) of rings between \( J_0 \) and \( K_G(Y \times Y) \). \( \psi(t_u \cdot v) = \overline{V(u)} \) follows from \( V(x)^* = V(w_0 x w_0^{-1}) \). The theorem is proved.

4.5. Let \( r_0 \) be the unique simple reflection of \( W' \) which doesn't belong to \( W_0 \). For any two-sided cell \( c \neq \{ e \} \) of \( W_0 \), there is a unique left cell \( \Gamma \) in \( c \) such that \( ur \leq u, \; u \in \Gamma, \; r \in S \) implies that \( r = r_0 \) (see [5]). If \( c = \{ e \} \), let \( \Gamma = c \). Then the element \( u \in \bigcup \Gamma_c \cap \Gamma_c^{-1} \) is just the shortest one in the double coset \( W_0 u W_0 = W_0 x W_0, \; x \in X^+ \). Thus \( \bigcup \Gamma_c \cap \Gamma_c^{-1} \) has a bijection with \( X^+ \). Let \( x_1, x_2, ..., x_k \) be basic weights; i.e., \( x_1, x_2, ..., x_k \) generate \( X \) and \( \langle x', x_i \rangle > 0 \) for all \( x \in A \) except one \( x \in A \) and \( \langle x', x_i \rangle = 1 \).

Proposition 4.6. In the setup of 4.5, we have \( u \in c_0 \) if and only if \( x_i^{-2} x \in X^+ \) for all basic weights \( x_i \).

Proof. Assume that \( u \in \Gamma_{c_0} \cap \Gamma_{c_0}^{-1} \), then \( u = w w_0 x' w^{-1} \), for some \( w \in \sigma, \; x' \in X^+ \). Since \( ur \leq u, \; r \in S \) implies that \( r = r_0 \), \( w \in \sigma \) is an element in \( \sigma \) with maximal length. Hence \( w = x_1 x_2 ... x_k w_0 \), and \( u = x_1^2 x_2^2 ... x_k^2 x' w_0 \). So \( x = x_1^2 x_2^2 ... x_k^2 x' \). The proposition is proved.

5. The Representations

In this section we give some discussions on the representations of \( J_0 = J_0 \otimes \mathbb{C}, \; K_G(Y \times Y) = K_G(Y \times Y) \otimes \mathbb{C}, \) and \( J_{\Gamma} \cap \Gamma^{-1} = J_{\Gamma} \cap \Gamma^{-1} \otimes \mathbb{C}, \; \Gamma \) a left cell in \( c_0 \).
5.1. We recall some results about the representations of $J_0$ and $K_G(Y \times Y)$ due to Lusztig. The reference is [4, III and IV].

It is known that the isomorphism classes of irreducible representations of $J_0$ has a bijection with the set of pairs $(s, \rho)$ modulo the action of $G$, where $s \in G$ is a semisimple element and $\rho$ is an irreducible representation of $A(s) = Z_G(s)/Z_G(s)''$ appearing in some $H^{2i}(\beta^i, \mathbb{C})$, here $\beta^i$ is the variety of Borel subgroups of $G$ containing $s$. Now $G$ is simply connected, almost simple, hence $A(s) = \{e\}$ and $\rho$ is always a unit representation. Thus the set of isomorphism classes of irreducible representations of $J_0$ has a bijection with the set of semisimple conjugacy classes of $G$. Let $E(s)$ be the irreducible module of $J_0$ corresponding to the conjugacy containing $s$. Then $E(s)$ can be described as the unique irreducible module of $J_0$ with $\phi(S_x)$ acting on it by scalar $\text{tr}(s, V(x))$.

Let $Y$ be as in 4.4. Let $\mathcal{F}$ be the algebra of $\mathbb{C}$-valued functions on $Y \times Y$, the multiplication in $\mathcal{F}$ is given by $f * f' (y_1, y_2) = \sum_{y \in Y} f(y_1, y) f'(y, y_2), f, f' \in \mathcal{F}, (y_1, y_2) \in Y \times Y$. For any semisimple element $s \in G$ we have an algebra homomorphism $h_s : K_G(Y \times Y) \to \mathcal{F}$, by associating to each $G$-vb. $V$ on $Y \times Y$ to the function $f(y, y') = \text{tr}(s, V_{y,y'})$. Let $F$ be the vector space of all functions $Y \to \mathbb{C}$. $F$ becomes an irreducible module of $\mathcal{F}$ if we define $f\mu(y) = \sum_{y' \in Y} f(y, y') \mu(y'), f \in \mathcal{F}, \mu \in F, y \in Y$, which in fact is the unique (up to isomorphism) irreducible of $\mathcal{F}$. Via homomorphism $h_s$, $F$ becomes an irreducible module $F(s)$ of $K_G(Y \times Y)$. Mas $s \to F(s)$ gives a bijection between the set of semisimple conjugacy classes of $G$ and the set of irreducible modules (up to isomorphism) of $K_G(Y \times Y)$.

**Proposition 5.2.** Each irreducible module of $J_0$ has dimension $|W_0|$. Via the isomorphism $\psi \otimes \text{id} : J_0 \to K_G(Y \times Y)$, $F(s)$ becomes an irreducible module of $J_0$ which is just $E(s)$.

**Proof.** The first assertion follows from $|Y| = |W_0|$. The second one follows from the fact that $\psi(\phi(S_x)) (x \in X^+) acts on $F(s)$ by scalar $\text{tr}(s, V(x))$.

5.3. **Remark.** Is it true that any irreducible module of $J_c \otimes \mathbb{C}$ has dimension $\leq |W_0|$ for every two-sided cell $c$ of $W$?

5.4. For any semisimple element $s \in G$, let $I_s$ be the irreducible module of $Z \otimes \mathbb{C}$ such that $S_x (x \in X^+)$ acts on it by scalar $\text{tr}(s, V(x))$. $I_s$ becomes an irreducible module of $J_{\Gamma \cap \Gamma^{-1}}$ via the isomorphism $\sum_{x \in X^+} S_x \otimes a_x \to \sum_{x \in X^+} \phi(S_x) t_d \otimes a_x$, $a_x \in \mathbb{C}$, here $\Gamma$ is a left cell in $c_0$, $d$ is the distinguished involution in $\Gamma$. $s \to I_s$ gives a bijection between the set of semisimple conjugacy classes of $G$ and the set of isomorphism classes of irreducible modules of $J_{\Gamma \cap \Gamma^{-1}}$. 

481/134/2-8
PROPOSITION 5.5. Let $\Gamma$ be a left cell in $c_0$, $d \in \Gamma$ be the distinguished involution, then $t_d E(s) = I_s$.

Proof. It is easy to see that $I_s$ must appear in some $t_d E(s')$ for some semisimple element $s'$ in $G$. But $t_d E(s')$ is a direct sum of some copies of $I_s$ if $t_d E(s') \neq 0$. Hence $t_d E(s) \neq 0$. Now $E(s)$ has dimension $|W_0|$ and the number of left cells in $c_0$ is also $|W_0|$. Therefore $\dim t_d E(s) = 1$ and $t_d E(s) = I_s$. The proposition is proved.

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REFERENCES