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Best proximity point theorems

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Abstract

Let us assume that A and B are non-empty subsets of a metric space. In view of the fact that a non-self mapping $T : A \longrightarrow B$ does not necessarily have a fixed point, it is of considerable significance to explore the existence of an element x that is as close to Tx as possible. In other words, when the fixed point equation $Tx = x$ has no solution, then it is attempted to determine an approximate solution x such that the error $d(x, Tx)$ is minimum. Indeed, best proximity point theorems investigate the existence of such optimal approximate solutions, known as best proximity points, of the fixed point equation $Tx = x$ when there is no solution. Because $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem ascertains an absolute minimum of the error $d(x, Tx)$ by stipulating an approximate solution x of the fixed point equation $Tx = x$ to satisfy the condition that $d(x, Tx) = d(A, B)$. This article establishes best proximity point theorems for proximal contractions, thereby extending Banach's contraction principle to the case of non-self mappings.

Keywords: Proximal contraction; Fixed point; Best proximity point; Optimal approximate solution

1. Introduction

The classical and well-known Banach's contraction principle states that every contraction on a complete metric space has a unique fixed point that is realizable as the limit of Picard iterates. Numerous interesting extensions and variants of the aforesaid result exist in the literature. However, the mappings involved in all these results are self-mappings. So, it is contemplated in this paper to derive some best proximity point theorems which furnish non-self mapping

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analogues of the aforesaid Banach's contraction principle. Consequently, the results established in this article guarantee the existence of optimal approximate solutions for certain fixed point equations when there is no solution.

Fixed point theory is an indispensable tool for solving the equation $Tx = x$ for a mapping T defined on a subset of a metric space, a normed linear space or a topological vector space. As a non-self mapping $T : A \longrightarrow B$ does not necessarily have a fixed point, one often tries to determine an element x which is in some sense closest to Tx. Best approximation theorems and best proximity point theorems are pertinent in this perspective. A classical best approximation theorem, due to Fan [7], asserts that if A is a non-empty compact convex subset of a Hausdorff locally convex topological vector space X with a semi-norm p and $T : A \longrightarrow X$ is a continuous mapping, then there is an element x in A satisfying the condition that $d_p(x, Tx) = d_p(Tx, A)$. There have been many subsequent extensions and variants of Fan's Theorem, including those by Prolla [11], Reich [12], Sehgal and Singh, [18,19]. Further, Vetrivel et al. [23] have furnished a unified approach to such interesting results.

On the other hand, despite the fact that best approximation theorems assure the existence of approximate solutions, such results need not produce optimal solutions. Best proximity point theorems provide sufficient conditions that ensure the existence of approximate solutions which are optimal as well. In fact, if there is no solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \longrightarrow B$, then it is desirable to determine an approximate solution x such that the error $d(x, Tx)$ is minimum. In light of the fact that $d(x, Tx) \geq d(A, B)$, an absolute optimal approximate solution is an element x for which the error $d(x, Tx)$ assumes the least possible value $d(A, B)$. As a result, a best proximity pair theorem furnishes sufficient conditions for the existence of an optimal approximate solution x, known as a best proximity point of the mapping T, satisfying the condition that $d(x, Tx) = d(A, B)$. Interestingly, best proximity theorems also serve as a natural generalization of fixed point theorems. Indeed, a best proximity point becomes a fixed point if the mapping under consideration is a self-mapping.

Analysis of several variants of contractions for the existence of a best proximity point can be seen in [1,5,6,8,22,24]. Many best proximity point theorems for set valued mappings have been established in [2,3,9,10,15–17,21,25–27]. Anthony Eldred et al. [4] have obtained best proximity point theorems for relatively non-expansive mappings. A best proximity point theorem for contractive non-self-mappings has been explored in [13]. Further, some common best proximity point theorems have been analyzed in [14,20].

On account of the preceding discussion, it is appropriate to elicit best proximity point theorems to facilitate the proposed generalizations of Banach's contraction principle to the case of non-self mappings. Indeed, this article focuses on best proximity point theorems for proximal contractions of the first and second kind, which serve as non-self mapping analogues of contraction self-mappings. Also, necessary and sufficient conditions are established for a non-self contraction mapping to have a best proximity point.

2. Preliminaries

Given two non-empty subsets A and B of a metric space, the following notions and notations are used in the sequel.

$$d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$$

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 2.1. A is said to be *approximatively compact with respect to* B if every sequence $\{x_n\}$ of A satisfying the condition that $d(y, x_n) \longrightarrow d(y, A)$ for some y in B has a convergent subsequence.

It is evident that every set is approximatively compact with respect to itself. If A intersects B, then $A \bigcap B$ is contained in both A_0 and B_0. Further, it can be seen that if A is compact and B is approximatively compact with respect to A, then the sets A_0 and B_0 are non-empty.

Definition 2.2. A mapping $T : A \longrightarrow B$ is said to be a *proximal contraction of first kind* if there exists a non-negative number $\alpha < 1$ such that, for all u_1, u_2, x_1, x_2 in A,

$$\left.\begin{array}{l} d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

It is easy to observe that a self-mapping that is a proximal contraction of the first kind reduces to a contraction.

Definition 2.3. A mapping $T : A \longrightarrow B$ is said to be a *strong proximal contraction of the first kind* if there exists a non-negative number $\alpha < 1$ such that, for all u_1, u_2, x_1, x_2 in A and for all $\beta \in [1, 2)$,

$$\left.\begin{array}{l} d(u_1, Tx_1) \leq \beta d(A, B) \\ d(u_2, Tx_2) \leq \beta d(A, B) \end{array}\right\} \Longrightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2) + (\beta - 1) d(A, B).$$

It is evident that a self-mapping that is a strong proximal contraction of the first kind reduces to a contraction.

Definition 2.4. A non-self mapping $T : A \longrightarrow B$ is said to be a *proximal contraction of the second kind* if there exists a non-negative real number $\alpha < 1$ such that

$$d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2)$$

whenever x_1, x_2, u_1 and u_2 are elements in A satisfying the condition that

$$d(u_1, Tx_1) = d(A, B) \quad \text{and} \quad d(u_2, Tx_2) = d(A, B).$$

The requirement for a self-mapping T to be a proximal contraction of second kind is that

$$d(T^2 x_1, T^2 x_2) \leq \alpha d(Tx_1, Tx_2)$$

for all x_1 and x_2 in the domain of T. Consequently, any contraction self-mapping is a proximal contraction of the second kind but the converse is not true. Consider R endowed with the Euclidean metric. Let the self-mapping $T : [0, 1] \longrightarrow [0, 1]$ be defined as

$$Tx = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise.} \end{cases}$$

Then, T is a proximal contraction of the second kind but not a contraction. Further, the preceding example exhibits that even a self-mapping that is a proximal contraction of second kind is not necessarily continuous.

Definition 2.5. Given $T : A \longrightarrow B$ and an isometry $g : A \longrightarrow A$, the mapping T is said to preserve isometric distance with respect to g if

$$d(Tgx_1, Tgx_2) = d(Tx_1, Tx_2)$$

for all x_1 and x_2 in A.

Definition 2.6. Given non-self mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$, the pair (S, T) is said to satisfy min – max condition if for all $x \in A$ and $y \in B$,

$$d(A, B) < d(x, y) \Longrightarrow \min(Sx, Ty) \neq \max(Sx, Ty)$$

where $\min(Sx, Ty)$ and $\max(Sx, Ty)$ are defined as

$$\begin{aligned} \min(Sx, Ty) &= \min\{d(x, y), d(x, Sx), d(y, Ty), d(Sx, Ty), d(x, STy), \\ &\quad d(y, TSx), d(Sx, TSx), d(Ty, STy), d(TSx, STy)\} \\ \max(Sx, Ty) &= \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx), d(Sx, Ty), \\ &\quad d(x, TSx), d(y, STy), d(x, STy), d(y, TSx), d(Sx, TSx), \\ &\quad d(Ty, STy), d(TSx, STy)\}. \end{aligned}$$

It can be observed that the min–max condition is satisfied by several classes of mappings.

Definition 2.7. Given non-self mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$, the pair (S, T) is said to be

(a) a cyclic contractive pair if $d(A, B) < d(x, y) \Longrightarrow d(Sx, Ty) < d(x, y)$
(b) a cyclic expansive pair if $d(A, B) < d(x, y) \Longrightarrow d(Sx, Ty) > d(x, y)$
(c) a cyclic inequality pair if $d(A, B) < d(x, y) \Longrightarrow d(Sx, Ty) \neq d(x, y)$

for all $x \in A$ and $y \in B$.

It is apparent that cyclic contractive pairs, cyclic expansive pairs and cyclic inequality pairs satisfy the min–max condition.

Definition 2.8. Given mappings $S : A \longrightarrow B$ and $T : B \longrightarrow A$, the pair (S, T) is said to form a cyclic contraction if there exists a non-negative number $\alpha < 1$ such that

$$d(Sx, Ty) \leq \alpha d(x, y) + (1 - \alpha) d(A, B)$$

for all $x \in A$ and $y \in B$.

It is easy to see that every cyclic contraction pair is cyclic contractive and hence satisfies the min–max condition.

3. Proximal contractions

The following best proximity point theorem extends Banach's contraction principle to the case of non-self mappings.

Theorem 3.1. *Let X be a complete metric space. Let A and B be non-empty, closed subsets of X such that A is approximatively compact with respect to B. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions.*

(a) *T is a continuous proximal contraction of second kind.*
(b) *g is an isometry.*
(c) *$T(A_0)$ is contained in B_0.*
(d) *A_0 is contained in $g(A_0)$.*
(e) *T preserves isometric distance with respect to g.*

Then, there exists an element x in A such that

$$d(gx, Tx) = d(A, B).$$

Moreover, if x^* is another element for which the preceding conclusion holds, then Tx and Tx^* are identical.

Proof. Let x_0 be a fixed element in A_0. Since $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$, there exists an element x_1 in A_0 such that

$$d(gx_1, Tx_0) = d(A, B).$$

Again, since Tx_1 is an element of $T(A_0)$ which is contained in B_0, and A_0 is contained in $g(A_0)$, it follows that there is an element x_2 in A_0 such that

$$d(gx_2, Tx_1) = d(A, B).$$

This process can be continued. Having chosen x_n in A_0, it is possible to find x_{n+1} in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B)$$

for every positive integer n because of the fact that $T(A_0)$ is contained in B_0 and A_0 is contained in $g(A_0)$. As T is a proximal contraction of the second kind,

$$d(Tgx_{n+1}, Tgx_n) \leq \alpha d(Tx_n, Tx_{n-1}).$$

Since T preserves isometric distance with respect to g,

$$d(Tx_{n+1}, Tx_n) \leq \alpha d(Tx_n, Tx_{n-1}).$$

So, it follows that $\{Tx_n\}$ is a Cauchy sequence and hence it converges to some element y in B.

Further,

$$\begin{aligned} d(y, A) \leq d(y, gx_n) &\leq d(y, Tx_{n-1}) + d(Tx_{n-1}, gx_n) \\ &= d(y, Tx_{n-1}) + d(A, B) \\ &\leq d(y, Tx_{n-1}) + d(y, A). \end{aligned}$$

Therefore, $d(y, gx_n) \longrightarrow d(y, A)$. In view of the fact that A is approximatively compact with respect to B, $\{gx_n\}$ has a subsequence $\{gx_{n_k}\}$ converging to some z in A. Therefore, it can be concluded that

$$d(z, y) = \lim_{k\to\infty} d(gx_{n_k}, Tx_{n_k-1}) = d(A, B).$$

Eventually, z is a member of A_0. Since A_0 is contained in $g(A_0)$, $z = gx$ for some x in A_0. As $g(x_{n_k}) \longrightarrow g(x)$ and g is an isometry, $x_{n_k} \longrightarrow x$. Since the mapping T is continuous, it follows that $Tx_{n_k} \longrightarrow Tx$. Consequently, y and Tx are identical. Thus, it follows that

$$d(gx, Tx) = \lim_{n\to\infty} d(gx_{n_k}, Tx_{n_k-1}) = d(A, B).$$

Suppose that there is another element x^* such that

$$d(gx^*, Tx^*) = d(A, B).$$

Since T is a proximal contraction of the second kind,

$$d(Tgx, Tgx^*) \leq \alpha d(Tx, Tx^*).$$

As T preserves isometric distance with respect to g, we have

$$d(Tx, Tx^*) \leq \alpha d(Tx, Tx^*)$$

which implies that $Tx = Tx^*$. This completes the proof of the theorem. □

If g is the identity mapping, then the preceding theorem yields the following corollary.

Corollary 3.2. *Let A and B be non-empty, closed subsets of a complete metric space such that A is approximatively compact with respect to B. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \longrightarrow B$ satisfy the following conditions.*

(a) *T is a continuous proximal contraction of the second kind.*
(b) *$T(A_0)$ is contained in B_0.*

Then, there exists an element x in A such that

$$d(x, Tx) = d(A, B).$$

Moreover, if x^* is another best proximity point of T, then Tx and Tx^* are identical.

The following result provides another generalization of Banach's contraction principle to the case of non-self mappings.

Theorem 3.3. *Let X be a complete metric space. Let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions.*

(a) *T is a continuous proximal contraction of the first kind.*
(b) *g is an isometry.*
(c) *$T(A_0)$ is contained in B_0.*
(d) *A_0 is contained in $g(A_0)$.*

Then, there exists a unique element x in A such that

$$d(gx, Tx) = d(A, B).$$

Proof. Proceeding as in Theorem 3.1, there exists a sequence $\{x_n\}$ in A satisfying the following condition.

$$d(gx_{n+1}, Tx_n) = d(A, B).$$

Since T is a proximal contraction of the first kind, we have

$$d(gx_{n+1}, gx_n) \leq \alpha d(x_n, x_{n-1}).$$

Since g is an isometry, it follows that

$$d(x_{n+1}, x_n) \leq \alpha d(x_n, x_{n-1}).$$

Therefore, $\{x_n\}$ is a Cauchy sequence and hence converges to some x in A. Since g and T are continuous, we have

$$d(gx, Tx) = \lim_{n\to\infty} d(gx_{n+1}, Tx_n) = d(A, B).$$

Suppose that there is another element x^* such that

$$d(gx^*, Tx^*) = d(A, B).$$

Since T is a proximal contraction of the first kind and g is an isometry, we have

$$d(x, x^*) = d(gx, gx^*) \le \alpha d(x, x^*)$$

which implies that x and x^* are identical. This completes the proof of the theorem. □

If g is the identity mapping, then the preceding theorem yields the following best proximity point theorem.

Corollary 3.4. *Let X be a complete metric space. Let A and B be non-empty, closed subsets of X. Further, suppose that A_0 and B_0 are non-empty. Let $T : A \longrightarrow B$ satisfy the following conditions.*

(a) *T is a continuous proximal contraction of the first kind.*
(b) *$T(A_0)$ is contained in B_0.*

Then, there exists a unique element x in A such that

$$d(x, Tx) = d(A, B).$$

The following result furnishes another generalization of Banach's contraction principle to the case of non-self mappings.

Theorem 3.5. *Let A and B be non-empty, closed subsets of a complete metric space such that $d(A, B) > 0$. Let $g : A \longrightarrow A$ and $T : A \longrightarrow B$ satisfy the following conditions.*

(a) *There is a sequence $\{x_n\}$ in A such that $d(gx_n, Tx_n) \longrightarrow d(A, B)$.*
(b) *T is a continuous, strong proximal contraction of the first kind.*
(c) *g is an isometry.*

Then, there exists a unique element x_0 in A such that

$$d(gx_0, Tx_0) = d(A, B).$$

Further, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to the element x_0.

Proof. For each positive integer k, let us define

$$A_k = \{x \in A : d(gx, Tx) \le (1 + 1/k)d(A, B)\}.$$

Since $d(gx_n, Tx_n) \longrightarrow d(A, B)$, there exists a member x_{n_k} of the sequence $\{x_n\}$ such that

$$d(gx_{n_k}, Tx_{n_k}) \le (1 + 1/k)d(A, B).$$

Therefore, A_k is non-empty for every k. Because of the fact that g and T are continuous, each A_k is closed. Also, it is evident that A_{k+1} is contained in A_k. If x, x^* are any two elements in A_k, then, as T is a strong proximal contraction of the first kind, we have

$$d(gx, gx^*) \le \alpha d(x, x^*) + (1/k)d(A, B)$$

for some $\alpha \in [0, 1)$. Since g is an isometry, it follows that

$$d(x, x^*) \leq \frac{1}{(1-\alpha)k} d(A, B).$$

Thus, $\operatorname{diam}(A_k) \longrightarrow 0$. Since X is a complete metric space, $\bigcap A_k$ contains just a single point, say x_0, which satisfies the condition that $d(gx_0, Tx_0) = d(A, B)$. Moreover, since g is an isometry and T is a strong proximal contraction of the first kind, it follows that

$$d(x_{n_k}, x_0) = d(gx_{n_k}, gx_0) \leq \alpha d(x_{n_k}, x_0) + (1/k) d(A, B).$$

Therefore, $d(x_{n_k}, x_0) \leq \frac{1}{(1-\alpha)k} d(A, B)$. Hence, the subsequence $\{x_{n_k}\}$ converges to the element x_0. This completes the proof of the theorem. □

The following result furnishes necessary and sufficient conditions for a contraction to have a best proximity point.

Theorem 3.6. *Let A and B be non-empty, closed subsets of a complete metric space. Let $S : A \longrightarrow B$ be a contraction.*

Then, $S : A \longrightarrow B$ has a best proximity point if and only if there is a non-expansive mapping $T : B \longrightarrow A$ such that the pair (S, T) satisfies the min–max condition.

Moreover, $d(x^\star, x^{\star\star}) \leq \left(\frac{2}{1-\alpha}\right) d(A, B)$ for any two best proximity points $x^\star$ and $x^{\star\star}$ of the mapping S.

Proof. Let S have a best proximity point x^*. Let $T : B \longrightarrow A$ be defined as $Ty = x^*$ for all y in B. Then, T is a non-expansive mapping. Further, it follows from the definition of T that

$$d(Ty, STy) = d(x^*, Sx^*) = d(A, B)$$

for all y in B. Consequently, $\min(Sx, Ty) = d(A, B)$ for all x in A and y in B.

Then, if $x \in A$ and $y \in B$ are such that $d(A, B) < d(x, y)$, then it is evident that

$$\min(Sx, Ty) = d(A, B) < d(x, y) \leq \max(Sx, Ty).$$

Thus, the pair (S, T) satisfies the min–max condition. Conversely, let us assume that there exists a non-expansive mapping T such that the pair (S, T) satisfies the min–max condition. Let x_0 be a fixed element in A. We can define a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = Sx_{2n}$$
$$x_{2n+2} = Tx_{2n+1}.$$

Since S is a contraction mapping and T is a non-expansive mapping, it can be shown by induction that

$$d(x_{2n}, x_{2n+2}) \leq \alpha^n d(x_0, x_2)$$
$$d(x_{2n+1}, x_{2n+3}) \leq \alpha^{n+1} d(x_0, x_2).$$

Eventually, it can be ascertained that the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are Cauchy sequences. Since the space is complete, $\{x_{2n}\}$ converges to some element $x^* \in A$ and $\{x_{2n+1}\}$ converges to some element $y^* \in B$. Since S is a continuous mapping, $\{Sx_{2n}\}$ converges to Sx^*, which signifies that $\{x_{2n+1}\}$ converges to Sx^*. Therefore, $Sx^* = y^*$. A similar argument asserts that $Ty^* = x^*$. Subsequently, $TSx^* = Ty^* = x^*$. Furthermore, $STy^* = Sx^* = y^*$. Therefore, it is easy to see that

$$\min(Sx^*, Ty^*) = d(x^*, y^*) = \max(Sx^*, Ty^*)$$

which mandates that $d(x^*, y^*) = d(A, B)$, as the pair (S, T) satisfies the min–max condition.

Hence,

$$d(x^*, Sx^*) = d(x^*, y^*) = d(A, B).$$
$$d(y^*, Ty^*) = d(x^*, y^*) = d(A, B).$$

If S has two best proximity points x^* and x^{**}, then

$$\begin{aligned} d(x^*, x^{**}) &\leq d(x^*, Sx^*) + d(Sx^*, Sx^{**}) + d(Sx^{**}, x^{**}) \\ &\leq \alpha d(x^*, x^{**}) + 2d(A, B). \end{aligned}$$

Therefore, $d(x^*, x^{**}) \leq \left(\frac{2}{1-\alpha}\right) d(A, B)$. This completes the proof of the theorem. □

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