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Perturbed cones for analysis of uncertain multi-criteria optimization problems

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Abstract

Partial ordering of two quantities $\mathbf{x}$ and $\mathbf{y}$ (i.e., the ability to declare that $\mathbf{x}$ is better than $\mathbf{y}$ with respect to some decision criteria) can be stated mathematically as: $\mathbf{x}$ is better than $\mathbf{y}$ *iff* $\mathbf{x} - \mathbf{y} \in \mathbf{K}$, where $\mathbf{K}$ is an ordering convex cone, not necessarily pointed. Cones can be very important in representing feasible domains (i.e., $\{\mathbf{Ax} \leqslant \mathbf{b}\} = \mathbf{M} + \mathbf{G}$, where $\mathbf{M}$ is a bounded convex hull of a finite number of points and $\mathbf{G}$ is a convex cone). We consider specific perturbations of the Cone of Feasible Directions, which lead to a better feasible solution with respect to some decision criteria. Such cones are introduced as a tool to mitigate and analyze the effects of input data uncertainty on the solution of a given problem. Properties of this cone provide a basis to prove necessary and sufficient conditions for stable/unstable unboundedness of the multi-criteria optimization problem.

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1. Notation

The majority of concepts employed in this article derive from linear algebra and convex analysis [22,27]. Throughout the article, $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ denote n-dimensional Euclidean space and the space of $(m \times n)$ real matrices respectively; $\mathbb{Z}$ denotes all integers; $\mathbb{N}$ represents all positive integers. For sets, matrices and vectors bold fonts are used, and scalars are shown in italics. In this work $\mathbf{C}$ denotes an $L \times n$ matrix composed of rows $\mathbf{c}_j$, $j = 1, \ldots, L$; $\mathbf{A}$ is an $m \times n$ matrix composed of rows $\mathbf{a}_j$, $j = 1, \ldots, m$; $\mathbf{A}/\mathbf{C}$ denotes the matrix formed from the rows of the matrices $\mathbf{A}$ and $\mathbf{C}$: $(\mathbf{A}/\mathbf{C})^\top = (\mathbf{a}_1^\top, \ldots, \mathbf{a}_m^\top, \mathbf{c}_1^\top, \ldots, \mathbf{c}_L^\top)$, where the symbol "$\top$" denotes transpose of vector or matrix; $r(\mathbf{C})$ is the rank of matrix $\mathbf{C}$; $\mathbb{C}$ is a linear operator corresponding to matrix $\mathbf{C}$, $\mathbb{C} : \mathbf{x} \mapsto \mathbf{y} = \mathbf{Cx}$. To be consistent with the existing literature in this area we adopt $\mathbf{xy}$ to represent a scalar product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The vector $\mathbf{e}^p$ denotes p-vector of ones $(1, \ldots, 1)^\top$.

For sets and subspaces: $(\mathbf{B})_L$ denotes the linear subspace of maximal dimension included in the set $\mathbf{B}$; $\mathbf{L}^\perp$ is the orthogonal complement of subspace $\mathbf{L}$. For the set $\mathbf{X}$, ri $\mathbf{X}$ denotes its relative interior. The polyhedral cone corresponding to a system of homogeneous linear inequalities $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} \geqslant \mathbf{0}\}$, where $\mathbf{0}$ is a vector of zeros, is denoted throughout the article by $\mathbf{K}(\mathbf{C})$ and its dual cone is denoted by $\mathbf{K}^\star(\mathbf{C}) \overset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^{L} \lambda_j \mathbf{c}_j,\ \lambda_j \geqslant 0\}$; and the recession cone of a set $\mathbf{M}$ is denoted by $\mathbf{O}^+\mathbf{M}$.

2. Introduction

In this paper we are concerned with analysis of the effects of uncertainty in the input data of optimization problems with multiple linear criteria. When uncertainty is present in the input data and decisions must be made contingent upon such uncertainty, methods are required to answer key questions, including:

- will the nominal solution change for perturbations to the input data (i.e., is the nominal solution robust to uncertainty in the input data);
- how large must changes in the input data be before a substantial change of the nominal solution results and how will these changes in solution be for given changes in the input data (i.e., can the robustness of the nominal solution be quantified);
- where is the effort in reducing input data uncertainty most effectively spent.

This work recognizes that uncertainty may arise from a number of sources including: time varying data, incorrect data, subjective data, incomplete data, and measurement errors; and that analysis of the effects of uncertainty in the input data is vital in a wide range of decision-making contexts, including operating policy development, resource deployment planning, risk mitigation. In optimization problems, small errors in the input data can lead to a solution that is significantly different from the nominal, uncorrupted solution. Since structured decision-making methods, such as optimization, are often used to make high level decisions that can have substantial economic impact, tools are needed to predict the influence of input perturbations on the solution.

Uncertainty analysis of continuous optimization problems with scalar objective functions is well understood and a rich literature is available (e.g. [8,9,13]). Sensitivity and stability results (i.e., effects of differential and finite changes, respectively), for the effects of uncertainty or changes in the input data for continuous problems take advantage of the underlying differentiability of such problems. Unfortunately, problems with decision variables that take on discrete values do not usually possess these differentiability properties and as a result, other analysis tools are required. The influence of input data perturbations for mixed-integer problems is less easily quantified but the importance of doing so can be illustrated by the following simple example.

Example 1. Consider

$$F(a) = \min\{ax_2 \mid x_1 + ax_2 \geqslant 2a, 0 \leqslant x_1 \leqslant 2a, 0 \leqslant x_2 \leqslant 2, x_1, x_2 \in \mathbb{Z}\},$$

where $a \geqslant 1$. If a is an integer, then the optimum is $F(a) = 0$ and the solution is $\mathbf{x}^\star(a) = (2a, 0)$. For any perturbation $\varepsilon > 0$: $F(a-\varepsilon) = a - \varepsilon$ and the solution is $\mathbf{x}^\star(a-\varepsilon)=(a, 1)$. In this case

$$\|\mathbf{x}^\star(a) - \mathbf{x}^\star(a-\varepsilon)\| = \sqrt{a^2+1} > a \quad \text{and} \quad |F(a) - F(a-\varepsilon)| = a - \varepsilon.$$

Thus an arbitrarily small perturbation of parameter a leads to a change of the solution, which is proportional to the magnitude of the parameter. Moreover, as it is shown in Fig. 1, the optimal solution of the perturbed problem is substantially different from the optimal solution of the original problem.

Multi-objective optimization problems attempt to “balance” two or more competing objectives simultaneously, and may contain continuous and/or integer variables. Such problems arise in a wide variety of situations, including: design and optimization of engineering systems, distribution of resources, budgeting, strategic

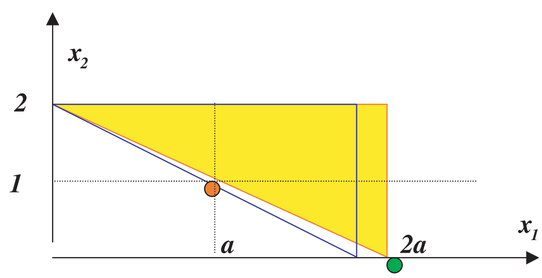

Fig. 1. Illustration to the Example 1.

stability and other economic, military, and social decision-making situations [10,26]. Comprehensive treatments of multi-objective optimization can be found in [6,12,25, 28]. Mixed-integer optimization problems differ from the thoroughly investigated continuous problems, since convexity and continuity, which play very important roles in optimization, do not extend to this case [14–20]. For example, the Pareto set of mixed-integer, linear, multi-objective programming problems can be neither closed, nor open [15,30], which poses problems for existing solution methods.

This work will focus on the use of perturbed cones for analysis of the effects of input data uncertainty on multiple criteria optimization problems of the form:

$$(\mathbf{C}, \mathbf{X}) : \max\{\mathbf{Cx} : \mathbf{x} \in \mathbf{X}\}, \tag{1}$$

where $\mathbf{X} = \mathbf{D} \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leqslant \mathbf{b}\}$ ($\mathbf{D} \subseteq \mathbb{R}^n$ is a set of arbitrary structure) represents the feasible set, and objective functions are represented by the rows of matrix $\mathbf{C} \in \mathbb{R}^{L\times n}$. Note that although we focus on the single type of decision-making problem, the ideas presented in this work can be extended to a broad class of problems, including: constraint satisfaction problems, semi-definite and cone optimization problems.

A key difference between scalar optimization and vector (multi-objective) optimization is ordering of the feasible space. In scalar optimization there is only one objective function and full ordering of the feasible domain with respect to the objective function is possible. If we consider any two feasible points, then we can determine whether one is better, or if they are equal, with respect to the objective function. In vector optimization, with two or more objective functions, two feasible points are not as easily compared. For example, one of the points may be better with respect to one objective, but may be worse with respect to another objective. Thus the feasible set is not fully ordered, but is partially ordered [12].

A considerable literature has evolved regarding the existence of solutions, their stability [5,11,31] and related questions of well-posedness in vector optimization [23]. In this work the effects of uncertainty in the input data on the solution of multiple criteria optimization problems are investigated in terms of cones. Cones are an important construct in mathematics, have been widely studied [4,21,27], and have found application in various engineering and scientific problems [2,24]. The use of cones specifically for vector (multi-objective) optimization is discussed in [11,28–30]. By definition [27], a subset $\mathbf{K}$ of $\mathbb{R}^n$ is called a cone, if it is closed under positive scalar multiplication (i.e., $\lambda\mathbf{x} \in \mathbf{K}$ when $\mathbf{x} \in \mathbf{K}$ and $\lambda > 0$). In other words a cone is a union of half-lines emanating from the origin. Cones can be used for analysis of both the feasible domain and the objective functions. Recession cones represent infinite directions of feasible domains [4,21,27]; ordering cones are basic in ordering the feasible domain with respect to the optimization objective(s).

We illustrate the value of the recession cone for stability analysis in the simplest case of polyhedral feasible set. A polyhedral set

$$\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leqslant \mathbf{b}\}$$

can be represented as a sum of a convex hull $\mathbf{M}$ of a finite number of points and a recession polyhedral cone $\mathbf{G} = \mathbf{O}^{+}\mathbf{X} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leqslant \mathbf{0}\}$ [1,3,4]:

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leqslant \mathbf{b}\} = \mathbf{M} + \mathbf{G}.$$

If the matrix $\mathbf{A}$ does not change, the recession cone $\mathbf{G}$ remains unchanged. Changes in the right-hand side vector $\mathbf{b}$ do not change the boundedness of the feasible domain: if the feasible domain is unbounded, it remains unbounded (or becomes empty); if it is bounded, it remains bounded (or becomes empty). Changes in matrix $\mathbf{A}$ perturb the recession cone. Under certain conditions arbitrarily small changes in matrix $\mathbf{A}$ can lead to large changes in the problem's structure.

Partial ordering of the feasible domain with respect to the optimization objective(s) can be interpreted in terms of a cone as well. From the mathematical point of view, $\mathbf{x}$ is better than $\mathbf{y}$ if and only if $\mathbf{x} - \mathbf{y} \in \mathbf{K}$, where $\mathbf{K}$ is a cone (the so-called ordering cone) [12,28]. In this paper ordering cones are defined as $\mathbf{K} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} \geqslant \mathbf{0}\}$. Although perturbations of ordering cones were considered in [17,29,30], there was no comprehensive study of specific perturbations of ordering cones. Such a comprehensive study of specific cone perturbations is a central contribution of our work. Further, the introduced perturbations are important in stability and uncertainty analysis of mixed-integer optimization problems, for which there are very few results in the current literature.

In the next section we represent particular perturbations, which are an extension of the work presented in [29], of the ordering cone $\mathbf{K}$ as a tool to analyze and predict changes in the solutions of multi-objective optimization problems with integer and/or continuous variables due to input data changes. In the fourth section we formulate and prove the properties of these cones, which can be used to:

- develop stability theory for multi-criteria optimization problems;
- formulate and prove necessary and sufficient conditions for different cases of stability;
- find equivalence conditions for different cases of stability;
- establish stability domains;
- estimate stability radii;
- develop regularization methods for ill-posed problems;
- estimate the measure of an ill-posed problem's initial data in the space of alternatives.

The preliminary applications of the proposed cone perturbation technique to stability analysis and regularization are introduced in the fifth section of the article. Necessary and sufficient conditions for the problem to be stable solvable/unbounded are provided. In particular, it is shown, that under certain conditions (e.g., $r(\mathbf{A}/\mathbf{C}) = n$) for any unstable problem there exists infinitely small perturbation of the original input data which defines a stable solvable problem. Under the same conditions there also exists infinitely small perturbation of the original input data, which defines a stable unbounded problem.

3. Preliminaries

In this section, the main characteristics of problem (1) are described in terms of cones. Specific cone perturbations are introduced as a framework to discuss uncertainty. Although the perturbations presented here are in terms of ordering cones, they apply equally well to the recession cone $\mathbf{G}$.

According to [28], the solution of problem (1) is a subset of one of the following sets: the set $\mathbf{\Pi}(\mathbf{C}, \mathbf{X})$ of all Pareto-optimal (efficient) solutions, the set $\mathbf{P}(\mathbf{C}, \mathbf{X})$ of all semi-efficient solutions, or the set $\mathbf{S}(\mathbf{C}, \mathbf{X})$ of all strictly efficient solutions. The point $\mathbf{x}^* \in \mathbf{X}$ is efficient (or Pareto-optimal) if there is no $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{Cx} \geqslant \mathbf{Cx}^*$ and $\mathbf{Cx} \neq \mathbf{Cx}^*$. Further, the point $\mathbf{x}^* \in \mathbf{X}$ is semi-efficient (weakly efficient, Slater-efficient) if there is no $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{Cx} > \mathbf{Cx}^*$. Finally, the point $\mathbf{x}^*$ is strictly efficient if there is no $\mathbf{x} \in \mathbf{X} : \mathbf{x} \neq \mathbf{x}^*$, such that $\mathbf{Cx} \geqslant \mathbf{Cx}^*$. The relationship between these sets can be expressed as

$$\mathbf{S}(\mathbf{C}, \mathbf{X}) \subseteq \mathbf{\Pi}(\mathbf{C}, \mathbf{X}) \subseteq \mathbf{P}(\mathbf{C}, \mathbf{X}).$$

Cones can be used to define semi-efficient, efficient and strictly efficient solutions, both as a language and framework to discuss uncertainty. Consider the convex cone $\mathbf{K} = \mathbf{K}(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} \geqslant \mathbf{0}\}$, which can be represented as the union of sets:

$$\mathbf{K} = \mathbf{K}_0 \cup \mathbf{K}_1 \cup \mathbf{K}_2,$$

where $\mathbf{K}_0 = \mathbf{K}_0(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} = \mathbf{0}\}$; $\mathbf{K}_1 = \mathbf{K}_1(\mathbf{C}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Cx} > \mathbf{0}\}$; $\mathbf{K}_2 = \mathbf{K}_2(\mathbf{C}) = \mathbf{K} \setminus (\mathbf{K}_0 \cup \mathbf{K}_1)$. Then,

$$\mathbf{x}^* \in \mathbf{\Pi}(\mathbf{C}, \mathbf{X}) \Leftrightarrow (\mathbf{x}^* + \mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{X} \subseteq \{\mathbf{x}^*\}; \tag{2}$$

$$\mathbf{x}^* \in \mathbf{P}(\mathbf{C}, \mathbf{X}) \Leftrightarrow (\mathbf{x}^* + \mathbf{K}_1) \cap \mathbf{X} \subseteq \{\mathbf{x}^*\}; \tag{3}$$

$$\mathbf{x}^* \in \mathbf{S}(\mathbf{C}, \mathbf{X}) \Leftrightarrow (\mathbf{x}^* + \mathbf{K}) \cap \mathbf{X} = \{\mathbf{x}^*\}. \tag{4}$$

Define $\mathbf{s} \in \mathbf{K}$ as feasible directions in the decision variable space. Consider that the point $\mathbf{x} \in \mathbf{X}$ lies on a line passing through any point $\mathbf{x}'$ such that $\mathbf{Cx}' \geqslant \mathbf{Cx}$, where $\mathbf{x}' = \mathbf{x} + \mathbf{s}$, with $\mathbf{s} \in \mathbf{K}$. Directions $\mathbf{s} \in \mathbf{K}_0$ may be called directions of equilibrium, since $\mathbf{Cx}' = \mathbf{Cx}$ (the solutions $\mathbf{x}$ and $\mathbf{x}'$ may be called equivalent); elements $\mathbf{s} \in \mathbf{K}_1$ may be called proper directions, since all the objective functions are improved (i.e., $\mathbf{Cx}' > \mathbf{Cx}$).

We can define the dual cone to $\mathbf{K}$ as

$$\mathbf{K}^* = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{k=1}^{L} \lambda_k \mathbf{c}_k, \lambda_k \geqslant 0, k = 1, \ldots, L \right\},$$

which will prove more useful for our purposes, but is equivalent to the more conventional definition

$$\mathbf{K}^* = \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{x}, \mathbf{y} \rangle \geqslant 0, \ \forall \mathbf{x} \in \mathbf{K}\},$$

for polyhedral cones. The equivalence of these two definitions follows from the Minkowski–Farkas Theorem [4,27].

Note that this definition of the dual cone differs from the notion of a polar cone, given in some of the literature, due to the sign of inequality (i.e., the inequality "$\leqslant$" used in defining polar cones is replaced with "$\geqslant$" for dual cone). Note that an important property of both type of cones is $(\mathbf{K}^\star)^\star = \mathbf{K}$. The relative interior of the set $\mathbf{K}^*$ can be defined as

$$\operatorname{ri}\mathbf{K}^* = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{k=1}^{L} \lambda_k \mathbf{c}_k, \lambda_k > 0, k = 1, \ldots, L\right\}.$$

Remark 1. If $r(\mathbf{C}) = n$ then the cone $\mathbf{K}$ is pointed [27]. Recall that $\mathbf{K}_0 = \{\mathbf{0}\} \Leftrightarrow r(\mathbf{C}) = n$ [22]. If $\mathbf{0} \in \operatorname{ri}\mathbf{K}^*$, then $\mathbf{K}^*$ is some subspace of the space $\mathbb{R}^n$ and the dual cone $\mathbf{K}$ is also a subspace of the space $\mathbb{R}^n$, $\mathbf{K} = \mathbf{K}_0, \mathbf{K}^* = \mathbf{K}_0^\perp$. Then $\dim\mathbf{K}^* = r(\mathbf{C})$ and $\dim\mathbf{K} = n - r(\mathbf{C})$.

Proposition 1 provides the basic framework for defining the structure of the problem (1) in terms of ordering cones.

Proposition 1. *The qualitative properties of efficient solutions of problem* (1) *can be describes as follows*:

(i) *If* $r(\mathbf{C}) = n$, *then* $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) = \mathbf{S}(\mathbf{C}, \mathbf{X})$;
(ii) *If* $\mathbf{0} \in \operatorname{ri}\mathbf{K}^*$, *then* $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) = \mathbf{X}$;
(iii) *If* $\mathbf{K}_1 = \emptyset$, *then* $\mathbf{P}(\mathbf{C}, \mathbf{X}) = \mathbf{X}$;
(iv) *If* $\exists\, \lambda_i \in \mathbb{R}, \lambda_i \geqslant 0, i = 1, \ldots, L$ *such that* $\sum_{k=1}^{L} \lambda_k \mathbf{c}_k = \mathbf{0}$ *and* $\sum_{k=1}^{L} \lambda_k > 0$, *then* $\mathbf{P}(\mathbf{C}, \mathbf{X}) = \mathbf{X}$;
(v) *If* $\mathbf{K}_2 = \emptyset$, *then* $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) = \mathbf{P}(\mathbf{C}, \mathbf{X})$.

Proof. For item (i) above, if $r(\mathbf{C}) = n$, from Remark 1 $\mathbf{K}_0 = \{\mathbf{0}\}$ and it follows from (2) and (4), that $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) = \mathbf{S}(\mathbf{C}, \mathbf{X})$.

For item (ii) above, if $\mathbf{0} \in \operatorname{ri}(K)^*$, it follows from Remark 1 that $\mathbf{K} = \mathbf{K_0}$ and $\mathbf{K}_1 \cup \mathbf{K}_2 = \emptyset$. Then $\forall \mathbf{x} \in \mathbf{X}$, $\mathbf{x} + \{\mathbf{0}\} \cup \mathbf{K}_1 \cup \mathbf{K}_2 = \mathbf{x}$ and using (2), we have $\mathbf{X} \subseteq \mathbf{\Pi}(\mathbf{C}, \mathbf{X})$.

Item (iii) above follows immediately from (3).

For item (iv) above, since item (iii) above is true, it is sufficient to show that if $\exists\, \lambda_i \in \mathbb{R}, \lambda_i \geqslant 0, i = 1, \ldots, L$ such that $\sum_{k=1}^{L} \lambda_k \mathbf{c}_k = \mathbf{0}$ and $\sum_{k=1}^{L} \lambda_k > 0$, then $\mathbf{K}_1 = \emptyset$. Assuming that $\exists \mathbf{x} \in \mathbf{K}_1$, then $\mathbf{c}_i\mathbf{x} > 0, i = 1, \ldots, L$ and therefore, $(\sum_{k=1}^{L} \lambda_k \mathbf{c}_k)\mathbf{x} = \sum_{k=1}^{L} \lambda_k(\mathbf{c}_k\mathbf{x}) > 0$. This contradicts the assumption that $\sum_{k=1}^{L} \times \lambda_k \mathbf{c}_k = 0$.

Item (v) follows immediately from (2) and (3). □

Consider the family of problems $\{(\mathbf{C}_\tau, \mathbf{X})\}$, that are based on problem (1) and where each row $\mathbf{c}_j^\tau$ of the matrix $\mathbf{C}_\tau$ has the form $\mathbf{c}_j^\tau = \mathbf{c}_j - \tau\mathbf{u}$, where $\mathbf{u} \in \operatorname{ri}\mathbf{K}^*$, i.e.,

$$\mathbf{u} = \sum_{k=1}^{L} \mu_k \mathbf{c}_k, \quad \mu_k > 0, \quad k = 1, \ldots, L. \tag{5}$$

Without loss of generality, we take $\sum_{k=1}^{L} \mu_k = 1$ and $\mathbf{u} \neq \mathbf{0}$.

We also consider the cones $\mathbf{K}^\tau = \mathbf{K}(\mathbf{C}_\tau) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}_\tau \mathbf{x} \geqslant 0\}$, which can be represented as the union of sets

$$\mathbf{K}^\tau = \mathbf{K}_0^\tau \cup \mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau,$$

where

$$\begin{aligned}
\mathbf{K}_0^\tau &= \mathbf{K}_0(\mathbf{C}_\tau) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}_\tau \mathbf{x} = \mathbf{0}\};\\
\mathbf{K}_1^\tau &= \mathbf{K}_1(\mathbf{C}_\tau) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{C}_\tau \mathbf{x} > \mathbf{0}\};\\
\mathbf{K}_2^\tau &= \mathbf{K}_2(\mathbf{C}_\tau) = \mathbf{K}^\tau \setminus (\mathbf{K}_0^\tau \cup \mathbf{K}_1^\tau).
\end{aligned}$$

Remark 2. Similar perturbations are introduced in [7]: each row $\mathbf{c}_j^\tau$ of the matrix $\mathbf{C}_\tau$ has the form $\mathbf{c}_j^\tau = \mathbf{c}_j - \tau \mathbf{u}_j$, were $\mathbf{u}_j \in \operatorname{ri} \mathbf{K}^*$, i.e.,

$$\mathbf{u}_j = \sum_{k=1}^{L} \mu_k^j \mathbf{c}_k, \quad \mu_k^j > 0, \quad k = 1, \ldots, L, \quad j = 1, \ldots, L. \tag{6}$$

Without loss of generality we assume $\sum_{k=1}^{L} \mu_k^j = 1,\ j = 1, \ldots, L$. Below we show (Section 4.1), that this generalization provides similar perturbations to those proposed in (5). Unfortunately, the monotonicity property, important in many applications, does not hold in case (6). This distinction and other properties of the generalized type of perturbations will be discussed in Section 4.2.

4. Properties of perturbed cones

In this section the properties defining the structure of perturbed ordering cones are developed. These properties permit the development of a regularization technique, which changes the partial ordering in the solution space such that efficient solutions of a perturbed problem will also be feasible and efficient solutions for the original problem and vice versa. The properties developed here are the building blocks for the regularization technique. Note that the developments in Sections 4.1 and 4.2 are in terms of ordering cones; however, they apply equally to the recession cones of feasible domains defined by systems of linear inequalities.

The properties of basic perturbations (5) and importance of their applications to the problem (1) are summarized below.

(1) Any feasible direction of the original problem (1) remains feasible for the perturbed problem, where the perturbation vector is defined by (5) and $\tau \leqslant 0$, since

$$\forall \tau \leqslant 0 : \mathbf{K} \subseteq \mathbf{K}^\tau.$$

This influences the perturbed problem in that every efficient solution remains efficient for the original problem. This property is a key to the development of the regularization technique, since solving the problem with a slightly enlarged ordering cone, which contains the original cone, leads to robust solutions of the original problem.

(2) Any feasible direction of the perturbed problem where $\tau \in [0, 1)$ is feasible for the original problem, since

$$\forall\tau \in [0, 1)\colon \mathbf{K}^{\tau} \subseteq \mathbf{K}$$

(i.e., the Pareto set of the perturbed problem with $\tau \in [0, 1)$ covers the Pareto set of the original problem).

(3) The size of the cone $\mathbf{K}^{\tau}$ changes monotonically with τ. In the case where $\tau < 1$, a cone corresponding to a smaller value τ contains a cone corresponding to larger value τ:

$$\tau' < \tau'' < 1 : \ \mathbf{K}^{\tau''} \subseteq \mathbf{K}^{\tau'} \subseteq \mathbf{K}^{-\infty} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{ux} \geqslant 0\};$$
$$\tau' < \tau'' < 1 : \ \mathbf{K}_i^{\tau''} \subseteq \mathbf{K}_i^{\tau'} \subseteq \mathbf{K}_i^{-\infty} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} > 0\}, \quad i = 1, 2.$$

A strictly (weakly) efficient solution of a perturbed problem, with the perturbation constructed as given in (5) and $\tau' < 1$, will be strictly (weakly) efficient for the perturbed problem, with same perturbation constructed as given in (5) and τ''' such that $\tau' < \tau'' < 1$.

The cone corresponding to $\tau = -\infty$ is the half-space and fully orders the feasible domain. The vector optimization problem is then reduced to an equivalent scalar optimization problem. This property provides the generalization of the well known result on convolution of objective functions (i.e., any solution of a scalar optimization problem with objective function $\mathbf{ux}$ and the original feasible domain provides efficient solutions of the original and perturbed vector optimization problems where $\tau < 1$).

In the case where $\tau > 1$, monotonicity changes its direction: a cone corresponding to a value of bigger $\tau > 1$ contains a cone corresponding to a value of smaller $\tau > 1$. The value of $\tau = +\infty$ defines the opposite sequence of inclusions. A strictly (weakly) efficient solution of a perturbed problem, with the perturbation constructed as given in (5) and $\tau'' > 1$, will be strictly (weakly) efficient for the perturbed problem, with same perturbation constructed as given in (5) and τ' such that $1 < \tau' < \tau''$:

$$1 < \tau' < \tau'' : \ \mathbf{K}^{\tau'} \subseteq \mathbf{K}^{\tau''} \subseteq \mathbf{K}^{\infty} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \leqslant 0\};$$
$$1 < \tau' < \tau'' : \ \mathbf{K}_i^{\tau'} \subseteq \mathbf{K}_i^{\tau''} \subseteq \mathbf{K}_i^{\infty} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} < 0\}, \quad i = 1, 2.$$

(4) For perturbations of the cone where $\tau < 0$, weakly efficient solutions of the perturbed problem will be efficient for the original problem, since

$$\forall\tau < 0 : \mathbf{K}_1 \cup \mathbf{K}_2 \subseteq \mathbf{K}_1^{\tau}.$$

If a weakly efficient solution of the perturbed problem is unique (i.e., there is no other feasible point with the same values of objective functions), then it is a strictly efficient solution of the original problem, since

$\forall\tau < 0 : \mathbf{K} \subseteq \mathbf{K}_0^\tau \cup \mathbf{K}_1^\tau.$

(5) If the direction **s** is feasible for both the original and perturbed problems with $\tau > 0$, then either this direction is the direction of equilibrium for the original problem or is a proper direction for the original problem, since

$\forall\tau > 0 : \mathbf{K}^\tau \cap \mathbf{K} \subseteq \mathbf{K}_0 \cup \mathbf{K}_1.$

(6) Equivalent solutions for the original and perturbed problems coincide in case $\tau \neq 1$. A strictly efficient solution of the original problem remains strictly efficient for the perturbed problem, with the perturbation constructed as given in (5) and $\tau \neq 1$, if it is still efficient under this perturbation, since

$\forall\tau \in \mathbb{R} : \mathbf{K}_0 \subseteq \mathbf{K}_0^\tau;$
$\forall\tau \in \mathbb{R} \setminus \{1\} : \mathbf{K}_0 = \mathbf{K}_0^\tau \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\};$
$\forall\tau \in \mathbb{R} \setminus \{1\} : \mathbf{K}_0^\tau = \mathbf{K}^\tau \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}.$

In case $\tau = 1$ all the solutions are equivalent, since

$\tau = 1 \Rightarrow \mathbf{K}^\tau = \mathbf{K}_0^\tau.$

(7) If $\exists\tau' < 1$ such that every feasible direction of the perturbed problem is the direction of equilibrium, then every feasible solution of the problem is efficient $\forall\tau \in [\tau', 1)$:

$\exists\tau' < 1 : \mathbf{K}_0^{\tau'} = \mathbf{K}^{\tau'} \Rightarrow \forall\tau \in [\tau', 1) : \mathbf{K}_0^\tau = \mathbf{K}^\tau.$

(8) If $\exists\tau' > 1$ such that every feasible direction of the perturbed problem is the direction of equilibrium, then every feasible solution of the problem is efficient $\forall\tau \in (1, \tau']$:

$\exists\tau' > 1 : \mathbf{K}_0^{\tau'} = \mathbf{K}^{\tau'} \Rightarrow \forall\tau \in (1, \tau'] : \mathbf{K}_0^\tau = \mathbf{K}^\tau.$

A comprehensive description of basic perturbations is provided in Section 4.1. The main distinctions between the basic case and generalized perturbations defined by (6) are discussed in Section 4.2.

4.1. Basic perturbations

In this section we consider perturbations defined by Eq. (5). We start with the simple case, since several properties hold that do not necessarily hold in general case, including: the *principle of complementary inclusion* and *monotonicity*. As we will show in Section 5, these properties are rather important in applications, such as: in the proof of necessary and sufficient condition for stability, and for development of a regularization technique. In the general case, loss of the monotonicity property may render the proposed regularization technique ineffective.

The following *principle of complementary inclusion* introduces $\tau = 1$ as a threshold, where the vector of the perturbations **u** dramatically changes the characteristics of perturbed problem. It will be shown below, how the sign of $(1 - \tau)$ influences the monotonicity of the proposed basic perturbations.

Theorem 1. $\forall \tau \in \mathbb{R} : (1 - \tau)\mathbf{u} \in (\mathbf{K}^\tau)^*$.

Proof. Let $\mathbf{u} \in \mathbf{K}^*$ be defined as in (5). Then

$$(1 - \tau)\mathbf{u} = \sum_{k=1}^{L} \mu_k \mathbf{c}_k - \tau \sum_{k=1}^{L} \mu_k \mathbf{c}_k = \sum_{k=1}^{L} \mu_k (\mathbf{c}_k - \tau \mathbf{u}) \tag{7}$$

which by definition requires $(1 - \tau)\mathbf{u} \in (\mathbf{K}^\tau)^*$. □

This "complementary inclusion" defines the supporting hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}$, which separates cones $\mathbf{K}^\tau$, $\tau < 1$, from those with $\tau > 1$.

Corollary 1. *If* $\tau < 1$, *then* $\mathbf{K}^\tau \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \geqslant 0\}$. *If* $\tau > 1$, *then* $\mathbf{K}^\tau \subseteq \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \leqslant 0\}$.

Proof. Since $(1 - \tau)\mathbf{u} \in (\mathbf{K}^\tau)^*$, from the definition of the dual cone we have $\forall \mathbf{x} \in \mathbf{K}^\tau : (1 - \tau)\mathbf{ux} \geqslant 0$. □

Corollary 1 immediately implies that each feasible direction of the original problem is feasible for perturbed problem $\forall \tau \leqslant 0$. This is a key property of the proposed perturbations. It will be shown in Section 4.2 that it holds in general case as well.

Theorem 2. $\forall \tau \leqslant 0 : \mathbf{K} \subseteq \mathbf{K}^\tau$.

Proof. Consider the arbitrary point $\mathbf{x} \in \mathbf{K}$. Since $\mathbf{c}_i\mathbf{x} \geqslant 0,\ i = 1, \ldots, L$, using Corollary 1, we obtain

$$\mathbf{ux} \geqslant 0 \Rightarrow \forall \tau \leqslant 0 : \mathbf{c}_i^\tau \mathbf{x} = \mathbf{c}_i\mathbf{x} - \tau\mathbf{ux} \geqslant 0,\ i = 1, \ldots, L \Rightarrow \mathbf{x} \in \mathbf{K}^\tau. \qquad \square$$

The result of Theorem 2 can be strengthened.

Theorem 3. $\forall \tau < 0 : \mathbf{K} \subseteq \mathbf{K}_0^\tau \cup \mathbf{K}_1^\tau$.

Proof. The inclusion $\mathbf{x} \in \mathbf{K}$ implies, as we have shown in Corollary 1, that $\mathbf{ux} \geqslant 0$. Two cases are possible:

(1) If $\mathbf{ux} = 0$, then $\forall \tau \in \mathbb{R} : \mathbf{Cx} = \mathbf{0}$ and $\mathbf{C}^\tau\mathbf{x} = \mathbf{0}$.
(2) If $\mathbf{ux} > 0$, then $\forall \tau < 0 : -\tau\mathbf{ux} > 0$ and $\mathbf{C}^\tau\mathbf{x} > \mathbf{0}$. □

The following theorem proves that directions of equilibrium, with the property $\mathbf{Cx} = \mathbf{Cy}$, remain equilibrium directions for any τ. Equilibrium directions belong to the supporting hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}$ when $\tau \neq 1$. When $\tau = 1$, each feasible direction of the perturbed problem is a direction of equilibrium.

Theorem 4. *$\forall \tau \in \mathbb{R} :\ \mathbf{K}_0 \subseteq \mathbf{K}_0^\tau$ and $\forall \tau \in \mathbb{R} \setminus \{1\} : \mathbf{K}_0 = \mathbf{K}_0^\tau \subseteq \{\mathbf{x} \in \mathbf{R}^n : \mathbf{ux} = 0\}$ then $\tau = 1 \Rightarrow \mathbf{K}_0^1 = \mathbf{K}^1$.*

Proof. For $\tau \in \mathbb{R}$,

$$\begin{aligned} \mathbf{x} \in \mathbf{K} \text{ and } \mathbf{ux} = 0 &\Leftrightarrow \mathbf{x} \in \mathbf{K}_0 \\ &\Rightarrow \mathbf{c}_i^\tau \mathbf{x} = \mathbf{c}_i \mathbf{x} - \tau \mathbf{ux} = 0,\ i = 1, \ldots, L \\ &\Rightarrow \mathbf{x} \in \mathbf{K}_0^\tau. \end{aligned}$$

Since (7) holds,

$$\forall \tau \in \mathbb{R} : \mathbf{x} \in \mathbf{K}_0^\tau \Rightarrow (1 - \tau)\mathbf{ux} = 0. \tag{8}$$

$\therefore$ for $\tau \neq 1$, we have $\mathbf{ux} = 0 \Rightarrow \forall i = 1, \ldots, L : \mathbf{c}_i \mathbf{x} = 0 \Rightarrow \mathbf{K}_0^\tau \subseteq \mathbf{K}_0$ and since $\forall \tau \in \mathbb{R} \setminus \{1\} : \mathbf{K}_0 \subseteq \mathbf{K}_0^\tau$, we obtain $\mathbf{K}_0 = \mathbf{K}_0^\tau$.

In the case where $\tau = 1$, we have $\mathbf{x} \in \mathbf{K}^\tau \Rightarrow \mathbf{c}_i^\tau \mathbf{x} \geqslant 0,\ i = 1, \ldots, L$. Since (7) holds, $\sum_{i=1}^L \mu_i \mathbf{c}_i^\tau \mathbf{x} = 0$. Hence, $\mathbf{c}_i^\tau \mathbf{x} = 0,\ i = 1, \ldots, L$ and $\mathbf{K}_0^1 = \mathbf{K}^1$. □

Remark 3. In the case where $\tau = 1$, the equality $\mathbf{K}_0 = \mathbf{K}_0^\tau$ does not necessarily hold.

Example 2. Consider $\mathbf{K} = \mathbf{K}(\mathbf{C})$, were $\mathbf{C} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. Then

$$\mathbf{K}_0 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0, x_1 + x_2 - x_3 = 0\}.$$

In other words,

$$\mathbf{K}_0 = \{\mathbf{x} \in \mathbb{R}^3 : x_1 = -x_2, x_3 = 0\}.$$

We determine the perturbation vector $\mathbf{u}$ according to (5), where $\mu_1 = \frac{2}{3}$, $\mu_2 = \frac{1}{3}$. In this case $\mathbf{u} = \frac{2}{3}\mathbf{c}_1 + \frac{1}{3}\mathbf{c}_2 = (1, 1, +\frac{1}{3})$. For $\tau = 1$,

$$\mathbf{C}_1 = \mathbf{C} - \begin{pmatrix} \mathbf{u} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{2}{3} \\ 0 & 0 & -\frac{4}{3} \end{pmatrix}.$$

Then $\mathbf{K}_0^1 = \{\mathbf{x} \in \mathbb{R}^3 : x_3 = 0\}$, $\mathbf{K}_0^1 \neq \mathbf{K}_0$.

Corollary 2. $\mathbf{K}_0 = \bigcap_{\tau \in \mathbb{R}} \mathbf{K}^\tau$.

Corollary 3. $\forall \tau \in \mathbb{R} \setminus \{1\} : \mathbf{K}_0^\tau = \mathbf{K}^\tau \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}$.

Proof. The inclusion $\mathbf{K}_0^\tau \subseteq \mathbf{K}^\tau \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}$ follows from (8). Let $\forall \mathbf{x} \in \mathbf{K}^\tau \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}$. Then, since (7) holds, $\forall i = 1, \ldots, L : \mathbf{c}_i^\tau \mathbf{x} \geqslant 0$ and $\mathbf{ux} = 0 \Rightarrow \mathbf{x} \in \mathbf{K}_0$. Thus, according to Theorem 4, we conclude $\mathbf{x} \in \mathbf{K}_0^\tau$. □

The following theorem further strengthens Theorem 2 and shows that each feasible direction, which improves at least one of the objectives of the original problem, is proper for the perturbed problem with $\tau < 0$ (i.e., it improves all the objectives of the perturbed problem).

Theorem 5. $\forall \tau < 0 : \mathbf{K}_1 \cup \mathbf{K}_2 \subseteq \mathbf{K}_1^{\tau}$.

Proof. Let $\mathbf{x} \in \mathbf{K}_1 \cup \mathbf{K}_2$. In this case, as we have shown above, $\forall \tau < 0 : -\tau \mathbf{ux} > 0 \Rightarrow \mathbf{x} \in \mathbf{K}_1^{\tau}$. □

The following theorem, which is important in applications, shows that each feasible direction of the perturbed problem with $\tau \in [0, 1)$ is feasible for the original problem. It will be shown in the next section that it also holds in the general case.

Theorem 6. $\forall \tau \in [0, 1) : \mathbf{K}^{\tau} \subseteq \mathbf{K}$.

Proof. Given $\forall \mathbf{x} \in \mathbf{K}^{\tau}$, show that $\mathbf{x} \in \mathbf{K}$. From Corollary 1 it follows that $\mathbf{ux} \geqslant 0$. Since $\mathbf{x} \in \mathbf{K}^{\tau}$, $\tau > 0$, and $\mathbf{ux} \geqslant 0$,

$$(\mathbf{c}_i - \tau \mathbf{u})\mathbf{x} \geqslant 0,\ i = 1, \ldots, L \Rightarrow \mathbf{c}_i \mathbf{x} \geqslant \tau \mathbf{ux} \geqslant 0,\ i = 1, \ldots, L. \qquad \square$$

The next four theorems describe the *monotonicity principle* of basic perturbations. In the next section, we will show that they do not necessarily hold in the general case.

Theorem 7. $1 < \tau' < \tau'' : \mathbf{K}^{\tau'} \subseteq \mathbf{K}^{\tau''} \subseteq \mathbf{K}^{\infty}\{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \leqslant 0\}$.

Proof. $\forall \mathbf{x} \in \mathbf{K}^{\tau'}, 1 < \tau' \Rightarrow \forall i = 1, \ldots, L : \mathbf{c}_i^{\tau'} \mathbf{x} \geqslant 0$. Then, using Corollary 1 and the inequalities $\mathbf{ux} \leqslant 0$, and $-\tau''\mathbf{ux} \geqslant -\tau'\mathbf{ux} \geqslant 0$, we have $\mathbf{x} \in \mathbf{K}^{\tau''}$. □

Theorem 8. $\tau' < \tau'' < 1 : \mathbf{K}^{\tau''} \subseteq \mathbf{K}^{\tau'} \subseteq \mathbf{K}^{-\infty} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \geqslant 0\}$.

Proof. $\forall \mathbf{x} \in \mathbf{K}^{\tau''}, \tau'' < 1 \Rightarrow \forall i = 1, \ldots, L : \mathbf{c}_i^{\tau''} \mathbf{x} \geqslant 0$. Then, using Corollary 1 and the inequalities $\mathbf{ux} \geqslant 0$ and $\tau''\mathbf{ux} \geqslant \tau'\mathbf{ux} \geqslant 0$, we have $\mathbf{x} \in \mathbf{K}^{\tau'}$. □

Theorem 9. $\tau' < \tau'' < 1 : \mathbf{K}_i^{\tau''} \subseteq \mathbf{K}_i^{\tau'} \subseteq \mathbf{K}_i^{-\infty} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} > 0\}, i = 1, 2$.

Proof. Let $\tau' < \tau'' < 1$. Then $\mathbf{x} \in \mathbf{K}_i^{\tau''} \Rightarrow \exists\, l = 1, \ldots, L$: $\mathbf{c}_l \mathbf{x} > \tau''\mathbf{ux}$. By Corollaries 1 and 3 we conclude $\mathbf{ux} > 0$. As $\tau' < \tau''$ implies $\tau'\mathbf{ux} < \tau''\mathbf{ux}$, we have

$$\mathbf{x} \in \mathbf{K}_i^{\tau''} \subseteq \mathbf{K}_i^{\tau'} \subseteq \mathbf{K}_i^{-\infty}. \qquad \square$$

Theorem 10. $1 < \tau' < \tau'' : \mathbf{K}_i^{\tau'} \subseteq \mathbf{K}_i^{\tau''} \subseteq \mathbf{K}_i^{\infty} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} < 0\}, i = 1, 2$.

Proof. Analogous to the proof of Theorem 9. □

The next theorem implies that a direction, feasible for both the original and perturbed problem, with $\tau > 0$, is either a direction of equilibrium or proper direction of the original problem.

Theorem 11. $\forall\tau > 0 : \mathbf{K}^\tau \cap \mathbf{K} \subseteq \mathbf{K}_0 \cup \mathbf{K}_1$.

Proof (*by contradiction*). Assume $\exists\tau > 0$, $\mathbf{x}' \in \mathbf{K}^\tau \cap \mathbf{K}$, $l' \in \{1, \ldots, L\}$, $\exists l'' \in \{1, \ldots, L\}$, $\mathbf{c}_{l'}\,\mathbf{x}' = 0$ and $\mathbf{c}_{l''}\mathbf{x}' > 0$. Then, using the inclusion $\mathbf{x}' \in \mathbf{K}$, we obtain

$$\mathbf{c}^\tau_{l'}\mathbf{x}' = (\mathbf{c}_{l'} - \tau\mathbf{u})\mathbf{x}' = \mathbf{c}_{l'}\mathbf{x}' - \tau\sum_{k=1}^{L}\mu_k\mathbf{c}_k\mathbf{x}' < 0$$

and $\therefore \mathbf{x}' \notin \mathbf{K}^\tau$, a contradiction. □

In the case $\tau > 1$, we can be more explicit: a direction, which is feasible for both the original and perturbed problems, is a direction of equilibrium for the original problem.

Theorem 12. $\forall\tau > 1 : \mathbf{K}^\tau \cap \mathbf{K} = \mathbf{K}_0$.

Proof. According to Corollary 2, $\mathbf{K}_0 \subseteq \mathbf{K}^\tau \cap \mathbf{K}$. Let $\mathbf{x} \in \mathbf{K}^\tau \cap \mathbf{K}$, then Corollary 1 asserts that $\mathbf{ux} \geqslant 0$ (since $\mathbf{x} \in \mathbf{K}$) and $\mathbf{ux} \leqslant 0$ (since $\mathbf{x} \in \mathbf{K}^\tau$). Therefore, $\mathbf{ux} = 0$ and according to Corollary 3 $\mathbf{K}^\tau \cap \mathbf{K} \subseteq \mathbf{K}_0$. □

The following theorem describes the structure of the perturbed cone in the case where $\tau = 1$ and shows that the hyperplane $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} = 0\}$ is not supporting for this case.

Theorem 13. $\forall\tau < 1 : \mathbf{K}^1 \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \geqslant 0\} \subseteq \mathbf{K}^\tau$; $\forall\tau > 1 : \mathbf{K}^1 \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \leqslant 0\} \subseteq \mathbf{K}^\tau$.

Proof. Consider $\forall\mathbf{x} \in \mathbf{K}^1 \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \geqslant 0\}$, $\tau < 1$. Then $\forall i = 1, \ldots, L$: $\mathbf{c}_i\mathbf{x} \geqslant \mathbf{ux} \geqslant \tau\mathbf{ux} \Rightarrow \mathbf{c}_i\mathbf{x} - \tau\mathbf{ux} \geqslant 0 \Rightarrow \mathbf{x} \in \mathbf{K}^\tau$. Consider $\forall\mathbf{x} \in \mathbf{K}^1 \cap \{\mathbf{x} \in \mathbb{R}^n : \mathbf{ux} \leqslant 0\}$, $\tau > 1$. Then $\forall i = 1, \ldots, L : \mathbf{c}_i\mathbf{x} \geqslant \mathbf{ux} \geqslant \tau\mathbf{ux} \Rightarrow: \forall i = 1, \ldots, L : \mathbf{c}_i\mathbf{x} - \tau\mathbf{ux} \geqslant 0 \Rightarrow \mathbf{x} \in \mathbf{K}^\tau$. □

The following theorem shows that if $\exists\tau' < 1$ such that every feasible direction of the perturbed problem is an equilibrium direction, then all feasible directions remain equilibrium directions $\forall\tau \in [\tau', 1)$.

Theorem 14. *If* $\exists\tau' < 1 : \mathbf{K}^{\tau'}_0 = \mathbf{K}^{\tau'}$, *then* $\forall\tau \in [\tau', 1) : \mathbf{K}^\tau_0 = \mathbf{K}^\tau$. *If* $\exists\tau' > 1 : \mathbf{K}^{\tau'}_0 = \mathbf{K}^{\tau'}$, *then* $\forall\tau \in (1, \tau'] : \mathbf{K}^\tau_0 = \mathbf{K}^\tau$.

Proof. Immediately follows from Theorems 4, 7 and 8. If $\tau \in [\tau', 1)$, then $\mathbf{K}_0^\tau = \mathbf{K}_0^{\tau'} = \mathbf{K}^{\tau'} \supseteq \mathbf{K}^\tau \Rightarrow \mathbf{K}_0^\tau = \mathbf{K}^\tau$. If $\tau \in (1, \tau']$, then $\mathbf{K}_0^\tau = \mathbf{K}_0^{\tau'} = \mathbf{K}^{\tau'} \subseteq \mathbf{K}^\tau \Rightarrow \mathbf{K}_0^\tau = \mathbf{K}^\tau$. □

Note that similar property holds for $\tau > 0$: if $\exists \tau' > 1$ such that every feasible direction of the perturbed problem is the direction of equilibrium, then all feasible directions remain equilibrium directions $\forall \tau \in (1, \tau']$.

4.2. General perturbations

In this section we will provide a comparative analysis of the cone perturbations defined by (5) and cone perturbations defined by (6). In the case defined by (5), all the rows of matrix $\mathbf{C}$ are perturbed by $\mathbf{u} \in \operatorname{ri} \mathbf{K}^\star$. In the general case (6), row j of matrix $\mathbf{C}$ is perturbed by a different $\mathbf{u}_j \in \operatorname{ri} \mathbf{K}^\star$, $j = 1, \ldots, L$. Perturbations are different for different rows. In what follows, the cone defined by (6) will be denoted by $\mathbf{K}^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$; and the cone, defined by (5), will be denoted by $\mathbf{K}^\tau(\mathbf{u})$.

Results for the general perturbation case cover basic perturbations; however, the monotonicity property does not hold for general perturbations. In what follows, we consider the general perturbation

$$\mathbf{C}_\tau = \mathbf{C} - \tau\mathbf{U}, \text{ where } \mathbf{U} \in \mathbb{R}^{L\times n}, \mathbf{U} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_L \end{pmatrix}.$$

In terms of matrices $\mathbf{U} = \mathbf{MC}$, where the $L \times L$ matrix $\mathbf{M}$ consists of entries μ_i^j, defined as in Remark 2:

$$\mathbf{M} = \begin{pmatrix} \mu_1^1 & \cdots & \mu_L^1 \\ \mu_1^2 & \cdots & \mu_L^2 \\ & \cdots & \\ \mu_1^L & \cdots & \mu_L^L \end{pmatrix}.$$

The matrix $\mathbf{M}$ is positive stochastic. Let $\Lambda = \{\lambda \in \mathbf{R} : \det(\lambda\mathbf{I} - \mathbf{M}) = 0\}$. Since $\Lambda \neq \emptyset$, According to [4,22], the dominant eigenvalue $\lambda(\mathbf{M}) = 1$ and $\max\{\lambda : \lambda \in \Lambda, \lambda > 0\} = 1$. Moreover, it follows from the Perron–Frobenius theorem, that matrix $\rho\mathbf{I} - \mathbf{M}$ has a nonnegative inverse iff $\rho > \lambda(\mathbf{M}) = 1$.

Therefore, in the case, where $\tau > 0$, the matrix $\mathbf{I} - \tau\mathbf{M}$ has nonnegative inverse iff $\tau < 1/\lambda(\mathbf{M}) = 1$, i.e.,

$$\begin{aligned}(\mathbf{C} - \tau\mathbf{U})\mathbf{x} \geqslant \mathbf{0} \Rightarrow \mathbf{M}(\mathbf{C} - \tau\mathbf{U})\mathbf{x} \geqslant \mathbf{0} &\Leftrightarrow (\mathbf{U} - \tau\mathbf{MU})\mathbf{x} \geqslant \mathbf{0} \\ &\Leftrightarrow (\mathbf{I} - \tau\mathbf{M})\mathbf{Ux} \geqslant \mathbf{0} \\ &\Leftrightarrow \mathbf{Ux} \geqslant \mathbf{0}.\end{aligned}$$

Therefore, for $\tau \in [0, 1)$:

$$\mathbf{K}^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L) \subseteq \mathbf{K}(\mathbf{U}) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \mathbf{Ux} \geqslant 0\} \Rightarrow \mathbf{K}^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L) \subseteq \mathbf{K}.$$

If the matrix $\mathbf{M}$ has a nonnegative inverse, then for $\tau \in [0, 1)$:

$$\mathbf{K}^{\tau}(\mathbf{u}_1, \ldots, \mathbf{u}_L) = \mathbf{K}(\mathbf{U}).$$

As mentioned in Remark 2, Theorem 6 can be generalized as follows.

Theorem 15. $\forall \tau \in [0, 1) : \mathbf{K}^{\tau}(\mathbf{u}_1, \ldots, \mathbf{u}_L) \subseteq \mathbf{K}$.

The following proof to this theorem is straightforward.

Proof. Let $y \in \mathbf{K}^{\tau}(\mathbf{u}_1, \ldots, \mathbf{u}_L)$, then $(\mathbf{c}_i - \tau\mathbf{u}_i)\mathbf{y} \geqslant 0$, $i = 1, \ldots, L$. We will show that $\mathbf{y} \in \mathbf{K}$. Let

$$\mathbf{u}_{i_0} = \min\{\mathbf{u}_i\mathbf{y} : i = 1, \ldots, L\}, \quad \mathbf{u}_{i_0} = \sum_{i=1}^{L} \mu_i^{i_0}\mathbf{c}_i.$$

Then for

$$\sum_{k=1}^{L} \mu_k^{i_0} = 1,\ \mu_k^{i_0} > 0, \quad k = 1, \ldots, L, \quad \tau > 0 \Rightarrow -\tau\mu_k^{i_0}\mathbf{u}_{i_0}\mathbf{y} \geqslant -\tau\mu_k^{i_0}\mathbf{u}_k\mathbf{y}.$$

If we add $\mu_k^{i_0}\mathbf{c}_k\mathbf{y}$ to both sides of these equations, we obtain

$$\mu_k^{i_0}\mathbf{c}_k\mathbf{y} - \tau\mu_k^{i_0}\mathbf{u}_{i_0}\mathbf{y} \geqslant \mu_k^{i_0}\mathbf{c}_k\mathbf{y} - \tau\mu_k^{i_0}\mathbf{u}_k\mathbf{y} = \mu_k^{i_0}(\mathbf{c}_k - \tau\mathbf{u}_k)\mathbf{y} \geqslant 0$$

or

$$\mu_k^{i_0}\mathbf{c}_k\mathbf{y} - \tau\mu_k^{i_0}\mathbf{u}_{i_0}\mathbf{y} \geqslant 0.$$

Then

$$\sum_{k=1}^{L} \mu_k^{i_0}\mathbf{c}_k\mathbf{y} - \tau\sum_{k=1}^{L} \mu_k^{i_0}\mathbf{u}_{i_0}\mathbf{y} \geqslant 0.$$

Since

$$\begin{aligned}\sum_{k=1}^{L} \mu_k^{i_0}\mathbf{c}_k\mathbf{y} - \tau\sum_{k=1}^{L} \mu_k^{i_0}\mathbf{u}_{i_0}\mathbf{y} &= \left(\sum_{k=1}^{L} \mu_k^{i_0}\mathbf{c}_k\right)\mathbf{y} - \tau\left(\sum_{k=1}^{L} \mu_k^{i_0}\right)\mathbf{u}_{i_0}\mathbf{y} \\ &= \mathbf{u}_{i_0}\mathbf{y} - \tau\mathbf{u}_{i_0}\mathbf{y} = (1-\tau)\mathbf{u}_{i_0}\mathbf{y} \geqslant 0,\end{aligned}$$

then $\mathbf{u}_{i_0}\mathbf{y} \geqslant 0$, $\mathbf{u}_k\mathbf{y} \geqslant 0$ and $\mathbf{c}_k\mathbf{y} \geqslant \tau\mathbf{u}_k\mathbf{y} \geqslant 0$. Therefore $\mathbf{c}_k\mathbf{y} \geqslant 0$ and $\mathbf{y} \in \mathbf{K}$. □

Theorem 2 can be further generalized as well.

Theorem 16. $\forall \tau \leqslant 0 : \mathbf{K} \subseteq \mathbf{K}^{\tau}(\mathbf{u}_1, \ldots, \mathbf{u}_L)$.

Proof. Consider the arbitrary point $\mathbf{x} \in \mathbf{K}$. Since $\forall \mathbf{x} \in \mathbf{K} \Rightarrow \mathbf{u}_k\mathbf{x} \geqslant 0$ and $-\tau > 0$, we have $-\tau\mathbf{c}_u\mathbf{x} \geqslant 0$, $k = 1, \ldots, L \geqslant 0$. Since $\mathbf{c}_k\mathbf{x} \geqslant 0$, $k = 1, \ldots, L$, we obtain $(\mathbf{c}_k - \tau\mathbf{u}_k)\mathbf{x} \geqslant 0$, $k = 1, \ldots, L$. This means, that $\mathbf{x} \in \mathbf{K}^{\tau}(\mathbf{u}_1, \ldots, \mathbf{u}_L)$. □

Theorem 3 can be generalized as follows:

Theorem 17. $\forall\tau < 0 : \mathbf{K} \subseteq \mathbf{K}_0^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L) \cup \mathbf{K}_1^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$.

Proof. The inclusion $\mathbf{x} \in \mathbf{K}_0$ implies, that $\mathbf{u}_k\mathbf{x} = 0$ for $k = 1, \ldots, L$ and $\mathbf{C}^\tau\mathbf{x} = \mathbf{0}$ for any τ, hence, $\mathbf{x} \in \mathbf{K}_0^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$. If $\mathbf{x} \in \mathbf{K} \setminus \mathbf{K}_0$, then $\mathbf{u}_k\mathbf{x} > 0$ for $k = 1, \ldots, L$, and $\forall\tau < 0 : -\tau\mathbf{u}\mathbf{x} > 0$ and $\mathbf{C}^\tau\mathbf{x} > \mathbf{0}$, hence, $\mathbf{x} \in \mathbf{K}_1^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$. □

The generalization of Theorem 4 is less straightforward, but is possible.

Theorem 18. $\forall\tau \in \mathbb{R}^n : \mathbf{K}_0 \subseteq \mathbf{K}_0^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$.

Proof

$$\mathbf{C}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{M}\mathbf{C}\mathbf{x} = \mathbf{0} \Rightarrow (\mathbf{C} - \tau\mathbf{U})\mathbf{x} = \mathbf{0}.$$

Therefore, for $\tau \in \mathbb{R}$:

$$\mathbf{K}_0 \subseteq \mathbf{K}_0(\mathbf{U}) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \mathbf{U}\mathbf{x} = 0\};$$

$$\mathbf{K}_0 \subseteq \mathbf{K}_0^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L). \qquad \square$$

Corollary 4

$$\mathbf{K}_0 \subseteq \mathbf{K}_0(\mathbf{U});$$

If matrix $\mathbf{M}$ *is invertible, then*

$$\mathbf{K}_0 = \mathbf{K}_0(\mathbf{U}).$$

Corollary 5. $\mathbf{K}_0 = \mathbf{K}_0^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$ *iff* $\frac{1}{\tau} \in \mathbb{R} \setminus \Lambda$.

In particular, it follows from the Perron–Frobenius theorem, that $\max\{|\lambda| : \lambda \in \Lambda\} \leqslant 1$. Therefore, the following corollary is also true.

Corollary 6. $\forall\tau \in (-1, 1) : \mathbf{K}_0^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L) = \mathbf{K}_0$.

Remark 4. As mentioned previously, the size of the cone $\mathbf{K}^\tau(\mathbf{u})$ monotonically changes with changes in τ. Thus a cone with the smaller τ contains the cone with bigger τ. This property of perturbed cones does not hold for the cone $\mathbf{K}^\tau(\mathbf{u}_1, \ldots, \mathbf{u}_L)$ if $\tau < 0$.

Example 3. Let $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{pmatrix}$. Then $\mathbf{M} = \mathbf{U}$ and

$$\bar{\mathbf{x}} = (-0.9;\ 1.1) \in \mathbf{K}^{-1}(\mathbf{u}_1, \ldots, \mathbf{u}_L);$$

$$\mathbf{K}^{-1}(\mathbf{u}_1, \ldots, \mathbf{u}_L) = \{\mathbf{x} \in \mathbb{R}^n : 1.1x_1 + 0.9x_2 \geqslant 0, 0.9x_1 + 1.1x_2 \geqslant 0\};$$

Nevertheless, $\bar{\mathbf{x}}$ does not belong to $\mathbf{K}^{-2}(\mathbf{u}_1, \ldots, \mathbf{u}_L) = \{\mathbf{x} \in \mathbb{R}^n : 1.2x_1 + 1.8x_2 \geqslant 0, 1.8x_1 + 1.2x_2 \geqslant 0\}$.

5. Unboundedness

The purpose of this section is to provide a preliminary stability analysis for problem (1) where the feasible domain is defined as:

$$\mathbf{X} = \mathbf{X}(\mathbf{A}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqslant \mathbf{b}\}.$$

Henceforth, we will refer to this linear, multi-objective optimization problem as problem $(\mathbf{C}, \mathbf{X})$. We will consider changes in the input data triple $(\mathbf{C}, \mathbf{A}, \mathbf{b})$. In this section, we assume $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} < \mathbf{b}\} \neq \emptyset$. Otherwise infinitely small changes in the right-hand side vector $\mathbf{b}$ can lead to an empty feasible set. Results from convex analysis and theory of linear inequalities which are used in this section, are provided in Appendix A.

Definition 1. The problem $(\mathbf{C}, \mathbf{X})$ is *unbounded* iff

$$\forall \mathbf{x} \in \mathbf{X}\ \exists \bar{\mathbf{x}} \in \mathbf{X} : \mathbf{C}\bar{\mathbf{x}} \geqslant \mathbf{C}\mathbf{x}\ \&\ \mathbf{C}\bar{\mathbf{x}} \neq \mathbf{C}\mathbf{x}.$$

Otherwise the problem is *solvable*, where solvable is taken to mean that at least one optimum exists.

Remark 5. The problem is unbounded if $\boldsymbol{\Pi}(\mathbf{C}, \mathbf{X}) = \emptyset$ and $X \neq \emptyset$. If $\exists \mathbf{x} \in \mathbf{X} : \nexists \bar{\mathbf{x}} \in \mathbf{X} : \mathbf{C}\bar{\mathbf{x}} \geqslant \mathbf{C}\mathbf{x}\ \&\ \mathbf{C}\bar{\mathbf{x}} \neq \mathbf{C}\mathbf{x}$, then $\mathbf{x} \in \boldsymbol{\Pi}(\mathbf{C}, \mathbf{X}) \neq \emptyset$.

5.1. Necessary and sufficient conditions of unboundedness

As before, let $\mathbf{G}$ be the recession cone of the feasible set $\mathbf{X}$, $\mathbf{G} = \mathbf{O}^+\mathbf{X}$ [27], and $\mathbf{K}$ be the ordering cone of the problem. The following theorem provides a necessary and sufficient *unboundedness condition*.

Theorem 19 (Unboundedness condition). *The problem* $(\mathbf{C}, \mathbf{X})$ *is unbounded iff*

$$(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} \neq \emptyset.$$

Proof. (*Necessity*) Assume that the problem is unbounded, i.e.,

$$\forall \mathbf{x} \in \mathbf{X}\ \exists \overline{\mathbf{x}} \in \mathbf{X} : \mathbf{C}\overline{\mathbf{x}} \geqslant \mathbf{C}\mathbf{x}\ \&\ \mathbf{C}\overline{\mathbf{x}} \neq \mathbf{C}\mathbf{x}.$$

By contradiction, let $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} = \emptyset$, i.e. $\mathbf{K} \cap \mathbf{G} = \mathbf{K}_0 \cap \mathbf{G}$. We will show that the set $\mathbf{R}(\mathbf{y}) = \{\mathbf{z} \in \mathbb{C}(\mathbf{X}) : \mathbf{z} \geqslant \mathbf{y}\}$ is bounded for every $\mathbf{y} \in \mathbf{Y} = \mathbb{C}(\mathbf{X})$. According to Theorem A.4 (see Appendix A), the set $\mathbf{R}(\mathbf{y})$ is bounded if and only if $\mathbf{O}^+\mathbf{R}(\mathbf{y}) = \{\mathbf{0}\}$. Since $\mathbf{R}(\mathbf{y}) = \mathbf{Y} \cap \{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \geqslant \mathbf{y}\}$, we have $\mathbf{O}^+\mathbf{R}(\mathbf{y}) = \mathbf{O}^+\mathbf{Y} \cap \mathbf{O}^+\{\mathbf{z} \in \mathbb{R}^\mathbf{n} : \mathbf{z} \geqslant \mathbf{y}\}$.

According to Theorem A.3 (see Appendix A),

$$\mathbf{O}^+\mathbb{C}^{-1}(\mathbf{Y}) = \mathbb{C}^{-1}(\mathbf{O}^+\mathbf{Y}).$$

Hence, $\mathbf{O}^+\mathbf{Y} = \mathbb{C}(\mathbf{O}^+\mathbf{X}) = \mathbb{C}(\mathbf{G})$ and

$$\begin{aligned}\mathbf{O}^+\mathbf{R}(\mathbf{y}) &= \mathbb{C}(\mathbf{G}) \cap \mathbf{O}^+\{\mathbf{z} \in \mathbb{R}^n : \mathbf{z} \geqslant \mathbf{y}\}\\ &= \{\mathbf{z} = \mathbb{C}(\mathbf{g}) : \mathbf{g} \in \mathbf{G}, \mathbf{z} \geqslant \mathbf{0}\}\\ &= \mathbb{C}(\mathbf{K} \cap \mathbf{G}) = \mathbb{C}(\mathbf{K}_0 \cap \mathbf{G}) = \{\mathbf{0}\}.\end{aligned}$$

We have, thus, shown, that $\mathbf{R}(\mathbf{y})$ is bounded and therefore $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) \neq \emptyset$.

(*Sufficiency*) Let $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} \neq \emptyset$ and $\mathbf{x} \in \mathbf{X}$. Then

$$\begin{aligned}((\mathbf{x} + (\mathbf{K}_1 \cup \mathbf{K}_2)) \cap \mathbf{X}) \setminus \{\mathbf{x}\} &= ((\mathbf{x} + (\mathbf{K}_1 \cup \mathbf{K}_2)) \cap (\mathbf{M} + \mathbf{G})) \setminus \{\mathbf{x}\}\\ &\supseteq ((\mathbf{x} + (\mathbf{K}_1 \cup \mathbf{K}_2)) \cap (\mathbf{x} + \mathbf{G})) \setminus \{\mathbf{x}\}\\ &= (\mathbf{x} + (\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G}) \setminus \{\mathbf{x}\} \neq \emptyset.\end{aligned}$$

Since $\mathbf{x} \in \mathbf{\Pi}(\mathbf{C}, \mathbf{X}) \Leftrightarrow (\mathbf{x} + (\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G}) \setminus \{\mathbf{x}\} = \emptyset$, we have $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) = \emptyset$. □

Remark 6. The following theorem is proved in [25] for linear vector problems: $\mathbf{\Pi}(\mathbf{C}, \mathbf{X}) \neq \emptyset \Leftrightarrow \operatorname{ri} \mathbf{K}^* \cap (-\mathbf{G})^* \neq \emptyset$. If Theorem 19 is true, then

$$\operatorname{ri} \mathbf{K}^* \cap (-\mathbf{G})^* \neq \emptyset \Leftrightarrow \mathbf{K} \cap \mathbf{G} = \mathbf{K}_0 \cap \mathbf{G}.$$

5.2. *Stable and unstable solvability*

Recall that one definition of the term *stability* refers to continuity of the problem's solution with respect to changes in the input data. Then, the term *stable solvability* is taken to mean that the original problem is solvable and for sufficiently small perturbations in the input data, the problem remains solvable. If the original problem is solvable, but there exist infinitely small perturbations in the input data for which the problem is no longer solvable, then the problem is called *unstable solvable*.

The following definition of stable solvability is analogous to the definition of stable solvability in the scalar case given by [1].

Definition 2. We say that problem $(\mathbf{C}, \mathbf{X})$ is *stable solvable*, if it is solvable and $\exists\delta > 0\ \forall\mathbf{C}(\delta) : \|\mathbf{C}(\delta) - \mathbf{C}\| < \delta, \forall\mathbf{A}(\delta) : \|\mathbf{A}(\delta) - \mathbf{A}\| < \delta, \forall\mathbf{b}(\delta) : \|\mathbf{b}(\delta) - \mathbf{b}\| < \delta$ problem $(\mathbf{C}(\delta), \mathbf{X}(\delta))$ is solvable.

Theorem 20. *If* $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$, *then the problem* $(\mathbf{C}, \mathbf{X})$ *is stable solvable.*

Proof. Since $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$, we have $(\mathbf{K} \cap \mathbf{G})^* = \mathbb{R}^n$. According to Lemma A.1 (see Appendix A) $(\mathbf{K}(\delta) \cap \mathbf{G}(\delta))^* = \mathbb{R}^n$, and thus $\mathbf{K}(\delta) \cap \mathbf{G}(\delta) = \{\mathbf{0}\}$. Hence,

$(\mathbf{K}_1(\delta) \cup \mathbf{K}_2(\delta)) \cap \mathbf{G}(\delta) = \emptyset$. According to the above Theorem 19 the problem $(\mathbf{C}(\delta), \mathbf{A}(\delta))$ is solvable. □

Remark 7. If $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$, then $r(\mathbf{A}/\mathbf{C}) = n$. The condition $r(\mathbf{A}/\mathbf{C}) = n$ does not necessarily lead to $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$.

Definition 3. We say that problem $(\mathbf{C}, \mathbf{X})$ is *unstable solvable*, if it is solvable and $\forall \delta > 0\, \exists \mathbf{C}(\delta) : \|\mathbf{C}(\delta) - \mathbf{C}\| < \delta, \exists \mathbf{A}(\delta) : \|\mathbf{A}(\delta) - \mathbf{A}\| < \delta, \exists\, \mathbf{b}(\delta) : \|\mathbf{b}(\delta) - \mathbf{b}\| < \delta$ problem $(\mathbf{C}(\delta), \mathbf{X}(\delta))$ is unsolvable.

Theorem 21. *If* $\mathbf{K} \cap \mathbf{G} \subseteq \mathbf{K}_0$ *and* $(\mathbf{K} \cap \mathbf{G}) \setminus \{\mathbf{0}\} \neq \emptyset$, *then the problem* $(\mathbf{C}, \mathbf{X})$ *is unstable solvable.*

Proof. Since $\mathbf{K} \cap \mathbf{G} \subseteq \mathbf{K}_0$, we have $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} = \emptyset$ and as a result of Theorem 19, the problem $(\mathbf{C}, \mathbf{X})$ is solvable. Since $(\mathbf{K} \cap \mathbf{G}) \setminus \{\mathbf{0}\} \neq \emptyset$, consider $\widetilde{\mathbf{x}} \in (\mathbf{K} \cap \mathbf{G}) \setminus \{\mathbf{0}\}$. Then $\mathbf{A}\widetilde{\mathbf{x}} = \mathbf{0}$, $C\widetilde{\mathbf{x}} = \mathbf{0}$ and $\widetilde{\mathbf{x}} \neq \mathbf{0}$. Consider

$$\mathbf{C}(\delta) : \mathbf{c}_i(\delta) = \mathbf{c}_i + \delta\widetilde{\mathbf{x}}, \quad i = 1, \ldots, l;$$

$$\mathbf{A}(\delta) : \mathbf{a}_i(\delta) = \mathbf{a}_i - \delta\widetilde{\mathbf{x}}, \quad i = 1, \ldots, l.$$

Then $\mathbf{C}(\delta)\widetilde{\mathbf{x}} > \mathbf{0}$, $\mathbf{A}(\delta)\widetilde{\mathbf{x}} < 0$. Hence, $\widetilde{\mathbf{x}} \in (\mathbf{K}_1(\delta) \cup \mathbf{K}_2(\delta)) \cap \mathbf{G}(\delta) \neq \emptyset$ and as a result of Theorem 19, problem $(\mathbf{C}(\delta), \mathbf{X}(\delta))$ is unbounded. □

The following corollary provides us with necessary and sufficient *condition of stable solvability*.

Corollary 7 (Condition of stable solvability). *The problem* $(\mathbf{C}, \mathbf{X})$ *is stable solvable iff* $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$.

Proof. According to Theorem 20, if $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$, then the problem is stable solvable.

Consider the case $(\mathbf{C}, \mathbf{X})$ is stable solvable and $\mathbf{K} \cap \mathbf{G} \neq \{\mathbf{0}\}$. According to Theorem 19 we have $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} = \emptyset$. Therefore $\mathbf{K} \cap \mathbf{G} = \mathbf{K}_0 \cap \mathbf{G}$ and $(\mathbf{K}_0 \cap \mathbf{G}) \setminus \{\mathbf{0}\} \neq \emptyset$, then, according to the Theorem 21 the problem $(\mathbf{C}, \mathbf{X})$ is unstable solvable. This contradicts the assumption. □

We can also provide necessary and sufficient *condition of unstable solvability*.

Corollary 8 (Condition of unstable solvability). *The problem is unstable solvable iff* $\mathbf{K} \cap \mathbf{G} \subseteq \mathbf{K}_0$ *and* $\mathbf{K}_0 \cap \mathbf{G} \neq \{\mathbf{0}\}$.

Proof. If the problem is solvable, then $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} = \emptyset$. There are two possibilities:

(a) $\mathbf{K}_0 \cap \mathbf{G} = \{\mathbf{0}\}$;
(b) $(\mathbf{K}_0 \cap \mathbf{G}) \setminus \{\mathbf{0}\} \neq \emptyset$.

In the first case, according to the Corollary 7 $(\mathbf{C}, \mathbf{X})$ is stable solvable. Thus, case (b) is valid. □

Theorem 22. *Let* $\mathbf{0} \in \mathrm{ri}(\mathbf{K} \cap \mathbf{G})^*$. *If* $r(\mathbf{A}/\mathbf{C}) = n$, *then the problem* $(\mathbf{C}, \mathbf{X})$ *is stable solvable. If* $r(\mathbf{A}/\mathbf{C}) < n$, *then the problem* $(\mathbf{C}, \mathbf{X})$ *is unstable solvable.*

Proof. Since $0 \in \mathrm{ri}(\mathbf{K} \cap \mathbf{G})^*$, $(\mathbf{K} \cap \mathbf{G})^*$ is some subspace of the space $\mathbb{R}^n$ and its dimension is $\dim (\mathbf{K} \cap \mathbf{G})^* = r(\mathbf{A}/\mathbf{C})$. Then, the dual cone $\mathbf{K} \cap \mathbf{G}$ is also a subspace of the space $\mathbb{R}^n$ and its dimension is $\dim \mathbf{K} \cap \mathbf{G} = n - r(\mathbf{A}/\mathbf{C})$. If $r(\mathbf{A}/\mathbf{C}) = n$, then $\mathbf{K} \cap \mathbf{G} = \{\mathbf{0}\}$. In this case the problem is stable solvable (Corollary 7). If $r(\mathbf{A}/\mathbf{C}) < n$, then $\mathbf{K} \cap \mathbf{G} = (\mathbf{K} \cap \mathbf{G})_L$. In this case the problem is unstable solvable (Corollary 8). □

5.3. Stable and unstable unboundedness

Definition 4. We say that the problem $(\mathbf{C}, \mathbf{X})$ is *stable unbounded*, if $\exists \delta > 0 \, \forall \mathbf{C}(\delta)$: $\|\mathbf{C}(\delta) - \mathbf{C}\| < \delta$, $\forall \mathbf{A}(\delta) : \|\mathbf{A}(\delta) - \mathbf{A}\| < \delta$, $\forall \mathbf{b}(\delta) : \|\mathbf{b}(\delta) - \mathbf{b}\| < \delta$ and the problem $(\mathbf{C}(\delta), \mathbf{X}(\delta))$ is unbounded.

Definition 5. We say that problem $(\mathbf{C}, \mathbf{X})$ is *unstable unbounded*, if it is unbounded and $\forall \delta > 0 \, \exists \mathbf{C}(\delta) : \|\mathbf{C}(\delta) - \mathbf{C}\| < \delta, \exists \mathbf{A}(\delta) : \|\mathbf{A}(\delta) - A\| < \delta$, $\exists \mathbf{b}(\delta) : \|\mathbf{b}(\delta) - \mathbf{b}\| < \delta$ and the problem $(\mathbf{C}(\delta), \mathbf{X}(\delta))$ is solvable.

Theorem 23. *If* $\mathbf{K}_1 \cap \mathbf{G} = \emptyset$ *and* $\mathbf{K}_2 \cap \mathbf{G} \neq \emptyset$, *then the problem* $(\mathbf{C}, \mathbf{X})$ *is unstable unbounded.*

Proof. By assumption $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} \neq \emptyset$, and, therefore, the problem $(\mathbf{C}, \mathbf{X})$ is unbounded. Consider the perturbed problem $(\mathbf{C}(\delta), \mathbf{X})$, where $\mathbf{C}(\delta) = \mathbf{C}_\tau$ is constructed as in (5) and

$$0 < \tau < \frac{\delta}{\|\bar{\mathbf{U}}\|}, \quad \bar{\mathbf{U}} = \mathbf{e}^L \mathbf{u}, \quad \mathbf{u} = \sum_{k=1}^{L} \mu_k \mathbf{c}_k, \quad \mu_k > 0, \ \ k = 1, \ldots, L. \tag{9}$$

Here and elsewhere, $\mathbf{e}^p = (1, \ldots, 1)^\top \in \mathbb{R}^p$. According to the properties of perturbed cones given in Theorem 6

$$\forall \tau \in [0, 1) : \mathbf{K}^\tau \subseteq \mathbf{K}$$

and in Theorem 11

$$\forall \tau > 0 : \mathbf{K}^\tau \cap \mathbf{K} \subseteq \mathbf{K}_0 \cup \mathbf{K}_1,$$

it follows that:

$$\mathbf{K}^\tau \subseteq \mathbf{K}_0 \cup \mathbf{K}_1.$$

According to the properties of perturbed cones (Theorem 4) $\forall \tau \in \mathbb{R} : \mathbf{K}_0 \subseteq \mathbf{K}_0^\tau$. Therefore $\mathbf{K}^\tau \subseteq \mathbf{K}_0^\tau \cup \mathbf{K}_1$. Hence $\mathbf{K}_1^\tau \subseteq \mathbf{K}_0^\tau \cup \mathbf{K}_1$ and $\mathbf{K}_2^\tau \subseteq \mathbf{K}_0^\tau \cup \mathbf{K}_1$. Then $\mathbf{K}_1^\tau \subseteq \mathbf{K}_1$ and $\mathbf{K}_2^\tau \subseteq \mathbf{K}_1$. Thus,

$$(\mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau) \cap \mathbf{G} = \emptyset.$$

The problem $(\mathbf{C}(\delta), \mathbf{X}(\delta))$, where $\mathbf{C}(\delta) = \mathbf{C}_\tau$ is constructed by analogy with (5), $0 < \tau < \frac{\delta}{\|\bar{\mathbf{U}}\|}$ and $\mathbf{X}(\delta) = \mathbf{X}$, is solvable. □

The following results provide a *necessary and sufficient condition of stable unboundedness*.

Theorem 24 (Condition of stable unboundedness). *The problem* $(\mathbf{C}, \mathbf{X})$ *is stable unbounded iff* $\mathbf{K}_1 \cap \mathbf{G}_1 \neq \emptyset$.

Proof. (*Necessity*) (by contradiction) Suppose problem $(\mathbf{C}, \mathbf{X})$ is stable unbounded, but $\mathbf{K}_1 \cap \mathbf{G}_1 = \emptyset$. Then, according to the proof of Theorem 23,

$$\forall \tau \in [0, 1) : (\mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau) \cap \mathbf{G}_1 = \emptyset.$$

Therefore, since the problem is stable unbounded, $\exists \tau^0 \in [0, 1) : \forall \tau \in [0, \tau^0)$ the following inclusion holds:

$$\emptyset \neq (\mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau) \cap \mathbf{G} \subseteq (\mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau) \cap (\mathbf{G}_0 \cup \mathbf{G}_2),$$

i.e.,

$$\forall \mathbf{x} \in \mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau \; \exists i = 1, \ldots, m : \mathbf{a}_i \mathbf{x} \geqslant 0.$$

Moreover, according to Theorems 4 and 9,

$$\forall \mathbf{x} \in \mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau : \; \mathbf{u}\mathbf{x} > 0.$$

Therefore, $\forall \mathbf{x} \in (\mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau) : \exists \; i = 1, \ldots, m : \; (\mathbf{a}_i + \tau \mathbf{u})\mathbf{x} > 0$ i.e. $(\mathbf{K}_1^\tau \cup \mathbf{K}_2^\tau) \cap \mathbf{G}^\tau = \emptyset$. Here $\mathbf{G}^\tau = \mathbf{K}(-(\mathbf{A} + \tau \bar{\mathbf{U}}))$. The problem $(\mathbf{C}_\tau, \mathbf{X}_\tau)$ with input data $(\mathbf{C}_\tau, \mathbf{A}_\tau, \mathbf{b})$ is solvable for $\tau \in (0, \tau^0)$. This is a contradiction with the assumption.

(*Sufficiency*) Since $\mathbf{K}_1 \cap \mathbf{G}_1 \neq \emptyset$, then $\exists \mathbf{g} \in \mathbf{K}_1 \cap \mathbf{G}_1 : \mathbf{C}\mathbf{g} > \mathbf{0}, \mathbf{A}\mathbf{g} < \mathbf{0}$ (here $\mathbf{G}_1$ denotes $-\mathbf{K}_1(\mathbf{A})$).

Let $\delta_0^1 = \min \delta_i^1, i = 1, \ldots, L$, where $\delta_i^1 = \mathbf{c}_i \mathbf{g} > 0$; $\delta_0^2 = \min \delta_i^2, i = 1, \ldots, m$, where $\delta_i^2 = -\mathbf{a}_i \mathbf{g} > 0$; $\delta_0 = \min\{\delta_0^1, \; \delta_0^2\}$; $0 < \delta < \frac{\delta_0}{\|\mathbf{g}\|}$. Consider arbitrary $\mathbf{C}(\delta)$ and $\mathbf{A}(\delta)$ such that

$$\|\mathbf{C}(\delta) - \mathbf{C}\| < \delta \quad \text{and} \quad \|\mathbf{A}(\delta) - \mathbf{A}\| < \delta.$$

Let $\mathbf{\Delta}_i = \mathbf{c}_i(\delta) - \mathbf{c}_i, \; i = 1, \ldots, L; \; \bar{\mathbf{\Delta}}_i = \mathbf{a}_i(\bar{\delta}) - \mathbf{a}_i, \; i = 1, \ldots, m$. Then $\forall i \in \{1, \ldots, L\}, \; \mathbf{c}_i(\delta)\mathbf{g} = \mathbf{c}_i \mathbf{g} + \mathbf{\Delta}_i \mathbf{g} \geqslant \mathbf{c}_i \mathbf{g} - |\mathbf{\Delta}_i \mathbf{g}| \geqslant \mathbf{c}_i \mathbf{g} - \|\mathbf{c}_i(\delta) - \mathbf{c}_i\| \times \|\mathbf{g}\| > \delta_i - \delta > \delta_i - \delta_0 > \mathbf{0}$.

Similarly $\forall i \in \{1, \ldots, m\}$, $\mathbf{a}_i(\delta)\mathbf{g} = \mathbf{a}_i\mathbf{g} + \bar{\mathbf{\Delta}}_i\mathbf{g} \leqslant \mathbf{a}_i\mathbf{g} + |\bar{\mathbf{\Delta}}_i\mathbf{g}| \leqslant \mathbf{a}_i\mathbf{g} + \|\mathbf{a}_i(\bar{\delta}) - \mathbf{a}_i\| \times \|\mathbf{g}\| < -\delta_i + \delta < -\delta_i + \delta_0 < \mathbf{0}$.

Thus, $\mathbf{K}_1(\delta) \cap \mathbf{G}_1(\delta) \neq \emptyset$.

Then, according to Theorem 19 the problem $(\mathbf{C}(\delta), \mathbf{X})$ is unbounded and the problem $(\mathbf{C}, \mathbf{X})$ is stable unbounded. □

Corollary 9 (Condition of unstable unboundedness). *A problem is unstable unbounded iff* $\mathbf{K}_1 \cap \mathbf{G}_1 = \emptyset$ *and* $(\mathbf{K}_1 \cup \mathbf{K}_2) \cap \mathbf{G} \neq \emptyset$.

Corollary 10. *If the problem is unstable unbounded, then* $\exists \tau^0 : \forall \tau \in (0, \tau^0)$ *the problem* $(\mathbf{C}_\tau, \mathbf{X}_\tau)$, *where*

$$\mathbf{C}_\tau = \mathbf{C} - \tau\bar{\mathbf{U}}, \quad \mathbf{A}_\tau = \mathbf{A} + \tau\bar{\mathbf{U}}, \quad \mathbf{b} = \mathbf{b}$$

is solvable.

Proof. Analogous to the proof of necessity for Theorem 24. □

5.4. Regularization approach for unbounded problems

We consider set of input data $\mathbf{\Omega} \subseteq \mathbb{R}^{L\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^m$ of all $(\mathbf{C}, \mathbf{A}, \mathbf{b}) \in \mathbb{R}^{L\times n} \times \mathbb{R}^{m\times n} \times \mathbb{R}^m$ such that $(\mathbf{C}, \mathbf{A}, \mathbf{b})$ defines a problem with $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} < \mathbf{b}\} \neq \emptyset$. The following theorems show that under certain conditions ($r(\mathbf{A}/\mathbf{C}) = n$), the set of all input data that define stable solvable/unbounded problems is dense everywhere in the set of input data $\mathbf{\Omega}$ (i.e., for any unstable problem there exists an infinitely small perturbation of input data which defines a stable problem).

Theorem 25. *For any input data* $(\mathbf{C}, \mathbf{A}, \mathbf{b})$ *of an unstable unbounded problem and* $\forall \varepsilon > 0\ \exists \tau < 0\ |\tau| < \varepsilon/\|\bar{\mathbf{U}}\|$:

$$\|\mathbf{C} - \mathbf{C}_\tau\| < \varepsilon, \quad \|\mathbf{A} - \mathbf{A}_\tau\| < \varepsilon; \tag{10}$$

$$\mathbf{C}_\tau = \mathbf{C} - \tau\bar{\mathbf{U}}, \quad \mathbf{A}_\tau = \mathbf{A} + \tau\bar{\mathbf{U}}, \quad \mathbf{b}_\tau = \mathbf{b}; \tag{11}$$

the problem $(\mathbf{C}_\tau, \mathbf{X}_\tau)$ *is stable unbounded.*

If $r(\mathbf{C}) = n$, *then* $\forall \varepsilon > 0\ \exists \tau > 0$, $\tau < \min\{1, \varepsilon/\|\bar{\mathbf{U}}\|\}$, *such that* $\mathbf{C}_\tau$, $\mathbf{A}_\tau$, $\mathbf{b}_\tau$ *are defined by* (11), *then the problem* $(\mathbf{C}_\tau, \mathbf{X}_\tau)$ *is stable solvable. If* $r(\mathbf{A}/\mathbf{C}) = n$ *and* $r(\mathbf{C}) \neq n$, *then* $\forall \varepsilon > 0\ \exists \tau > 0$, $\tau < \min\{1, \varepsilon/\|\mathbf{W}\|\}$, *such that* $\mathbf{A}_\tau/\mathbf{C}_\tau = (\mathbf{A}/\mathbf{C}) + \tau\mathbf{W}$, $\mathbf{W} = \mathbf{e}^{m+L}(\mathbf{e}^{m+L})^{\mathrm{T}}(\mathbf{A}/-\mathbf{C})$, $\mathbf{b}_\tau = \mathbf{b} + \tau\mathbf{e}^m$, *then the problem* $(\mathbf{C}_\tau, \mathbf{X}_\tau)$ *is also stable solvable. Note that in the latter case, we perturb the cone* $\mathbf{K}(\mathbf{C}/-\mathbf{A})$.

An analogous theorem can be proven for unstable solvable problem in case $r(\mathbf{A}/\mathbf{C}) = n$.

Theorem 26. *For the input data* $(\mathbf{C}, \mathbf{A}, \mathbf{b})$ *of an unstable solvable problem and*

$$\forall \varepsilon > 0\ \exists \tau < 0, \quad |\tau| < \varepsilon/\|\mathbf{W}\|$$

there exists stable unbounded problem with the input data defined as in Theorem 25. *Further*, *for the same problem*, *there exists* τ,

$$\min\{1, \varepsilon/\|\mathbf{W}\|, \varepsilon\} > \tau > 0,$$

such that problem $(\mathbf{C}_\tau, \mathbf{X}_\tau)$ *is stable solvable.*

The developments of this section can be used to formulate a method for analyzing a linear, multi-objective optimization problem to determine whether the problem is ill-formed and if so, to identify the source of ill-posedness (i.e., the input data is of poor quality or the optimization problem is poorly structured). The approach we propose is:

(1) formulate the linear, multi-objective optimization problem and determine the input data values,
(2) define the input perturbation using (9) and Theorem 25.
(3) perform the following tests:
   (a) test for solvability:
      (i) if for any arbitrarily chosen value of $\tau < 0$ a solution exists for the perturbed problem, then the original problem is solvable; next, determine whether the problem, as posed, is stable solvable by checking $r(\mathbf{A}/\mathbf{C})$ (see Theorem 25).
         (A) if $r(\mathbf{A}/\mathbf{C}) = n$ the problem is stable.
         (B) if $r(\mathbf{A}/\mathbf{C}) < n$ the problem may be unstable and the structure of optimization problem should be modified.
      (ii) otherwise, the original problem may be unbounded and a further test is required.
   (b) test for unboundedness:
      (i) if for any arbitrarily chosen value of $\tau \in [0, 1)$ a solution does not exist for the perturbed problem, then the original problem is stable unbounded and substantial changes to the input data of the original problem are required;
      (ii) otherwise, the original problem is unstable and minor changes are required in the input data of the original problem.

To illustrate the proposed regularization procedure, consider the following example.

Example 4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \leqslant \mathbf{b}\}$$

is unbounded and the solution of the problem

$$\max\{\mathbf{Cx}:\ \mathbf{Ax} \leqslant \mathbf{b}\}$$

does not exist. Note that this problem is also unstable. Since $r(\mathbf{A}/\mathbf{C}) = 2$ and perturbing the problem with $\mathbf{u} = 0.5\mathbf{c}_1 + 0.5\mathbf{c}_2 = (0.5, 0.5)$ and $\tau = -10^{-2}$,

$$\mathbf{A}_\tau = \begin{bmatrix} 0.95 & -0.05 \\ -1.05 & -0.05 \\ -0.95 & -0.95 \end{bmatrix}, \quad \mathbf{C}_\tau = \begin{bmatrix} 1.05 & 0.05 \\ 0.05 & 1.05 \end{bmatrix},$$

we find that no solution exists for the perturbed problem. Then, using $\tau = 10^{-2}$,

$$\mathbf{A}_\tau = \begin{bmatrix} 1.05 & 0.05 \\ -0.95 & 0.05 \\ -0.95 & -0.95 \end{bmatrix}, \quad \mathbf{C}_\tau = \begin{bmatrix} 0.95 & -0.05 \\ -0.05 & 0.95 \end{bmatrix},$$

we find that the the feasible domain becomes bounded and the perturbed problem becomes stable solvable. Then, it is clear for the original problem, that small changes to the input data will render it stable solvable.

6. Conclusions

We have developed a comprehensive theory of perturbed cones, required for developing a stability theory for mixed-integer, multiple criteria optimization problems. Basic and general perturbations have been treated separately, since additional properties can be proven for basic perturbations, including: principles of complementary inclusion and monotonicity. Further, it has been shown that Pareto set of the perturbed problem covers only robustly efficient solutions of the original problem and under the assumption $\tau \in [0, 1)$, the theory covers the Pareto set of the original problem. Although in the latter case, the Pareto set of the perturbed problem may contain inefficient solutions of the original problem. These two properties do not necessarily hold in general case. As examples of application of basic perturbations: necessary and sufficient conditions for stable solvability (unboundedness) of linear vector optimization problems have been proven; density of the the input data of stable solvable/unsolvable problems has been discussed; and a regularization approach has been developed for analyzing unstable unbounded problems.

Appendix A

Recall some basic theorems from [27] used in the proof of Theorem 19.

Theorem A.1. *If* $\mathbf{S}$ *is nonempty convex set, then* $\mathbf{O}^+\mathbf{S}$ *is convex cone, which contains* $\mathbf{0}$ *element, and*

$$\mathbf{O}^{+}\mathbf{S} = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{S} + \mathbf{v} \subseteq \mathbf{S}\}.$$

Theorem A.2. *Let* $\mathbf{S}$ *be nonempty closed convex set and* $\mathbf{v} \neq \mathbf{0}$. *If there exists at least one element* $\mathbf{x}$ *such that* $\{\mathbf{x} + \lambda\mathbf{v} : \forall\lambda \geqslant \mathbf{0}\} \subseteq \mathbf{S}$, *then* $\mathbf{v} \in \mathbf{O}^{+}\mathbf{S}$.

Theorem A.3. *If* $\mathbb{A}$ *is a linear mapping from* $\mathbb{R}^n$ *into* $\mathbb{R}^m$ *and* $\mathbf{D}$ *is closed convex set from* $\mathbb{R}^m$, *such that* $\mathbb{A}^{-1}\mathbf{D} \neq \emptyset$, *then*

$$\mathbf{O}^{+}\mathbb{A}^{-1}(\mathbf{D}) = \mathbb{A}^{-1}(\mathbf{O}^{+}\mathbf{D}).$$

Theorem A.4. *Nonempty closed convex set* $\mathbf{S}$ *is bounded if and only if* $\mathbf{O}^{+}\mathbf{S} = \{\mathbf{0}\}$.

The following Lemma A.1 [1] is used in the proof of Theorem 20.

Lemma A.1. *If cone* $\mathbf{S} = \operatorname{con}(\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_r)$ *and* $\mathbf{S} = \mathbb{R}^n$, *then there exists* $\delta_0 > 0$, *such that for every* δ, $0 < \delta \leqslant \delta_0$, *the perturbed cone* $\mathbf{S}(\delta) = \operatorname{con}(\mathbf{g}_1(\delta),\ \mathbf{g}_2(\delta), \ldots, \mathbf{g}_r(\delta)) = \mathbb{R}^n$, *where* $\mathbf{g}_i(\delta) : \|\mathbf{g}_i(\delta) - \mathbf{g}_i\| < \delta$.

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