The Possible Inertias for a Hermitian Matrix and Its Principal Submatrices*

Jerome Dancis
Department of Mathematics
University of Maryland
College Park, Maryland 20742

Submitted by Chandler Davis

ABSTRACT

Definition: A Hermitian matrix $H$ is a Hermitian extension of a given set of Hermitian matrices $\{H_{ii}, i = 1, \ldots, m\}$ if these $\{H_{ii}\}$ are the block diagonals of $H$. Let $(\pi, \nu, \delta_i) = \text{In} H_{ii}$, the inertia of each $H_{ii}$. Special case: Given Hermitian matrices $\{H_{ii}, i = 1, \ldots, m\}$ and given nonnegative integers $\pi$, $\nu$, and $\delta$ such that

$$\pi + \nu + \delta = \sum (\pi_i + \nu_i + \delta_i),$$

then a Hermitian extension $H$ exists such that

$$\text{Ker} H \supset \oplus \text{Ker} H_{ii} \quad \text{and} \quad \text{In} H = (\pi, \nu, \delta)$$

if and only if

$$\delta \geq \sum \delta_i \quad \text{and} \quad \pi \geq \max \pi_i \quad \text{and} \quad \nu \geq \max \nu_i.$$

We also present a simple extension theorem for the general case ($\text{Ker} H \supset \oplus \text{Ker} H_{ii}$).

1. INTRODUCTION

The purpose of this paper is to construct Hermitian matrices with prescribed inertias and a prescribed block diagonal. A result is a list of the

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possible inertias of all the Hermitian matrices, with a given block diagonal, in terms of the inertias of the (given disjoint) blocks on the diagonal. This paper is a proof of Theorem 1.3, which appears after we state two extreme cases.

**General hypotheses.** Given positive integers \( m, n_1, n_2, \ldots, n_m, \ n = \sum n_i \), and \( \pi, \nu, \delta \) such that \( n = \pi + \nu + \delta \). Given \( n_i \times n_i \) Hermitian matrices \( H_{ii}, \ i = 1, 2, \ldots, m. \) Let \( (\pi_i, \nu_i, \delta_i) = \text{In} \ H_{ii} \) denote the inertia of each \( H_{ii} \), that is, the numbers of positive, negative, and zero eigenvalues of \( H_{ii} \).

**Definition.** An \( n \times n \) Hermitian matrix \( H \) is a Hermitian extension of the (given) \( \{ H_{ii} \} \) if \( H \) may be written as an \( m \times m \) block matrix \( H = (X_{ij}) \) such that each \( X_{ii} = H_{ii} \).

Let \( (\pi, \nu, \delta) = \text{In} \ H \).

**Corollary 1.1.** A Hermitian extension \( H \) of \( \{ H_{ii} \} \) exists such that

\[
\text{Ker} \ H \supseteq \bigoplus_{i=1}^{m} \text{Ker} \ H_{ii}
\]

if and only if

\[
\delta \geq \sum_{i=1}^{m} \delta_i, \quad \pi \geq \max \pi_i \quad \text{and} \quad \nu \geq \max \nu_i.
\]

**Definition.** We consider the domain of \( H \) as the direct sum \( C^n = \bigoplus C^{n_i} \), where \( C^{n_i} \) is the domain of \( H_{ii} \). We will associate each \( C^{n_i} \) with \( 0 \oplus C^{n_i} \oplus 0 \subset C^n \). The subspace \( V_i \subset \text{Ker} \ H_{ii} \) that is preserved by an extension \( H \) is

\[
V_i = \text{Ker} \ H|C^{n_i}.
\]

**Corollary 1.2.** A Hermitian extension \( H \) of \( \{ H_{ii} \} \) exists which preserves no subspaces of \( \{ \text{Ker} \ H_{ii} \} \), that is,

\[
\text{Ker} \ H|C^{n_i} = 0, \quad i = 1, 2, \ldots, m,
\]

if and only if

\[
\pi \geq \max(\pi_i + \delta_i) \quad \text{and} \quad \nu \geq \max(\nu_i + \delta_i) \quad \text{and} \quad \delta \geq 0.
\]
The inequalities for $\pi$, $\nu$, and $\delta$ in the general case (Theorem 1.3) will fall between the "extreme" inequalities of Corollaries 1.1 and 1.2.

**Extension Theorem 1.3.** Choose subspaces $V_i \subset \text{Ker } H_i$; let $d_i = \text{Dim } V_i$. A Hermitian extension $H$ of $\{H_i\}$ exists which preserves

$$V_i \subset \text{Ker } H_{ii}, \quad i = 1, 2, \ldots, m,$$

if and only if

$$\delta \geq \sum d_i, \quad \pi \geq \max(\pi_i + \delta_i - d_i), \quad \text{and} \quad \nu \geq \max(\nu_i + \delta_i - d_i).$$

**Lemma 1.4.** Let $H_{ii}$ be any principal submatrix of any Hermitian matrix $H$. Suppose that $\text{In } H = (\pi, \nu, \delta)$ and $\text{In } H_{ii} = (\pi_i, \nu_i, \delta_i)$. Let

$$d = \text{Dim Ker}(H|\text{domain of } H_{ii}).$$

Then

$$\pi \geq \pi_i + \delta_i - d \quad \text{and} \quad \nu \geq \nu_i + \delta_i - d.$$ 

**Remark.** Lemma 1.4 is the main result of our paper [4].

*Proof of the "only if" part.* That the subspaces $\{V_i\}$ are preserved in the kernel of $H$ implies that the direct sum

$$\bigoplus V_i \subset \text{Ker } H.$$

Therefore,

$$\delta = \text{Dim Ker } H \geq \sum \text{Dim } V_i = \sum d_i.$$ 

The other inequalities

$$\pi \geq \max(\pi_i + \delta_i - d_i) \quad \text{and} \quad \nu \geq \max(\nu_i + \delta_i - d_i)$$

were established by Lemma 1.4.
Remark. The "only if" part of Corollary 1.2 is a corollary of Lemma 1.4. The "only if" part of Corollary 1.1 is a simple consequence of Cauchy's interlacing inequalities.

Notation. A vector or a matrix diagonal $v \in \{ -1,0,1 \}^n$ shall mean that $v \in \mathbb{R}^n$ and the entries of $v$ all come from the set $\{ -1,0,1 \}$.

Existence Theorem 1.5. Given positive integers $m$, $n_1$, $n_2$, ..., $n_m$, and $n = \sum_{i=1}^{m} n_i$; given diagonal matrices $D_1$, $D_2$, ..., $D_m$, such that the main diagonal of each $D_i$ is in $\{ -1,0,1 \}^n$. Let $(\pi_i, \nu_i, \delta_i) = \text{In } D_i$. Given non-negative integers $\pi$, $\nu$, and $\delta$ such that

$$\pi + \nu + \delta = n,$$

$$\pi \geq \pi_i + \delta_i \text{ and } \nu \geq \nu_i + \delta_i, \quad i = 1,2,\ldots,m.$$  

Then there is a real symmetric matrix $M$ which is a Hermitian extension of the $\{ D_i \}$ such that

(i) $\text{In } M = (\pi, \nu, \delta)$;

(ii) when we consider $\mathbb{R}^n$ as the direct sum $\mathbb{R}^n = \oplus \mathbb{R}^{n_i}$, then $\text{Ker } M|\mathbb{R}^n = 0$.

Remark. This theorem is a special case of Corollary 1.2.

This paper was inspired by the related results of Bryan Cain and E. Marques de Sá in [2] and [3]. Their main result is Theorem 2.1 of [3], which we will quote as Theorem 11.4.

Professor Cain's proof starts with the case $m = 2$, that is, only two Hermitian blocks $H_{11}$ and $H_{22}$ are given. This two-block case is the main result of Cain and Marques de Sá's paper [2]. This paper [2] uses results on the Schur complement and results from Marques de Sá's paper [1].

It is possible that our Corollary 1.1 may also be a consequence of their work (Theorem 11.4.)

These papers, [1], [2], [3] are not prerequisites for reading this paper.

Corollary 1.1 with $\delta = 0$ is a special case of Corollary 6 of [5].

Contents. In Sections 2, 3, 4, and 5 we will establish many special cases of the Existence Theorem 1.5 when all the $D_i$ are just $1 \times 1$ matrices. The results of Sections 2, 3, 4, and 5 are summarized as Lemma 1.7. In Sections 6, 7, and 8 we will use all these special cases in order to establish the Existence Theorem 1.5. In Section 9, we will use the Existence Theorem 1.5.
in order to establish the "if" part of Corollary 1.2. In Section 10, we will use Corollary 1.2 in order to establish the "if" part of Extension Theorem 1.3. In Section 11, we will present a simple proof of Condition III, parts (3) and (4), of Bryan Cain's Theorem 11.4.

**Notation.** We will let $M(r, s, t; \pi, \nu, \delta)$ denote a real symmetric matrix whose main diagonal is $r$ ones, $s$ minus ones, and $t$ zeros and whose inertia is $(\pi, \nu, \delta)$ (these matrices will be listed in Sections 2, 3, 4 and 5). Also, these matrices will have no zero rows and will satisfy (when $rs > 0$) a special zero condition to be stated later.

**Remark 1.6.** This notation is consistent with some arithmetic, namely,

$$- M(r, s, t; \pi, \nu, \delta) = M(s, r, t; \nu, \pi, \delta), \quad (1.1)$$

and for the direct sum

$$M(r_1, s_1, t_1; \pi_1, \nu_1, \delta_1) \oplus M(r_2, s_2, t_2; \pi_2, \nu_2, \delta_2) = M(r_1 + r_2, s_1 + s_2, t_1 + t_2; \pi_1 + \pi_2, \nu_1 + \nu_2, \delta_1 + \delta_2). \quad (1.2)$$

**Notation.** $0^{m \times n}$, $1^{m \times n}$, and $2^{m \times n}$ will denote $m \times n$ submatrices in which all the entries are 0, 1, and 2, respectively. In particular, $1^{m \times m}$ will be written $I_m$.

**Lemma 1.7.** The matrices $M(r, s, t; \pi, \nu, \delta)$ exist whenever

$$r + s + t = \pi + \nu + \delta$$

and

(a) when $t \geq 1$ and $r + s + t \geq 2$, then $\pi \geq 1$ and $\nu \geq 1$;
(b) when $r \geq 1$, then $\pi \geq 1$;
(c) when $s \geq 1$, then $\nu \geq 1$.

**Remark.** This lemma is most of the special cases of the Existence Theorem 1.5 when all the diagonal submatrices $D_1, D_2, \ldots, D_m$ are $1 \times 1$ matrices. The matrices $M(r, s, t; \pi, \nu, \delta)$ of Lemma 1.7 will be listed in the proofs of Lemmas 2.1, 3.1, 4.1, and 5.1.
We now present two sublemmas which we will use in the next four sections.

**Sublemma 1.8.** Let \( A \) be any real symmetric \((n-1)\times(n-1)\) matrix, and let \( Av_i = \lambda_i v_i \) for \( i = 1, 2, \ldots, n - 1 \), where \( \{v_i\} \) is a basis of (real) orthogonal eigenvectors. Suppose \( \lambda_1 \neq 0 \). Suppose \( v_i^Tv_i = 2\lambda_1^2 \). Let \( M_1 \) be the \( n \times n \) matrix

\[
M_1 = \begin{pmatrix} A & v_1 \\ v_1^T & 0 \end{pmatrix}.
\]

Then the \( n \) eigenvalues of \( M_1 \) are

\[
2\lambda_1, \ -\lambda_1, \ \text{and} \ \lambda_2, \lambda_3, \ldots, \lambda_{n-1}.
\]

This is elementary.

**Corollary 1.9.** When \( \lambda_1 > 0 \), \( \text{Im} \ M_1 = \text{Im} \ A + (0, 0, 0, \ldots, 0) \).

**Sublemma 1.10.** Let \( A \) and \( C \) be \( r \times r \) and \( t \times t \) real symmetric matrices, and let

\[
Av_i = \lambda_i v_i \quad \text{for} \quad i = 1, 2, \ldots, r \quad \text{and} \quad Cw_j = \mu_j w_j \quad \text{for} \quad j = 1, 2, \ldots, t,
\]

where \( \{v_i\} \) and \( \{w_j\} \) are orthogonal eigenbases. Suppose \( \lambda_1, \mu_1 > 0 \), and suppose that

\[
v_i^Tv_i = \lambda_1 \quad \text{and} \quad w_1^T w_1 = \mu_1.
\]

Let \( M_2, M_3 \) be the \((r+t)\times(r+t)\) symmetric matrices with these \(2\times2\) block forms:

\[
M_2 = \begin{pmatrix} A & v_1w_1^T \\ w_1v_1^T & C \end{pmatrix}, \quad M_3 = \begin{pmatrix} A & 2v_1w_1^T \\ 2w_1v_1^T & C \end{pmatrix}.
\]

Then the \( r + t \) eigenvalues of \( M_2 \) are

\[
\lambda_1 + \mu_1, \ 0, \ \text{and} \ \lambda_2, \lambda_3, \ldots, \lambda_r \ \text{and} \ \mu_2, \mu_3, \ldots, \mu_t;
\]
$M_3$ has a positive and a negative eigenvalue in addition to $\lambda_2, \ldots, \lambda_r$ and $\mu_2, \ldots, \mu_r$.

**Proof.** The eigenvectors of $M_3$ associated with the list of eigenvalues given will be these vectors in $R \oplus R'$:

\[
\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} \mu_1 v_1 \\ -\lambda_1 w_1 \end{pmatrix}, \text{ and } \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} v_r \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ w_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ w_r \end{pmatrix}.
\]

The proof for $M_3$ is similar; the two new eigenvalues are the roots of $x^2 - (\lambda_1 + \mu_1)x = 3\lambda_1 \mu_1$.

**Remark.** The statement of Sublemma 1.10 was inspired by some of the ideas in Cain and Marques de Sá's paper [2].

**Corollary 1.11.** When $\lambda_1 > 0$ and $\mu_1 > 0$,

\[
\begin{align*}
\text{In } M_2 &= \text{In } A + \text{In } C + (-1, 0, 1), \\
\text{In } M_3 &= \text{In } A + \text{In } C + (-1, 1, 0).
\end{align*}
\]

2. MATRICES WITH ONLY ZEROS ON THE MAIN DIAGONAL

The purpose of this section is to establish Lemma 2.1.

**Lemma 2.1.** Given integers $\pi \geq 1$, $\nu \geq 1$, and $\delta \geq 0$ and

\[ t = \pi + \nu + \delta \geq 2, \]

there is a real symmetric matrix $M(0,0,t; \pi, \nu, \delta)$ such that

(a) its inertia is $(\pi, \nu, \delta)$,

(b) all the main diagonal elements are zero, and

(c) it does not have a zero row.

**Proof.** Let $\Gamma_m$, $m \geq 2$, denote the $m \times m$ matrix

\[
\Gamma_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \end{pmatrix}.
\]
It has zero main diagonal and has no zero row or column. Evidently its characteristic polynomial is \( \lambda^n - (m - 1)\lambda^{m - 2} \), whence

\[
\text{In } \Gamma_m = (1, 1, m - 2).
\]

Starting with \( \Gamma_{g-2} \) as \( M(0, 0, \delta + 2; 1, 1, \delta) \), we may use Corollary 1.9 successively \( \pi + \nu - 2 \) times in order to construct the matrices \( M(0, 0, t; \pi, \nu, \delta) \). For example, when \( j \geq 1 \) and \( k \geq 2 \), if we have

\[
A = M(0, 0, j - 1 + k + \delta; j, k - 1, \delta),
\]

and \( v_1 \) is an eigenvector associated with a positive eigenvalue \( \lambda_1 \) of \( A \) such that \( v_1^Tv_1 = 2\lambda_1^2 \), then the matrix \( M_1 \) supplied by Corollary 1.9 may be used as

\[
M(0, 0, j + k + \delta; j, k, \delta).
\]

In this inductive manner, Lemma 2.1 is established.

3. REAL SYMMETRIC MATRICES WITH ONLY ONES ON THE MAIN DIAGONAL

The purpose of this section is to present a direct proof of Lemma 3.1. Probably, Lemma 3.1 is also a consequence of Professor Cain's result ([3]).

**Lemma 3.1.** Given integers \( n \geq 1 \), \( \pi \geq 1 \), \( \nu \geq 0 \), and \( \delta \geq 0 \) such that \( n = \pi + \nu + \delta \), there is a real symmetric matrix \( M(n, 0, 0; \pi, \nu, \delta) \) with inertia \((\pi, \nu, \delta)\) and with \( n \) ones as the main diagonal.

**Proof.** For the \((\nu + 1) \times (\nu + 1)\) matrix

\[
2J_{\nu+1} - I_{\nu+1} = \begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & \\
2 & & 1
\end{pmatrix},
\]

the eigenvalues are \( \{2\nu + 1, -1, -1, \ldots, -1\} \) and therefore \( \text{In}(2J_{\nu+1} - I_{\nu+1}) = \).
\( I_{\nu+1} = (1, \nu, 0) \). We set

\[ M(n, 0, 0; 1, \nu, 0) = 2J_{\nu+1} - I_{\nu+1}. \quad (3.1) \]

Since inertias are summed over direct sums, we may set

\[ M(n, 0, 0; \pi, \nu, \delta) = \frac{1}{2} (2J_{\nu+1} - I_{\nu+1}) \oplus J_{\delta+1} \quad (3.2) \]

when \( \nu \geq 2 \). We can set

\[ M(n, 0, 0; \pi, 0, \delta) = I_{\pi} \oplus J_{\delta+1} \quad (3.3) \]

when \( \nu = 0 \) and \( \nu \geq 1 \).

The remaining case is \( \pi = 1 \) and \( \nu \geq 1 \), \( \delta \geq 1 \). For this case we will use Sublemma 1.10 with

\[ A = 2J_{\nu+1} - I_{\nu+1} \quad \text{and} \quad C = J_{\delta} \]

with \( A_{1} = 2\nu + 1 > 0 \) and \( \mu_{1} = \delta > 0 \). Then, for the matrix \( M_{2} \) of Sublemma 1.10, Corollary 1.11 says that

\[ \text{In } M_{2} = (1, \nu, 0) + (1, 0, \delta - 1) + (-1, 0, 1) \]

\[ = (1, \nu, \delta). \]

Therefore we may set

\[ M(n, 0, 0; 1, \nu, \delta) = M_{2}. \quad (3.4) \]

Lemma 3.1 is established by (3.1), (3.2), (3.3), and (3.4).

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4. MATRICES WITH ONLY ONES AND ZEROS ON THE MAIN DIAGONAL

The purpose of this section is to establish Lemma 4.1 using Lemmas 2.1 and 3.1.
Lemma 4.1. Given integers \( r \geq 1 \) and \( t \geq 1 \) and integers \( \pi \geq 1, \nu \geq 1, \) and \( \delta \geq 0 \) such that

\[
r + t = \pi + \nu + \delta,
\]

then there is a real symmetric matrix \( M(r,0,t;\pi,\nu,\delta) \) with no zero rows whose main diagonal consists of \( r \) ones and \( t \) zeros and whose inertia is \( (\pi, \nu, \delta) \).

We will establish this lemma case by case.

Proof for the case \( \pi \geq 2 \) and \( \nu \geq 1 \) and \( t \geq 2 \). We observe, by simply adding two matrices provided by Lemmas 2.1 and 3.1, that we may set

\[
M(r,0,t;\pi,\nu,\delta) = M(r,0,0;\pi_1,\nu_1,\delta_1) \oplus M(0,0,t;\pi_3,\nu_3,\delta_3),
\]

where \( \pi_1, \nu_1, \delta_1, \pi_3, \nu_3, \) and \( \delta_3 \) are chosen (nonuniquely) such that:

\[
\pi_1 + \nu_1 + \delta_1 = r \quad \text{and} \quad \pi_3 + \nu_3 + \delta_3 = t;
\]

\[
\pi = \pi_1 + \pi_3, \quad \nu = \nu_1 + \nu_3, \quad \text{and} \quad \delta = \delta_1 + \delta_3;
\]

\[
\pi_1 \geq 1, \quad \pi_3 \geq 1, \quad \text{and} \quad \nu_3 \geq 1.
\]

The last two lines require that \( \pi \geq 2 \) and \( \nu \geq 1 \), and Lemma 2.1 requires that \( t \geq 2 \).

Proof for the case \( \pi = 1 = \nu \) (and \( t \geq 1 \)). We will show that the desired matrix can have this \( 2 \times 2 \) block form:

\[
M(r,0,t;1,1,n-2) = \begin{pmatrix}
J_r & 1^{r \times t} \\
1^{t \times r} & 0^{t \times t}
\end{pmatrix},
\]

where \( 1^{r \times t} \) is an \( r \times t \) block consisting only of ones. Clearly the rank of this matrix is 2, and hence it has zero as an eigenvalue of order \( n - 2 \). But it is plainly not semidefinite. Therefore its inertia is \( (1,1,n-2) \) as desired.

The case \( \pi = 1 \) and \( \nu \geq 2 \) and the case \( t = 1 \) are the most difficult. For these cases we need Sublemmas 1.8 and 1.10.

Proof for the case \( t = 1 \). Let \( A \) be the \( (n-1) \times (n-1) \) matrix provided by Lemma 3.1:

\[
A = M(r,0,0;\pi,\nu-1,\delta) \quad \text{for} \quad \pi \geq 1 \quad \text{and} \quad \nu-1 \geq 0.
\]
Since this $A$ has a positive eigenvalue, we may assume that $\lambda_1 > 0$. Then Sublemma 1.8 for this matrix $A$ will provide (by Corollary 1.9) a matrix $M_1$ of the type

$$M_1 = M(r, 0, 1; \pi, \nu, \delta).$$

**Proof for the case** $\pi = 1$, $\nu \geq 2$ and $t \geq 2$. Let $A$ and $C$ be $r \times r$ and $t \times t$ matrices provided by Lemmas 2.1 and 3.1,

$$A = M(r, 0, 0; 1, \nu_1, \delta_1), \quad r = 1 + \nu_1 + \delta_1,$$

$$C = M(0, 0, t; 1, \nu_0, \delta_0), \quad 2 \leq t = 1 + \nu_0 + \delta_0 \text{ and } \nu_0 \geq 1,$$

when $\nu_1$, $\delta_1$, $\nu_0$, and $\delta_0$ satisfy the two equations

$$\nu = \nu_1 + \nu_0 + 1 \quad \text{and} \quad \delta = \delta_1 + \delta_0. \quad (4.1)$$

Since each of $A$ and $C$ has a positive eigenvalue, we may assume that $\lambda_1$ and $\mu_1$ are these two positive eigenvalues.

Then, for these matrices $A$ and $C$, Sublemma 1.10 provides on $(r + t) \times (r + t)$ matrix $M_3$. Furthermore Corollary 1.11 says that

$$\ln M_3 = \ln A + \ln C + \left( -1, 1, 0 \right) = (1, \nu_1 + \nu_0 + 1, \delta_1 + \delta_0).$$

Thus, using (4.1),

$$\ln M_3 = (1, \nu, \delta)$$

Therefore, since the diagonals of $A$ and $C$ are just $r$ ones and $t$ zeros respectively, we may set

$$M(r, 0, t; 1, \nu, \delta) = M_3.$$

We have listed the matrices $M(r, 0, t; \pi, \nu, \delta)$ for all the possible cases of Lemma 4.1. Therefore, Lemma 4.1 is established.

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5. **MATRICES WITH ONLY ONES, ZEROS, AND MINUS ONES ON THE MAIN DIAGONAL**

The purpose of this section is to use Lemmas 2.1, 3.1, and 4.1 in order to establish Lemma 5.1.
We shall use the following technical condition:

**Definition.** A (symmetric) matrix $M = \{ m_{ij} \}$ satisfies the *special zero conditions* if

(i) $M$ has no zero rows, and
(ii) whenever $m_{ii} = 1$ and $m_{jj} = -1$, then $m_{ij} = 0$.

**Lemma 5.1.** Given integers $r \geq 1$, $s \geq 1$, $t \geq 1$ and integers $\pi \geq 1$, $\nu \geq 1$, and $\delta \geq 0$ such that

$$r + s + t = \pi + \nu + \delta.$$ 

There is a real symmetric matrix $M(r, s, t; \pi, \nu, \delta)$ which satisfies the special zero conditions.

We will establish Lemma 5.1 in a case-by-case manner.

**Proof for the case $\pi \geq 2$, $\nu \geq 2$, and $t \geq 2$.** We simply use the many possible combinations of matrices provided by Lemmas 2.1 and 3.1. The desired matrices will be

$$M(r, 0, 0; \pi_1, \nu_1, \delta_1) \oplus [ - M(s, 0, 0; \nu_2, \pi_2, \delta_2) ] \oplus M(0, 0, \pi_3, \nu_3, \delta_3)$$

when

$$\pi_1 + \nu_1 + \delta_1 = r, \quad \pi_2 + \nu_2 + \delta_2 = s, \quad \text{and} \quad \pi_3 + \nu_3 + \delta_3 = t,$$

$$\sum \pi_i = \pi, \quad \sum \nu_i = \nu, \quad \text{and} \quad \sum \delta_i = \delta.$$ 

The other conditions that are required by Lemmas 2.1 and 3.1 are that

$$\pi_1 \geq 1, \quad \nu_2 \geq 1, \quad \pi_3 \geq 1, \quad \text{and} \quad \nu_3 \geq 1.$$ 

These inequalities only require that

$$\pi = \sum \pi_i \geq 2 \quad \text{and} \quad \nu = \sum \nu_i \geq 2.$$ 

The special zero conditions are trivially satisfied.
**Proof for the case** $\pi = 1 = v$ and $t \geq 1$. The desired matrix has this $3 \times 3$ block form:

\[
M(r, s, t; 1, 1, n - 2) = \begin{pmatrix}
J_r & 0^{r \times s} & 1^{r \times t} \\
0^{s \times r} & -J_s & 1^{s \times t} \\
1^{r \times s} & -1^{s \times s} & 0^{1 \times t}
\end{pmatrix},
\]

with $r + s + t = n$.

**Claim.** The rank of this $n \times n$ matrix is just 2.

**Proof.** In each block column, all the columns are identical. Thus, the rank is $\leq 3$. We note that the columns in the third block column are simply the sum of two other columns, one from the middle block column and one from the first block column. Thus the rank must be 2.

**Proof of the case** $\pi = 1 = v$ and $t \geq 1$, continued. Since $t \geq 1$, this matrix is plainly not semidefinite. This together with having rank 2 forces the inertia to be $(1, 1, n - 2)$ as desired. Clearly there are no zero rows, and the special zero conditions are satisfied.

**Proof for the case** $\pi = 1$, $v \geq s + 1$, and $t \geq 1$. Here the desired matrix is simply a direct sum of (minus) an $s \times s$ identity matrix and a matrix from Section 4 for the case $\pi = 1$ therein:

\[
M(r, s, t; 1, v, \delta) = [-I_v] \oplus M(r, 0, t; 1, v - s, \delta),
\]

when

\[
v \geq s + 1 \quad \text{and} \quad r + t = 1 + v - s + \delta.
\]

**Proof for the case** $\pi = 1$, $1 \leq v \leq s$, and $t \geq 1$. Here the desired matrix is a direct sum of (minus) an identity matrix and the matrix for the case $\pi = 1 = v$:

\[
M(r, s, t; 1, v, \delta) = [-I_v] \oplus M(r, s - v + 1, t; 1, 1, \delta)
\]

when

\[
v \leq s \quad \text{and} \quad r + s + t = 1 + v + \delta.
\]

Since each summand satisfies the special zero condition, this matrix also satisfies it.
Remark. The last two cases together are the case for $\pi = 1$, $\nu \geq 1$, and $\tau \geq 1$. The negatives of these matrices [as noted in (1.1)] give the case for $\nu = 1$, $\pi > 1$, and $\tau > 1$. The case for $\pi > 1$, $\nu > 1$, and $\tau > 1$ has already been presented.

Thus all the cases for $\tau \geq 2$ have been presented. All that remains is the case for $\tau = 1$, $\pi > 1$, and $\nu > 1$.

Proof for the case $t = 1$, $\pi > 1$, and $\nu > 1$. The matrices for this case are simply direct sums of matrices provided by Lemma 3.1 and the case $t = 1$ of Lemma 4.1; namely, we set

$$M(r, s, 1; \pi, \nu, \delta) = M(r, 0, 1; \pi_1, \nu_1, \delta_1) \oplus -M(s, 0, 0; \nu_2, \pi_2, \delta_2),$$

when $r \geq 1$, $s \geq 1$, $\pi > 1$, and $\nu > 1$; when $\pi_1$, $\nu_1$, $\delta_1$, $\pi_2$, $\nu_2$, and $\delta_2$ are chosen (nonuniquely) so that $\pi_1 \geq 1$, $\nu_1 \geq 1$, and $\nu_2 \geq 1$; and when

$$\pi = \pi_1 + \pi_2, \quad \nu = \nu_1 + \nu_2, \quad \text{and} \quad \delta = \delta_1 + \delta_2;$$

since the $+1$'s and the $-1$'s on the main diagonal are in different blocks, the special zero condition is preserved. Having listed all possible cases, we have established Lemma 5.1.

6. An Extension of Lemma 1.7

The purpose of this section is to establish a simple extension of Lemma 1.7, namely Corollary 6.1.

Proof of Lemma 1.7. All the matrices needed for Lemma 1.7 either have been listed in the proofs of Lemmas 2.1, 3.1, 4.1, and 5.1 or may be obtained from those listed by using Equation (1.1). This establishes Lemma 1.7.

The next result is simply a direct sum of the many matrices provided by Lemmas 2.1, 3.1, 4.1, and 5.1.

Corollary 6.1. Given nonnegative integers $p, q, r$ ($p + q + r > 0$) and given positive integers

$$m_1, m_2, \ldots, m_{p + q + r} \quad \text{and} \quad n = \sum_{j=1}^{p + q + r} m_j.$$
Suppose \( m_p, q, r, \ldots, m_p \cdot q \cdot r \geq 2 \). Set:

(i) \( \nu_i = 1_{m_j} = (1, 1, \ldots, 1) \in \mathbb{R}^{m_j}, \ j = 1, 2, \ldots, p; \)

(ii) \( \nu_i = -1_{m_j} = (-1, -1, \ldots, -1) \in \mathbb{R}^{m_j}, \ j = p + 1, \ldots, p + q; \)

and let

(iii) \( u_i \in \{-1, 0, 1\}^{m_j}, \ j = p + q + 1, \ldots, p + q + r, \)

such that each \( u_i \) has at least one zero entry. Let \( \pi, \nu, \delta \) be any nonnegative integers such that \( \pi > \nu + \delta \) and

\[
\pi > p + r \quad \text{and} \quad \nu > q + r.
\]

Then there are real symmetric \( m_i \times m_i \) matrices, \( S_1, S_2, \ldots, S_{p \cdot q \cdot r}, \) where the diagonal of each \( S_j \) is \( \nu_i, \) \( \nu_i, \) or \( u_i \) as the case may be, such that their direct sum

\[
S = \bigoplus_{j=1}^{p \cdot q \cdot r} S_j
\]

has inertia \( \text{In} \ S = (\pi, \nu, \delta), \) and \( S \) satisfies the special zero conditions.

**Proof.** For \( j = 1, 2, \ldots, p, \) the \( S_j \) will be provided by Lemma 3.1 and therefore

\[
S_j = M(\nu_i, 0, 0; \nu_i, \pi_i, \delta_i) \quad \text{and} \quad \text{In} \ S_j \geq (1, 0, 0). \tag{6.2}
\]

For \( j = p + 1, \ldots, p + q, \) the \( S_j \) will also be provided by Lemma 3.1; they are

\[
S_j = -M(\nu_i, 0, 0; \nu_i, \pi_i, \delta_i) \quad \text{and} \quad \text{In} \ S_j \geq (0, 1, 0). \tag{6.3}
\]

For \( j = p + q + 1, \ldots, p + q + r, \) the \( S_j \) will be provided by Lemmas 2.1, 4.1, and 5.1: let \( r_j, s_j, \) and \( t_j \) be the numbers of ones, minus ones, and zeros in \( u_j; \) then

\[
S_j = M(r_j, s_j, t_j; \nu_i, \pi_i, \delta_i) \quad \text{and} \quad \text{In} \ S_j \geq (1, 1, 0). \tag{6.4}
\]
Since \( S = \Theta S \), the inequalities for the inertias, (6.2), (6.3), and (6.4), imply that

\[
\sum_{j=1}^{p+q+r} \text{In} S_j \geq \sum_{j=1}^{p} (1,0,0) + \sum_{j=p+1}^{q} (0,1,0) + \sum_{j=p+q+1}^{p+q+r} (1,1,0)
\]

\[
= (p + r, q + r, 0).
\]

Therefore

\[
\text{In} S = \sum \text{In} S_j \geq (p + r, q + r, 0).
\]

This inequality is the reason for (6.1).

We need only choose the \( \pi_j \)'s, \( \nu_j \)'s, and \( \delta_j \)'s; then the \( S_j \)'s will be specified by Lemmas 2.1, 3.1, 4.1, and 5.1. But the \( \pi_j \)'s, \( \nu_j \)'s, and \( \delta_j \)'s may be chosen almost freely; they must satisfy only these few conditions:

\[
\pi_j + \nu_j + \delta_j = m_j, \quad j = 1, 2, \ldots, p + q + r,
\]

\[
\sum \pi_j = \pi, \quad \sum \nu_j = \nu, \quad \text{and} \quad \sum \delta_j = \delta,
\]

\[
\pi_j \geq 1 \quad \text{when} \quad j = 1, 2, \ldots, p \quad \text{and} \quad p + q + 1, \ldots, p + q + r,
\]

\[
\nu_j \geq 1 \quad \text{when} \quad j = p + 1, p + 2, \ldots, p + q + r.
\]

There is no problem choosing the \( \pi_j \)'s, \( \nu_j \)'s, and \( \delta_j \)'s since \( \pi, \nu, \) and \( \delta \) must satisfy the conditions

\[
\pi + \nu + \delta = n = \sum m_j, \quad \pi \geq p + r, \quad \text{and} \quad \nu \geq q + r.
\]

In this way the \( S_j \)'s and hence \( S \) are constructed, and hence Corollary 6.1 is established.

7. A COMBINATORIAL RESULT

The purpose of this section is to establish Theorem 7.1, a combinatorial result. This result is the key to my proof of the existence and extension theorems. (Theorem 7.1 will provide the permutation stated by the hypotheses of Lemma 8.4.)
Theorem 7.1. Suppose we have $m$ decks of cards (decks $1, 2, \ldots, m$). The cards are only aces, kings, and blanks. Each deck has its own numbers of these three types of cards.

Let $r$ be the maximum number of blank cards in any single deck.

Let $p + r$ be the maximum number of aces and blank cards in any single deck.

Let $q + r$ be the maximum number of kings and blank cards in any single deck.

Suppose that the total number of cards in all the decks together is \( p + q + 2r \).

Suppose we have $p$ red boxes, $q$ white boxes, and $r$ blue boxes. Then it is possible to deal all the cards from all the decks into these $p + q + r$ boxes in such a way that:

(a) from each deck is dealt only:
   (i) an ace or $\emptyset$ (no card) to each red box,
   (ii) a king or $\emptyset$ to each white box,
   (iii) a blank card or an ace-king pair or an ace or a king or $\emptyset$ to each blue box;

(b) no box remains empty;

(c) each blue box receives at least two cards including a blank card.

Proof. We shall present a method for dealing out all the cards. Without loss of generality, suppose that deck 1 has $r$ blank cards (the maximum number of blank cards).

We start by dealing from deck 1 a single blank card to each blue box.

Now we deal from deck 2 to the blue boxes, according to condition (a) (iii). We deal all the blank cards, then as many ace-king pairs as possible, and afterwards (if needed) the aces or kings (whichever remain) until each blue box has received a single card or an ace-king pair, or until deck 2 is empty.

We now deal from each of the $m - 2$ other decks into the blue boxes in the same manner as for deck 2.

Observation 7.2. If there is a blue box with only one card, then all of decks $2, 3, \ldots, m$ have been completely dealt into (a proper subset of) the blue boxes.

Lemma 7.3. Let $\pi_i$, $v_i$, and $\delta_i$, $i = 2, 3, \ldots, m$, be the numbers of aces, kings, and blank cards in deck $i$ at the beginning. After dealing to the blue
boxes, in the manner stated, for deck \(i\).

\[
\text{number of blank cards remaining} = 0, \\
\text{number of aces} = \max \{ \pi_i + \delta_i - r, 0 \} \leq p, \\
\text{number of kings} = \max \{ \nu_i + \delta_i - r, 0 \} \leq q.
\]

**Proof.** The definitions of \(p\), \(q\), and \(r\) in Theorem 7.1 imply that

\[
\delta_i \leq r, \pi_i + \delta_i \leq p + r \text{ and } \nu_i + \delta_i \leq q + r, i = 1, 2, \ldots, m.
\]

These inequalities establish the (right-hand-side) inequalities in Lemma 7.3.

Since the blank cards were dealt first and since \(\delta_i \leq r\), all the blank cards have been dealt out. When \(\pi_i + \delta_i \leq r\), then all the aces and blank cards have been dealt to the blue boxes, and hence no aces remain in deck \(i\). When \(\pi_i + \delta_i > r\), exactly \(r\) aces and blank cards have been dealt. Therefore, exactly \(\pi_i + \delta_i - r\) aces remain in deck \(i\), as stated in Lemma 7.3.

Thus, Lemma 7.3 is established.

---

**Proof of Theorem 7.1, continued.** Let \(A\) and \(K\) be two (special) natural numbers such that, for decks \(A\) and \(K\),

\[
\pi_A + \delta_A = p + r \text{ and } \nu_K + \delta_K = q + r.
\]

Then lemma 7.3 implies that, after dealing to the blue boxes:

- deck \(A\) has exactly \(p\) aces (and maybe some kings), and
- deck \(K\) has exactly \(q\) kings.

Therefore, we can deal, from decks \(A\) and \(K\), an ace to each of the \(p\) red boxes and a king to each of the \(q\) white boxes. Thus condition (b) is satisfied.

Lemma 7.3 says that there remain, in each deck, \(\leq p\) aces and \(\leq q\) kings. Therefore, for each deck, we may deal of its aces and kings into the red and white boxes, respectively, at most one card to a box.

All that remains is condition (c). We observe that each blue box receives at least one card from each of decks \(1\), \(A\), and \(K\).

When decks \(1\), \(A\), and \(K\) are three or even two distinct decks, then each blue box has already received at least two cards and condition (c) is satisfied.

Let us consider the remaining case where decks \(1\), \(A\), and \(K\) are one and the same deck. If there is a blue box with only one card, Observation 7.2 implies that only the \(p\) aces and \(q\) kings from deck \(1 = \text{deck } A\) were dealt
into the red and white boxes. By hypothesis, there are at least $p + q + 2r$
cards. Therefore, there are $\geq 2r$ other cards, in the blue boxes, which
"should" be enough to get two cards into each blue box. We may remove
some of the "extra" cards from some of the blue boxes, being careful to leave
undisturbed the blank card received from deck 1 and any second card in each
box. Therefore, none of these extra cards started in deck 1. We may then
distribute a single extra card to each blue box which has only the blank card
(it received it from deck 1). In this manner the $\geq 2r$ cards in the blue boxes
may be rearranged to satisfy condition (c) without undoing condition (a), part
(iii).

Having distributed all the cards and having satisfied conditions (a), (b),
and (c), Theorem 7.1 is established.

8. PROOF OF THE EXISTENCE THEOREM

The purpose of this section is to establish the Existence Theorem 1.5. The
main tools will be Corollary 6.1, the combinatorial result (Theorem 7.1), and
a technical result, namely, Lemma 8.4.

We shall prepare for the proof of Lemma 8.4 by first stating a technical
condition and some sublemmas.

DEFINITION. A matrix $B$ satisfies the special matrix conditions if

(i) there is no zero column in $B$, and
(ii) each row of $B$ contains at most one nonzero element.

SUBLEMMA 8.1. If $B$ satisfies the special matrix conditions, then $\text{Ker } B = 0$.

Proof. Condition (ii) implies that no column of $B$ is contained in the
span of the other columns. This, together with condition (i), implies that the
columns of $B$ form a linearly independent set. Therefore $\text{Ker } B = 0$, and
Sublemma 8.1 is established.

 Observation 8.2. If $B$ is an $n \times m$ matrix which satisfies the special
matrix conditions and if $D_1$ and $D_2$ are two invertible $n \times n$ and $m \times m$
diagonal matrices, then the matrix $D_1 B D_2$ also satisfies the special matrix
conditions.
**Sublemma 8.3.** Let us consider real matrices with the block forms

\[
N = \begin{pmatrix} 0^d \times d & 0 \\ 0 & F \end{pmatrix}, \quad B_2 = \begin{pmatrix} B & B_1 \end{pmatrix},
\]

where \( B \) has \( d \) columns. Suppose that \( \ker F = 0 \) and \( \ker B = 0 \). Then

\[
\ker N \cap \ker B_2 = 0.
\]

This is easy.

**Lemma 8.4.** Given positive integers \( m, n_1, n_2, \ldots, n_m \), and \( n = \sum_{i=1}^{m} n_i \). Given vectors

\[
x_i \in \{-1, 0, 1\}^{n_i}, \quad i = 1, 2, \ldots, m.
\]

Let \( X \) be the concatenation of these \( m \) vectors, namely

\[
X = (X_1, X_2, \ldots, X_m).
\]

(8.1)

Let \( P \) be an \( n \times n \) permutation matrix and let \( Y = PX \). Suppose that this vector \( Y \) may be partitioned as a set of nonempty vectors

\[
Y = (v_1, v_2, \ldots, v_p, w_1, 1, \ldots, w_p, u_{p+1}, 1, \ldots, u_{p+q}, 1).
\]

(8.2)

which satisfy the hypothesis of Corollary 6.1. Suppose \( P \), considered as a function from the ordered set \( X = \{x_{11}, x_{22}, \ldots, x_{nn}\} \) (partitioned as in (8.1)) into the ordered set \( Y = \{y_{11}, y_{22}, \ldots, y_{nn}\} \) (partitioned as in (8.2)), is such that:

(i) from each \( X_i \) to any \( v_j \), \( P \) sends only \( \{1\} \) or \( \emptyset \);
(ii) from each \( X_i \) to any \( w_j \), \( P \) sends only \( \{-1\} \) or \( \emptyset \);
(iii) from each \( X_i \) to any \( u_j \), \( P \) sends only \( \{0\}, \{1, -1\}, \{1\}, \{-1\} \), or \( \emptyset \).

Let \( S \) be the real symmetric matrix supplied by Corollary 6.1. Let \( S = PSP^{-1} \). Then this matrix \( S \) has the block form \( S = \{N_{ij}\} \), where

(a) each \( N_{ij} \) is an \( m_i \times m_j \) matrix,
(b) each \( N_{ii} \) is a diagonal matrix and its main diagonal is the given vector \( X_i \).
Also

(c) \( \text{In } S_p = \text{In } S \);

(d) when we consider \( R^n \) as a direct sum, \( R^n = \bigoplus R^n_i \), then

\[
(\text{Ker } S_p) \cap (0 \times R^n_i \times 0) = 0. \quad i = 1, 2, \ldots, m.
\]

Remark. Given such \( X \) and \( Y \), the existence of such \( P \) is guaranteed by Theorem 7.1. The interpretation is as follows: ace, king, blank correspond to 1, \(-1,0\) respectively; deck \( i \) to vector \( X_i \); red, white, blue boxes to \( v_i \)'s, \( w_i \)'s, \( u_i \)'s respectively.

Proof. Assume the hypothesis of this lemma. Since \( Y = PX \) is the diagonal vector of \( S \) and since \( P \) is a permutation matrix, it is easily checked that the diagonal of \( S_p = PSP^{-1} \) is precisely \( X \). Therefore the diagonal of \( N_i \), will be the \( i \)th part of \( X \), namely \( X_i \).

We will now explain why each \( N_i \) is a diagonal matrix. Let \( S = \{ y_{jk} \} \) and \( S_p = \{ x_{jk} \} \). Let us consider two diagonal elements \( x_{jj} \) and \( x_{kk} \) in \( N_i \) \( (j \neq k) \). Suppose that the permutation matrix moves the \( j \)th and \( k \)th coordinates to the \( j \)'th and \( k \)'th coordinates. So \( x_{jj} \) and \( x_{kk} \) become \( y_{j'j} \) and \( y_{k'k} \) on the diagonal, \( Y \), of \( S \). In general, \( y_{j'j} \) and \( y_{k'k} \) may be in different vectors (of the \( v_i \)'s, \( w_i \)'s, and \( u_i \)'s), and therefore they will be in different \( S_{ij} \)'s. Since \( S = \bigoplus S_i \), this implies that \( y_{j'j'} = 0 \). Of course \( x_{jk} = y_{j'k'} \), and hence \( x_{jk} = 0 \) as desired.

It remains to consider the case that \( y_{j'j} \) and \( y_{k'k'} \) are in the same block. The restrictions placed on the permutation \( P \) permit this only when \( x_{jj} = 1 \) and \( x_{kk} = -1 \) (or vice versa). But then \( y_{j'j'} = 1 \) and \( y_{k'k'} = -1 \) also. Each \( S_i \) satisfies the special zero conditions, and hence for this situation \( y_{j'k'} = 0 \). Again \( x_{jk} = y_{j'k'} \), and hence \( x_{jk} = 0 \). Having shown that off-diagonal terms (of the \( N_i \)) are zero, we conclude that each \( N_i \) is a diagonal matrix. Thus (b) is verified.

Since \( P \) is a permutation matrix and \( S \) has no zero rows, \( S_p \) will also have no zero rows and \( \text{In } S_p = \text{In } S \). Thus (c) is verified.

In order to verify (d) we need a sublemma:

Sublemma 8.5. Let \( \delta_1 \) be the number of zeros in \( X_1 \). Suppose \( X_1 = (0, 0, \ldots, 0, 1, \ldots, 1, -1, \ldots, -1) \) starts with its \( \delta_1 \) zeros. Consider \( S_p \) as a \( 2 \times 2 \) block matrix

\[
S_p = \begin{pmatrix} 0 & B^T \\ B & C \end{pmatrix},
\]
where $0$ is a $n_1 \times \delta_1$ zero submatrix of $N_{11}$, and $B$ is an $(n - n_1) \times \delta_1$ matrix. Then $B$ satisfies the special matrix conditions.

Proof. We have already shown that $S_p$ has no zero rows and therefore (by symmetry) it has no zero columns. Therefore $B$ has no zero columns and part (i) of the definition is established.

Given two different zeros of $X_1$, the permutation $P$ has sent them to two different vectors, say $u_j$ and $u_k$, $j \neq k$.

An element $x_{jk}$ in $B$ (and hence in $S_p$) is nonzero if and only if the corresponding element $y_{jk'}$ ($= x_{jk}$) is nonzero in $S$. Since $S = \oplus S_i$ is a direct sum, an element $y_{jk'}$ can be nonzero only if coordinate indices $j'$ and $k'$ both belong to the same vector (a $r$, $u$, or $w$) associated with one of the blocks (for example $S_1$).

Let us fix $k$ (and its $k'$) and choose two indices $j_1 < j_2 \leq \delta_1$. According to hypothesis (iii) of the lemma, the permutation $P$ sent the $j_1$th zero and the $j_2$th zero to two different vectors, say $u_*$ and $u_{**}$. Therefore $j_1'$ and $j_2'$ (the corresponding indices after permutation $P$) cannot both be in the same vector as $k'$. Therefore, either $y_{j_1'k'} = 0$ or $y_{j_2'k'} = 0$. Therefore either $x_{j_1'k} = 0$ or $x_{j_2'k} = 0$. That is, there do not exist two nonzero elements in the $k$th row of $B$. Part (ii) of the definition is established.

Proof of part (d) for the first block $N_{11}$. The matrix $S_p$ may be written in a $2 \times 2$ block form

$$S_p = \begin{bmatrix} N_{11} & C \\ B_2 & D \end{bmatrix},$$

where $N_{11}$ is the $n_1 \times n_1$ diagonal matrix with diagonal $X_1$. [$B_2$ is an $(n - n_1) \times n_1$ submatrix.] Therefore,

$$\left( \ker S_p \right) \cap (R^{n_1} \times 0) = (\ker N_{11}) \cap \ker B_2. \quad (8.3)$$

We may partition $B_2$ as $B_2 = (B, B_1)$, where $B$ is an $(n - n_1) \times \delta_1$ submatrix. Thus this $B$ is also the $B$ of Sublemma 8.5, and hence it satisfies the special matrix conditions. Therefore, Sublemma 8.1 implies that $\ker B = 0$.

Therefore $B_2$ and $N_{11}$ have the form of the matrices $B_2$ and $N$ of Sublemma 8.3. Therefore

$$\ker N_{11} \cap \ker B_2 = 0.$$
This together with Equation (8.3) implies that
\[ \text{Ker} S_p \cap (\mathbb{R}^n \times O) = O. \]

This establishes part (d) for the first block. Of course, there is nothing special about the first block. Therefore part (d) is established.

This completes the proof of Lemma 8.1.

Proof of the Existence Theorem 1.5. Let \( X_1, X_2, \ldots, X_m \) be the main diagonals of the diagonal matrices \( D_1, D_2, \ldots, D_m \) given in the Existence Theorem. The hypotheses of the Existence Theorem imply that
\[ n \geq \pi + \nu \geq \pi_i + \delta_i + \nu_i + \delta_i, \quad i, j = 1, 2, \ldots, \]
and hence
\[ n \geq \max_i (\pi_i + \delta_i) + \max_i (\nu_i + \delta_i). \]

Let \( r = \max_i \delta_i, \quad p + r = \max_i (\pi_i + \delta_i), \quad q + r = \max_i (\nu_i + \delta_i), \) so that
\[ n \geq (p + r) + (q + r) = p + q + 2r. \]

Therefore, for the catenation \( X = (X_1, X_2, \ldots, X_m) \), the permutation \( P \) needed for Lemma 8.4 will be produced by Theorem 7.1 (See the Remark above.) Then the matrix \( S_p \) provided by Lemma 8.4 is the desired matrix \( M \) of the Existence Theorem 1.5.

Thus the Existence Theorem is established.

9. PROOF OF THE "IF" PART OF COROLLARY 1.2

Assume the hypotheses of Corollary 1.2. In particular, we are given \( m \) Hermitian matrices \( \{ H_{ij} \} \), and we choose any integers \( \pi, \nu \) and \( \delta \) which satisfy the inequalities stated.

The spectral theorem for Hermitian matrices says that there is a unitary matrix \( U \) such that the matrix \( E_i \) defined by \( E_i = U^* H_{ii} U \) is a real diagonal matrix and the main diagonal of \( E_i \) is just the eigenvalues of \( H_{ii} \) (with the zeros listed first) for \( i = 1, 2, \ldots, m \). Hence \( \text{In} H_{ii} = \text{In} E_i \). Let
\[(0, 0, \ldots, 0, \lambda_1 + \delta, \lambda_2 - \delta, \ldots, \lambda_n)\]
be the main diagonal of $E_i$. Let

$$b_i = \begin{cases} \sqrt{|\lambda_i|}, & \lambda_i \neq 0, \\ 1, & i \leq \delta_i. \end{cases}$$

For each $i$, let $D'_i$ be the invertible, diagonal matrix with real main diagonal $(b_1, b_2, \ldots, b_n)$. Set

$$D_i = D'_i E_i D'_i, \quad i = 1, 2, \ldots, m.$$ 

Then each $D_i$ is a real diagonal matrix: its main diagonal is in $\{-1, 0, 1\}^n$, and $\ln D_i = \ln E_i, i = 1, 2, \ldots, m$.

We set

$$U = \bigoplus_{i=1}^{m} U_i, \quad E = \bigoplus_{i=1}^{m} E_i, \quad D = \bigoplus_{i=1}^{m} D_i, \quad \text{and} \quad D' = \bigoplus_{i=1}^{m} D'_i.$$ 

We observe that

$$E = (D')^{-1} D (D')^{-1} \quad \text{and} \quad \bigoplus_{i=1}^{m} H_{ii} = U E U^*.$$  \hspace{1cm} (9.1)

The inequalities for $\pi$, $\nu$, and $\delta$ are the same in Corollary 1.2 and in the Existence Theorem. Therefore, for the $\pi$, $\nu$, $\delta$ chosen here and for the vectors $X_i = \text{main diag } D_i$, there is a matrix $S_p$ with $\ln S_p = (\pi, \nu, \delta)$.

We set

$$S_L = (D')^{-1} S_p (D')^{-1} \quad \text{and} \quad H = U S_L U^*.$$ 

We will show that this matrix $H$ satisfies all the conclusions of Corollary 1.2.

We note that (by Sylvester's theorem)

$$\ln H = \ln S_L = \ln S_p = (\pi, D, \delta)$$

and that the matrix $S_p$ is obtained from Lemma 8.4. Because of (9.1), it is easily checked, by block multiplication, that this $H$ is an extension of $H_{11}, \ldots, H_{mm}$.

All that remains is to check that $0 = \text{Ker } H(C^n \times 0)$. Without loss of generality, take $i = 1$. We will use the following easily verified fact.
Lemma 9.1. Given a block matrix
\[ H = \begin{pmatrix} H_{11} & G_{2}^* \\ G_{2} & K_3 \end{pmatrix} = \begin{pmatrix} U_1^* & 0 \\ 0 & Q^* \end{pmatrix} \times \begin{pmatrix} E & A_2^* \\ A_2 & K_2 \end{pmatrix} \times \begin{pmatrix} U_1 & 0 \\ 0 & Q \end{pmatrix}, \]
where \( U_1 \) and \( Q \) are invertible matrices, and \( H_{11}, U_1, \) and \( E \) are \( n_1 \times n_1 \) matrices. Suppose that
\[ \ker E \cap \ker A_2 = 0. \]
Then
\[ \ker H \mid (C^{n_1} \times 0) = 0. \]

Our aim is to apply this to the \( H \) at hand. In order to concentrate our attention on the subspace \( C^{n_1} \), we write the matrices \( S_p, S_F, \) and \( H \) in the \( 2 \times 2 \) block form
\[ S_p = \begin{pmatrix} N_{11} & B_2^T \\ B_2 & K_1 \end{pmatrix}, \quad S_F = \begin{pmatrix} E_1 & A_2^T \\ A_2 & K_2 \end{pmatrix}, \quad H = \begin{pmatrix} H_{11} & G_{2}^* \\ G_{2} & K_3 \end{pmatrix}. \]

We now focus on the left block columns of \( S_F \) and \( S_p \). We know that
\[ \begin{pmatrix} N_{11} \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D}_1 \end{pmatrix}, \]
that \( B \) satisfies the special matrix conditions and \( \hat{D}_1 \) is an invertible diagonal matrix, and that \( E_1 = D_1'N_{11}D_1' \). Hence, the submatrix \( \begin{pmatrix} E_1 \\ A_2 \end{pmatrix} \) has the block form
\[ \begin{pmatrix} E_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & E_1 \end{pmatrix} \]
(9.2)

where
\[ \hat{E}_1 = (D_1'0 \times R^{n_1-\delta_1})\hat{D}_1(D_1'0 \times R^{n_1-\delta_1}) \]
is also an invertible (diagonal) matrix, and hence

$$\text{Ker} \, \mathcal{E}_1 = 0. \quad (9.3)$$

Also, since $$S_x = D_S D_x'$$, we see that

$$A_2 = \left( \bigoplus_{j=2}^n D_j' \right) \times B_2 \times D_1'. \quad (9.3)$$

Therefore $$A$$ [of $$(A, A_1) - A_2$$] is obtained from $$B$$ [of $$(B, B_1) - B_2$$] by multiplication by two invertible diagonal matrices. Since $$B$$ satisfies the special matrix conditions, Observation 8.2 implies that $$A$$ also satisfies them. Then Sublemma 8.1 implies that $$\text{Ker} \, A = 0$$.

We have shown both that $$\text{Ker} \, A = 0$$ and, in (9.3), that $$\text{Ker} \, \mathcal{E}_1 = 0$$. Therefore Sublemma 8.3 and (9.2) imply that $$\text{Ker} \, E_1 \cap \text{Ker} \, A_2 = 0$$. This together with Lemma 9.1 implies that $$\text{Ker} \, H(C^n \times 0) = 0$$.

The proof of Corollary 1.2 is complete.

10. THE EXTENSION THEOREM

In this section, we shall use Corollary 1.2 in order to establish the Extension Theorem (and Corollary 1.1).

We shall prove the case for just two blocks $$H_{11}$$ and $$H_{22}$$, because this simplifies the notation, without omitting anything important.

Proof of the “if” part of the Extension Theorem 1.3 when $$m = 2$$. Assume the hypotheses of this theorem. We are given $$H_{11}$$ and $$H_{22}$$ with domains $$C^{n_1}$$ and $$C^{n_2}$$. We are given subspaces $$V_1 \subset \text{Ker} \, H_{11}$$ and $$V_2 \subset \text{Ker} \, H_{22}$$.

Therefore we may write

$$C^{n_1} = V_1^\perp \oplus V_1 \quad \text{and} \quad C^{n_2} = V_2 \oplus V_2^\perp$$

as orthogonal direct sums. With respect to these direct sums, the Hermitian operators may be written as

$$H_{11} = 0 \oplus H_{11}' \quad \text{and} \quad H_{22} = H_{22}' \oplus 0$$
where each
\[ H_i': V_i \to V_i \]
(since eigenspaces are orthogonal).

We set \( \delta_i' = \delta_i - d_i \) for both \( i = 1 \) and \( 2 \), and we set \( \delta' = \delta - \sum d_i \). We check that

\[ \text{In } H_{ii}' = \text{In } H_{ii} \quad (0, 0, d_i) = (\pi_i, r_i, \delta_i'). \]

Also we check
\[
\pi \geq \max(\pi_i + \delta_i - d_i) = \max(\pi_i + \delta_i'), \\
\nu \geq \max(\nu_i + \delta_i - d_i) = \max(\nu_i + \delta_i'), \\
\pi + \nu + \delta' = \sum \pi_i + \sum \nu_i + \sum \delta_i'.
\]

Therefore, Corollary 1.2 may be used to provide a Hermitian extension

\[ H': V_1 \oplus V_2 \to V_1 \oplus V_2 \]

of \( H_{11}' \) and \( H_{22}' \), with \( \text{In } H' = (\pi, \nu, \delta') \), and \( H' \) has the block form

\[
\begin{pmatrix}
H_{11}' & X_{21}' \\
X_{12}' & H_{22}'
\end{pmatrix}.
\]

Also

\[ \text{Ker } H' \cap (V_1 \times 0) = 0 = \text{Ker } H' \cap (0 \times V_2) \quad (10.1) \]

Let \( \hat{H}: V_1 \oplus V_1 \oplus V_2 \oplus V_2 \) be the Hermitian matrix defined by this block form:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & H' & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

We calculate
\[ \text{In } \hat{H} = \text{In } H' + (0, 0, d_1 + d_2) = (\pi, \nu, \delta). \]
Also, we see using (10.1) that

\[(\text{Ker} \, \hat{H} \cap (C^{n_1} \times 0)) - V_1 \oplus [\text{Ker} \, H' \cap (V_1^\perp \times 0)] = V_1 \times 0,\]

\[(\text{Ker} \, H) \cap (0 \times C^{n_2}) = [\text{Ker} \, H' \cap (0 \times V_2^\perp)] \oplus V_2 = 0 \times V_2.\]

Let \( U_1 \) and \( U_2 \) be unitary change-of-basis matrices from the standard coordinate systems of \( C^{n_1} \) and \( C^{n_2} \) to the \( V_1 \oplus V_1^\perp \) and \( V_2 \oplus V_3 \) coordinate systems. Then

\[H_{11} = U_1^* (0 \oplus H'_{11}) U_1,\]

\[H_{22} = U_2^* (H'^{-1}_{22} \oplus 0) U_2.\]

Let \( U = U_1 \oplus U_2 \).

The matrix \( H = U^* \hat{H} U \) satisfies the conclusions of this theorem.

Corollary 1.1 is simply the special case of the Extension Theorem 1.3 when all the \( V_i \) are zero.

11. A CONNECTION WITH CAIN’S AND MARQUES DE SÁ’S RESULTS

In this section, we present an alternate proof of condition (III), parts (3) and (4), of Theorem 11.4. Our proof is based on our earlier work, namely Lemma 1.4.

**Corollary 11.1.** Let \( H \) be a Hermitian extension of \( \{H_{ii}\} \). Let \( I \subset \{1, 2, \ldots, m\} \) be an indexing set with \( k \) elements. Then

\[kv + \delta \geq \sum_{i \in I} (v_i + \delta_i) \quad \text{and} \quad k\pi + \delta \geq \sum_{i \in I} (\pi_i + \delta_i). \quad (11.1)\]

**Proof.** Lemma 1.4 (applied to the \( k \, H_{ii}, \quad i \in I \)) implies that

\[kv = \sum_{i \in I} v_i \geq \sum_{i \in I} (v_i + \delta_i - d_i) = \sum_{i \in I} (v_i + \delta_i) - \sum_{i \in I} d_i.\]

Next we use the condition \( \delta \geq \sum_{i=1}^m d_i \geq \sum_{i=1}^m d_i \) from the “only if” part of
the Extension Theorem 1.3. Therefore,

\[ kv \geq \sum_{i \in I} (v_i + \delta_i) - \delta. \]

The inequality for \( k\pi + \delta \) is established in the same manner.

Remark. Poincaré’s inequalities and Cauchy’s interlacing theorem only imply that

\[ kv + k\delta \geq \sum_{i \in I} (v_i + \delta_i) \quad \text{and} \quad k\pi + k\delta \geq \sum_{i \in I} (\pi_i + \delta_i). \]

Therefore, Corollary 11.1 is an improvement.

The next lemma and remark state that the inequalities of Corollary 11.1 and the inequalities of condition (III), parts (3) and (4), are equivalent.

Lemma 11.2. For any positive integer \( m \), any indexing set \( I \subset \{1, 2, \ldots, m\} \) with \( k \) integers, and any numbers

\[ \{n_i = \pi_i + v_i + \delta_i, \; i = 1, 2, \ldots, m\} \]

and \( n = \pi + v + \delta = \sum_{i=1}^{m} n_i \), the following two conditions are equivalent:

(a) \( kv + \delta \geq \sum_{i \in I} (v_i + \delta_i) \),
(b) \( \pi - (k - 1)v \leq \sum_{i \in I} \pi_i + \sum_{i \in I} n_i \)

Proof. Given (a), make the substitutions \( \delta = n - \pi - v \) and \( v_i + \delta_i = n_i - \pi_i \). This yields

\[ n - \pi + (k - 1)v \geq \sum_{i \in I} n_i - \sum_{i \in I} \pi_i. \]

We note that \( n = \sum_{i=1}^{m} n_i = \sum_{i \in I} n_i + \sum_{i \notin I} n_i \). Therefore,

\[ \pi - (k - 1)v \leq \sum_{i \in I} \pi_i + \sum_{i \notin I} n_i, \]

Thus (b) is established. All our steps are reversible. Therefore (b) \( \Rightarrow \) (a).

Thus Lemma 11.2 is established.
Remark 11.3. In the same manner the equivalence of

(a) \( k \pi + \delta \geq \sum_{i \in I} (\pi_i + \delta_i) \) and
(b) \( v - (k - 1) \pi \leq \sum_{i \in I} \nu_i + \sum_{i \in I} n_i \)

may be established.

We now state the main result of Professor Cain’s paper ([3]).
Given real numbers \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \), for \( k = 1, \ldots, m \) set

\[
\ell_k (m, x_*, y_*) = \min \left\{ \sum_{i \notin I} x_i + \sum_{i \in I} y_i : I \subseteq \{1, \ldots, m\} \text{ and } \#(I) = k \right\}.
\]

Here \( \#(I) \) is the number of elements in \( I \).

Theorem 11.4 (Bryan Cain [3]). Let \( m, n_1, \ldots, n_m \) be positive integers, and set \( n = n_1 + \cdots + n_m \). The following are equivalent:

(I) Given \( n_i \times n_i \) Hermitian matrices \( H_i \) with \( \text{In } H_i = (\pi_i, \nu_i, \delta_i) \) for \( i = 1, \ldots, m \), there exists an \( n \times n \) Hermitian matrix \( H = (X_{ij})_{i,j=1}^{m} \) where the \( X_{ij} \) are \( n_i \times n_i \) blocks satisfying \( X_{ii} = H_i \) and \( \text{In } H = (\pi, \nu, n - \pi - \nu) \).

(II) There exists an \( n \times n \) Hermitian matrix \( H = (X_{ij})_{i,j=1}^{m} \) where the \( X_{ij} \) are \( n_i \times n_i \) blocks satisfying \( \text{In } X_{ii} = (\pi_i, \nu_i, \delta_i) \) and \( \text{In } H = (\pi, \nu, n - \pi - \nu) \).

(III) For \( i = 1, \ldots, m \) the numbers \( \pi_i, \nu_i, \delta_i \) are nonnegative integers satisfying \( \pi_i + \nu_i + \delta_i = n_i \) and \( \pi, \nu \) are nonnegative integers which satisfy

(1) \( \pi + \nu \leq n \),
(2) \( \max\{\pi_1, \ldots, \pi_m\} \leq \pi \), \( \max\{\nu_1, \ldots, \nu_m\} \leq \nu \),
(3) \( \pi - (k - 1)\nu \leq \ell_k (m, n_*, \pi_*) \) for \( k = 1, \ldots, m \),
(4) \( \nu - (k - 1)\pi \leq \ell_k (m, n_*, \nu_*) \) for \( k = 1, \ldots, m \).

Finally we note that conditions (b) of Lemma 11.2 and Remark 11.3 are equivalent to condition (III), parts (3) and (4), of Theorem 11.4. Therefore Corollary 11.1, Lemma 11.2, and Remark 11.3 provide an alternate proof that condition (III), parts (3) and (4), of Theorem 11.4 follow from the other conditions. For the case \( m = 2 \) and \( I = \{1,2\} \), these inequalities were first stated as condition III, parts 4 and 5, of the Theorem of Cain and Marques de Sá’s paper [2].
Note Added in Proof: If the matrices \( \{ H_{ii} \} \) and hence also \( H \) of the Extension Theorem 1.3 are merely diagonalizable matrices with real eigenvalues, instead of Hermitian matrices, then the "if" part of Extension Theorem 1.3 remains valid.

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REFERENCES


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