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PROJECTIVE MODULES AND TENSOR PRODUCTS *

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1. Introduction

With a view towards extending the results of block theory and towards rationalizing the many cohomological calculations for finite groups of Lie type, we have been studying projective modules [1–4]. An unexpected and recurring theme in this work and that of others has been the use of tensor products of irreducible modules. In this paper we shall establish a few simple results which hopefully shed some light on this phenomenon.

We now fix a finite group G and an algebraically closed field F of prime characteristic p. All FG-modules are assumed to be right modules and to be finitely generated Recall [3] that an indecomposable FG-module is said to be irreducibly generated if it is isomorphic with a direct summand of a tensor product of a finite number of irreducible FG-modules.

Theorem 1. *If G has no non-identity normal p-subgroup then every indecomposable projective FG-module is irreducibly generated.*

In other words, if we can determine the structure of all the irreducibly generated modules in this case then we have also dealt with the projective modules. Usually much less is required; in fact, as we shall see below, if S is an irreducible and projective FG-module then it suffices to decompose the tensor products of S with each of the irreducible FG-modules.

More generally, let S be any non-zero projective FG-module. Let $x_1, ..., x_s$ be representatives of the conjugacy classes of p'-elements of G. Let $V_1, ..., V_s$ be irreducible FG-modules, one of each isomorphism type. Let $P_1, ..., P_s$ be their projective covers so these are indecomposable projective FG-modules, one of each isomorphism

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type, with V_i a homomorphic image of P_i. Let Φ be the Brauer character of S.

Since S is projective the tensor product of V_i with S is also projective and is therefore a direct sum of indecomposable projective FG-modules. We can therefore write, schematically,

$$V_i \otimes S \cong a_{i1}P_1 \oplus a_{i2}P_2 \oplus \cdots \oplus a_{is}P_s$$

where the a_{ij} are non-negative integers. Thus, $A = (a_{ij})$ is an s by s matrix determined by S.

Theorem 2. *With the above notation, the following assertions hold:*
1) No column of A is zero;
2) The rank of A equals the number of conjugacy classes of G where Φ does not vanish;
3) The determinant of A equals the product $\prod_{i=1}^{s} \Phi(x_i)/|C(x_i)|_p$;
4) If S is self dual then A is symmetric.

The term $|C(x_i)|_p$ is the order of the Sylow p-subgroup of the centralizer of x_i. The dual of an FG-module M is the dual vector space with the usual right module structure. Notice also that the first statement is simply a concise way of saying that every module P_j is isomorphic with a summand of some tensor product $V_i \otimes S$.

The above result holds equally well for "virtual" projective modules. Therefore, since $K_0(FG)$ has a basis consisting of the isomorphism types of indecomposable projective FG-modules [11], $G_0(FG)$ has a basis consisting of the isomorphism types of irreducible FG-modules and tensor products yield a $G_0(FG)$-module structure for $K_0(FG)$, we can deduce a consequence:

Corollary. *$K_0(FG)$ is a free $G_0(FG)$-module.*

Early work in this direction was done by Jeyakumar [9] and Ballard [5]. This result was obtained for finite Chevalley groups, with the Steinberg module as generator, by Lusztig [10] and in full generality independently by Feit [8].

We shall apply these ideas to finite groups of Lie type and do so here for the rank one case. For our purposes, we shall define G to be of Lie type, rank one and characteristic p provided the following assertions hold:

(a) U is a Sylow p-subgroup of G;
(b) B is the normalizer of U;
(c) H is a p'-group with $B = HU$;
(d) w is an element of $N(H)$ with $G = B \cup BwB$ and $H = B \cap B^w$.

For such a group there is a natural choice for the module S. We have the usual result:

Lemma. *The induced module $F \otimes_{FB} FG$ is the direct sum $F \oplus S$, where S is an irreducible and projective FG-module of dimension the order of U.*

For such a group we shall introduce some further s by s matrices. We set $T = (t_{ij})$ where

$$t_{ij} = \dim_F \operatorname{Hom}_{FB}(V_i|B, V_j|B)\,,$$

$M = (m_{ij})$ where

$$m_{ij} = \dim_F \operatorname{Hom}_{FH}(V_i|H, V_j|H)\,,$$

and $C = (c_{ij})$, the Cartan matrix, so

$$c_{ij} = \dim_F \operatorname{Hom}_{FG}(P_i, P_j)\,.$$

Here, the vertical bars denote restrictions.

Theorem 3. *If G is of Lie type, rank one and characteristic p then*

$$A + I = T$$

$$ACA + A = M\,.$$

The matrix A is as above. As we shall see Theorem 2 implies that A is non-singular so we have a formula for C: $C = (T - I)^{-1} M(T - I)^{-1} - (T - I)^{-1}$. This shows C is determined if we "know" the restrictions of the irreducible FG-modules to B. A similar but more complicated result holds for all groups of Lie type relating the Cartan matrix and the restrictions of the irreducible modules to the parabolic subgroups.

2. Proofs

We first establish Theorem 1. Since G has no non-identity normal p-subgroup, it follows that the direct sum

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

is a faithful FG-module. Hence, by a result of Bryant and Kovacs [6], there is a positive integer n such that

$$U = V^0 \oplus V^1 \oplus V^2 \oplus \cdots \oplus V^n$$

contains a free submodule, where $V^0 = F$ and $V^i = V \quad V \otimes \cdots \otimes V$ is the ith tensor power of V for positive integers i. However, free modules are projective and hence injective so U contains a free summand. Therefore, each indecomposable projective FG-module is isomorphic with a summand of U. But U is also a direct sum of tensor products of irreducible FG-modules so our claim holds.

We now turn to the second theorem and prove each statement in turn. If $1 \leqslant j \leqslant s$ and S^* is the dual of S then $S^* \otimes V_j$ has a non-zero socle and so there is i, $1 \leqslant i \leqslant s$, with $\mathrm{Hom}_{FG}(V_i, S^* \otimes V_j) \neq 0$. Hence, $\mathrm{Hom}_{FG}(V_i \otimes S, V_j) \neq 0$ as these two vector spaces are naturally isomorphic. But P_j is the only one of the P_k which has a non-zero homomorphism to V_j. Thus, $V_i \otimes S$ has a summand isomorphic with P_j, $a_{ij} \neq 0$ and the jth column of A is not zero.

Let R be the complex algebra of all complex-valued functions from the set of p'-elements of G to the complex numbers which are constant on conjugacy classes. There are several natural bases for R: the Brauer characters $\varphi_1, ..., \varphi_s$ of $V_1, ..., V_s$ are a basis; the Brauer characters $\Phi_1, ..., \Phi_s$ of $P_1, ..., P_s$ are a basis; if e_i, $1 \leqslant i \leqslant s$, is the function that is one on conjugates of x_i and zero elsewhere then $e_1, ..., e_s$ is a basis.

The definition of the matrix A implies that for all i, $1 \leqslant i \leqslant s$,

$$\varphi_i \Phi = \sum_j a_{ij} \Phi_j$$

so that the rank of A equals the rank of the linear transformation of R given by multiplication by Φ. But this equals the dimension of the ideal of R generated by Φ. However, $e_i \Phi = \Phi(x_i) e_i$ so this ideal has dimension equal to the number of conjugacy classes where Φ does not vanish. Therefore, A has the stated rank.

Since $\Phi_j = \Sigma_k c_{jk} \varphi_k$ we deduce from above that

$$\varphi_i \Phi = \sum_{j,k} a_{ij} c_{jk} \varphi_k$$

so that

$$\varphi_i(x_t) \Phi(x_t) = \sum_{j,k} a_{ij} c_{jk} \varphi_k(x_t)$$

for every t, $1 \leqslant t \leqslant s$. Hence, if $\varphi = (\varphi_i(x_t))$ is an s by s matrix and K is the diagonal matrix with entries $\Phi(x_1), ..., \Phi(x_s)$ then

$$\varphi K = A C \varphi .$$

But φ is invertible, C has determinant $\prod_i |C(x_i)|_p$ by a theorem of Brauer [7] so our assertion about the determinant of A is true.

Since $\mathrm{Hom}_{FG}(P_j, V_j) \cong F$ while $\mathrm{Hom}_{FG}(P_j, P_k) = 0$ if $j \neq k$ it follows that $a_{ij} = \dim_F \mathrm{Hom}_{FG}(V_i \otimes S, V_j)$. But, if S is self dual then

$$\mathrm{Hom}_{FG}(V_i \otimes S, V_j) \cong \mathrm{Hom}_{FG}(V_i, S^* \otimes V_j)$$

$$\cong \mathrm{Hom}_{FG}(V_i, S \otimes V_j) .$$

However, the socle of P_k is isomorphic with V_k so $\mathrm{Hom}_{FG}(V_k, P_k) \cong F$ while $\mathrm{Hom}_{FG}(V_k, P_t) = 0$ if $k \neq t$. Thus, $\dim_F \mathrm{Hom}_{FG}(V_i, S \otimes V_j) = a_{ji}$ and so A is symmetric.

We consider the corollary next. The group $K_0(FG)$ can be identified with the

Abelian group of all integral linear combinations of $\Phi_1, ..., \Phi_s$, while the ring $G_0(FG)$ can be identified with the ring of all integral linear combinations of $\varphi_1, ..., \varphi_s$. With these identifications the module structure is given by multiplication of functions.
To each Φ in $K_0(FG)$ there is a matrix A_Φ associated, just as above, and all the results of Theorem 2 hold by the very same arguments. It suffices to show that there is Φ in $K_0(FG)$ with the determinant of A_Φ equal to 1 (or -1). For then the map of $G_0(FG)$ to $K_0(FG)$ which sends each ψ in $G_0(FG)$ to $\psi\Phi$ is a module isomorphism. But there is an element β of $K_0(FG)$ with $\beta(x_i) = |C(x_i)|_p$ for each i, $1 \leqslant i \leqslant s$, by a theorem of Brauer [7]. Hence, A_β has determinant $\prod_i \beta(x_i)/|C(x_i)|_p$, by Theorem 2, which is just one. This proves the corollary.

For the rest of this section we assume that our group G is of Lie type, rank one and characteristic p and we use all the notation given above. First, we prove the lemma. We begin by noting that $|G : B| = 1 + |U|$. Indeed, there are $|B : H|$ cosets of B in the double coset BwB since $B \cap B^w = H$ by assumption. Hence, $|G : B| = 1 + |B : H|$. But $B = UH$ and the subgroups U and H have coprime orders so that $|B : H| = |U|$ and $|G : B|$ is as claimed.

By assumption, G is doubly transitive on the cosets of B in G. Hence, if C denotes the complex numbers, it follows that $C \otimes_{CB} CG = C \oplus X$, where X is an irreducible CG-module. Since $\dim_C X = |U| = |G|_p$, X is the unique irreducible CG-module in a p-block of defect zero. Therefore, "reducing X modulo p" yields an irreducible and projective module. In particular, $F \otimes_{FB} FG$ has the stated structure.

Before proceeding, we note, as promised, that the matrix A, determined by S, is invertible. Let π be the character of the permutation representation of G on the cosets of B. Since w is in $N(H)$ and not in B, it follows that $\pi(h) \geqslant 2$ for any element of H. Moreover, if x is a p'-element of G not conjugate to an element of H then $\pi(x) = 0$. Therefore, $\Phi(h) \geqslant 1$ and $\Phi(x) = -1$, so A has a non-zero determinant by Theorem 2.

(We digress here for a moment. Suppose, as is most often the case, that if $h \neq 1$ then $h^g \in B$, $g \in G$ only if $g \in B \cup Bw$. It follows that $\pi(h) = 2$ so $\Phi(h) = 1$. Now A is an invertible integral matrix with determinant +1 or -1.)

This leaves only Theorem 3 to deal with. We first introduce some notation and remind the reader of a few elementary facts. Let L be a subgroup of G and let U and V be FG and FL modules, respectively. Denote the induced module $V \otimes_{FL} FG$ by V^G and the restriction of U to L by $U|L$. Recall that $U \otimes V^G$ and $(U|L \otimes V)^G$ are naturally isomorphic as are $\mathrm{Hom}_{FG}(V^G, U)$ and $\mathrm{Hom}_{FL}(V, U|L)$. Also, as is well known, since G is finite, $\mathrm{Hom}_{FG}(U, V^G)$ and $\mathrm{Hom}_{FL}(U|L, V)$ are isomorphic [12]. If $L = B$ then $V^G|B \cong V \oplus (V|H)^B$ by Mackey's theorem [7].

Also recall from above that

$$a_{ij} = \dim_F \mathrm{Hom}_{FG}(V_i \otimes S, V_j)$$

and that since P_k has socle isomorphic with V_k we also have

$$a_{ij} = \dim_F \operatorname{Hom}_{FG}(V_j, V_i \otimes S)\,.$$

If δ_{ij} is the usual Kronecker delta, then

$$\begin{aligned} a_{ij} + \delta_{ij} &= \dim_F \operatorname{Hom}_{FG}(V_i \otimes S, V_j) + \dim_F \operatorname{Hom}_{FG}(V_i, V_j) \\ &= \dim_F \operatorname{Hom}_{FG}(V_i \otimes (F|B)^G, V_j) \\ &= \dim_F \operatorname{Hom}_{FG}((V_i|B)^G, V_j) \\ &= \dim_F \operatorname{Hom}_{FB}(V_i|B, V_j|B) = t_{ij} \end{aligned}$$

so $A + I = T$ as claimed.

Our construction of S forces S to be self dual so that A is symmetric, by Thereom 2. Thus,

$$\begin{aligned} \dim_F \operatorname{Hom}_{FG}(V_i \otimes S, V_j \otimes S) &= \sum_{k,t} a_{ik}\, a_{jt} \dim_F \operatorname{Hom}_{FG}(P_k, P_t) \\ &= \sum_{k,t} a_{ik}\, c_{kt}\, a_{tj} = (ACA)_{ij}\,, \end{aligned}$$

the ij entry of ACA. Hence,

$$\begin{aligned} &\dim_F \operatorname{Hom}_{FG}(V_i \otimes (F|B)^G, V_j \otimes (F|B)^G) \\ &\quad = \dim_F \operatorname{Hom}_{FG}(V_i \oplus (V_i \otimes S), V_j \oplus (V_j \otimes S)) \\ &\quad = \delta_{ij} + a_{ij} + a_{ji} + (ACA)_{ij} \\ &\quad = (I + 2A + ACA)_{ij}\,. \end{aligned}$$

But also

$$\begin{aligned} &\dim_F \operatorname{Hom}_{FG}(V_i \otimes (F|B)^G, V_j \otimes (F|B)^G) \\ &\quad = \dim_F \operatorname{Hom}_{FG}((V_i|B)^G, (V_j|B)^G) \\ &\quad = \dim_F \operatorname{Hom}_{FB}(V_i|B, (V_j|B)^G|B) \\ &\quad = \dim_F \operatorname{Hom}_{FB}(V_i|B, V_j|B \oplus (V_j|H)^B) \\ &\quad = \dim_F \operatorname{Hom}_{FB}(V_i|B, V_j|B) + \dim_F \operatorname{Hom}_{FH}(V_i|H, V_j|H) \\ &\quad = t_{ij} + m_{ij}\,. \end{aligned}$$

Hence, $T + M = I + 2A + ACA$, so since $A + I = T$ it follows that $M = A + ACA$. All the results are now established.

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