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A Garside-theoretic approach to the reducibility problem in braid groups

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Abstract

Let D_n denote the n-punctured disk in the complex plane, where the punctures are on the real axis. An n-braid α is said to be *reducible* if there exists an essential curve system $\mathcal{C}$ in D_n, called a *reduction system* of α, such that $\alpha * \mathcal{C} = \mathcal{C}$ where $\alpha * \mathcal{C}$ denotes the action of the braid α on the curve system $\mathcal{C}$. A curve system $\mathcal{C}$ in D_n is said to be *standard* if each of its components is isotopic to a round circle centered at the real axis.

In this paper, we study the characteristics of the braids sending a curve system to a standard curve system, and then the characteristics of the conjugacy classes of reducible braids. For an essential curve system $\mathcal{C}$ in D_n, we define the *standardizer* of $\mathcal{C}$ as $\mathrm{St}(\mathcal{C}) = \{P \in B_n^+ \colon P * \mathcal{C} \text{ is standard}\}$ and show that $\mathrm{St}(\mathcal{C})$ is a sublattice of B_n^+. In particular, there exists a unique minimal element in $\mathrm{St}(\mathcal{C})$. Exploiting the minimal elements of standardizers together with canonical reduction systems of reducible braids, we define the outermost component of reducible braids, and then show that, for the reducible braids whose outermost component is simpler than the whole braid (including split braids), each element of its ultra summit set has a standard reduction system. This implies that, for such braids, finding a reduction system is as easy as finding a single element of the ultra summit set.

Keywords: Braid group; Reducible braid; Dynamical type; Conjugacy problem

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1. Introduction

Let $D_n = \{z \in \mathbb{C}\colon |z| \leqslant n+1\} \setminus \{1, \ldots, n\}$, the n-punctured disk in the complex plane with punctures lying on the real axis. The n-braid group B_n acts on the set of curve systems in D_n. For an n-braid α and a curve system $\mathcal{C}$ in D_n, let $\alpha * \mathcal{C}$ denote the action of α on $\mathcal{C}$. An n-braid α is said to be *reducible* if $\alpha * \mathcal{C} = \mathcal{C}$ for some essential curve system $\mathcal{C}$ in D_n, called a *reduction system* of α. In this paper, we are interested in the *reducibility problem*: given a braid, decide whether it is reducible or not and find a reduction system if it is reducible.

1.1. Motivation and some of previous works

The Nielsen–Thurston classification theorem [Thu88] states that an irreducible automorphism of an orientable surface with negative Euler characteristic is either periodic or pseudo-Anosov up to isotopy. Recall that an orientation preserving self-diffeomorphism f of a surface S is said to be

- *periodic* if f^k is isotopic to the identity for some $k \neq 0$;
- *reducible* if there exist pairwise disjoint simple closed curves $C_1, \ldots, C_k$ in S, isotopic to neither a point nor a puncture nor a boundary component, such that $f(\mathcal{C})$ is isotopic to $\mathcal{C}$, where $\mathcal{C} = C_1 \cup \cdots \cup C_k$;
- *pseudo-Anosov* if there exist a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) and a real $\lambda > 1$ such that $f(F^s, \mu^s) = (F^s, \lambda^{-1}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$.

There have been several approaches to the problem of deciding dynamical types of surface automorphisms. Bestvina and Handel [BH95] made the train track algorithm that, given a surface automorphism, decides its dynamical type and finds its dynamical structure: a pair of transverse measured foliations for a pseudo-Anosov automorphism; a reduction system for a reducible automorphism. Benardete, Gutiérrez and Nitecki [BGN95] solved the reducibility problem in braid groups. (It is known that a periodic n-braid is conjugate to either $(\sigma_1\sigma_2\cdots\sigma_{n-1})^l$ or $(\sigma_1(\sigma_1\sigma_2\cdots\sigma_{n-1}))^l$ for some integer l [Ker19,Eil34,BDM02]. This implies that α is a periodic n-braid if and only if either α^n or α^{n-1} is equal to Δ^{2m} for some integer m. Hence, it is easy to decide the periodicity of braids. Therefore, in order to decide the dynamical type of a given braid, it suffices to decide the reducibility.) Humphries [Hum91] solved the problem of recognizing split braids.

With the above results, solving the reducibility problem and the problem of recognizing split braids seems at least as hard as solving the conjugacy problem. When using the train track algorithm, one needs to describe a given n-braid as a graph map of the n-bouquet, and the length of this description grows exponentially with respect to the length of the braid word on Artin generators. The other two solutions need to use the algorithms solving the conjugacy problem in braid groups.

Another motivation for this work is the close relationship between the reducibility problem and the conjugacy problem. The approach to the conjugacy problem in braid groups can be divided into two steps: solving the reducibility problem and solving the conjugacy problem for irreducible braids. See [BGG06a, §1.4] for a more precise description of this strategy. The conjugacy problem for periodic braids is easy to solve. There are two different polynomial-time solutions to this case by Birman, Gebhardt and González-Meneses [BGG06b] and by the authors [LL07b]. For the conjugacy problem for pseudo-Anosov mapping classes, there are several

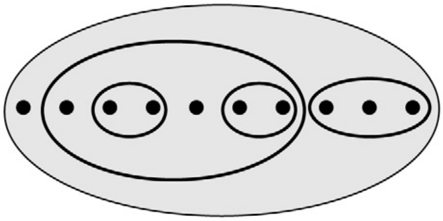


Fig. 1. A standard curve system in D_{10}.

results. In [Los93], Los solved the problem for pseudo-Anosov braids by using combinatorial efficient representatives. Recently, Fehrenbach and Los [FL07] proposed an algorithm that finds roots and symmetries of pseudo-Anosov mapping classes together with a new solution to the conjugacy problem. Mazur and Minsky [MM99,MM00] showed that, fixing a mapping class group and a finite set of generators for this group, there exists a constant K such that if α and β are conjugate pseudo-Anosov mapping classes then there is a conjugating element γ with $|\gamma| \leqslant K(|\alpha| + |\beta|)$, where $|\cdot|$ denotes the word length. In order to extend the results on irreducible braids to general braids, we need to solve the reducibility problem more efficiently.

For the last ten years, no serious progress has been made in the reducibility problem. On the other hand, recently, there have been several new contributions to Garside-theoretic approach to braid groups, for example [Deh02,FG03,Geb05,Lee07]. Exploiting them, we study the characteristics of the conjugacy classes of reducible braids. Our approach uses neither the train track algorithm nor the complete conjugacy algorithm. We hope that our results are useful in obtaining a more efficient solution to the reducibility problem in braid groups.

1.2. Our results

Before stating our results, we recall some notions and results from the Garside theory in braid groups.

- Let B_n^+ be the submonoid of B_n generated by $\sigma_1, \ldots, \sigma_{n-1}$. The partial order $\leqslant_R$ on B_n^+ is defined as follows: for $P, Q \in B_n^+$, $P \leqslant_R Q$ if $Q = SP$ for some $S \in B_n^+$. The poset $(B_n^+, \leqslant_R)$ is a lattice, i.e., there exist the gcd $P \wedge_R Q$ and the lcm $P \vee_R Q$ of $P, Q \in B_n^+$.
- For $\alpha \in B_n$, there are integer-valued invariants $\inf(\alpha)$ and $\sup(\alpha)$. Let $[\alpha]$ denote the conjugacy class of $\alpha \in B_n$. The following are conjugacy invariants.

$$\inf{}_s(\alpha) = \max\bigl\{\inf(\beta)\colon\ \beta \in [\alpha]\bigr\}, \qquad t_{\inf}(\alpha) = \lim_{m\to\infty} \inf\bigl(\alpha^m\bigr)/m,$$
$$\sup{}_s(\alpha) = \min\bigl\{\sup(\beta)\colon\ \beta \in [\alpha]\bigr\}, \qquad t_{\sup}(\alpha) = \lim_{m\to\infty} \sup\bigl(\alpha^m\bigr)/m.$$

- In the conjugacy class $[\alpha]$, there are finite, nonempty, computable subsets, the super summit set $[\alpha]^S$, the ultra summit set $[\alpha]^U$ and the stable super summit set $[\alpha]^{St}$. They depend only on the conjugacy class, and $[\alpha]^U, [\alpha]^{St} \subset [\alpha]^S$.

We call an essential curve system (see Definition 3.1) in D_n a *standard* curve system if each component is isotopic to a round circle centered at the real axis as in Fig. 1. For an essential curve system $\mathcal{C}$ in D_n, we define the *standardizer* of $\mathcal{C}$ as the set

$$\mathrm{St}(\mathcal{C}) = \{P \in B_n^+ \colon\ P * \mathcal{C} \text{ is standard}\}$$

where $P * \mathcal{C}$ denotes the left action of the positive braid P on the curve system $\mathcal{C}$, and then show the following.

Theorem 4.2. *For an essential curve system $\mathcal{C}$ in D_n, its standardizer* $\mathrm{St}(\mathcal{C})$ *is closed under $\wedge_R$ and $\vee_R$, and hence a sublattice of B_n^+. Therefore* $\mathrm{St}(\mathcal{C})$ *contains a unique $\leqslant_R$-minimal element.*

Theorem 4.9. *Let α be a reducible n-braid with a reduction system $\mathcal{C}$. Let P be the $\leqslant_R$-minimal element of* $\mathrm{St}(\mathcal{C})$*. Then the following hold.*

(i) $\inf(\alpha) \leqslant \inf(P\alpha P^{-1}) \leqslant \sup(P\alpha P^{-1}) \leqslant \sup(\alpha)$.
(ii) *If $\alpha \in [\alpha]^S$, then $P\alpha P^{-1} \in [\alpha]^S$.*
(iii) *If $\alpha \in [\alpha]^U$, then $P\alpha P^{-1} \in [\alpha]^U$.*
(iv) *If $\alpha \in [\alpha]^{St}$, then $P\alpha P^{-1} \in [\alpha]^{St}$.*

Theorem 4.2 is essential in our approach to the reducibility problem, as the closedness under $\wedge_R$ of $\{P \in B_n^+ \colon\ P\beta P^{-1} \in [\alpha]^S\}$ and $\{P \in B_n^+ \colon\ P\beta P^{-1} \in [\alpha]^U\}$ for $\beta \in [\alpha]^S$ plays an important role in solving the conjugacy problem [FG03,Geb05]. Theorem 4.9 shows that standardizing a reduction system $\mathcal{C}$ of a braid by the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$ preserves the membership of the super summit set, ultra summit set and stable super summit set.

It is known by Birman, Lubotzky and McCarthy [BLM83] and Ivanov [Iva92] that a reducible surface automorphism admits a unique *canonical reduction system*. For $\alpha \in B_n$, let $\mathcal{R}_{\mathrm{ext}}(\alpha)$ be the collection of the outermost components of the canonical reduction system of α. Let P be the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{R}_{\mathrm{ext}}(\alpha))$. Since $\mathcal{R}_{\mathrm{ext}}(P\alpha P^{-1}) = P * \mathcal{R}_{\mathrm{ext}}(\alpha)$ is standard, the outermost component of $D_n \setminus \mathcal{R}_{\mathrm{ext}}(P\alpha P^{-1})$ is naturally identified with the k-punctured disk D_k for some $k \leqslant n$. We define the *outermost component* α_{ext} of α as the k-braid obtained by restricting the braid $P\alpha P^{-1}$ to the outermost component of $D_n \setminus \mathcal{R}_{\mathrm{ext}}(P\alpha P^{-1})$. See Section 5 for the precise definition. The following is the main result of this paper. (In the statement, $[\alpha]^U_{\mathbf{d}}$ denotes the ultra summit set of α with respect to decycling. See the next section for the precise definition.)

Theorem 7.4. *Let α be a non-periodic reducible n-braid.*

(i) *If $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$, then each element of $[\alpha]^U$ has a standard reduction system.*
(ii) *If $\sup_s(\alpha_{\mathrm{ext}}) < \sup_s(\alpha)$, then each element of $[\alpha]^U_{\mathbf{d}}$ has a standard reduction system.*
(iii) *If α is a split braid, then each element of $[\alpha]^U \cup [\alpha]^U_{\mathbf{d}}$ has a standard reduction system.*
(iv) *If α_{ext} is periodic, then there exists $1 \leqslant q < n$ such that each element of $[\alpha^q]^U \cup [\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.*
(v) *If $t_{\inf}(\alpha_{\mathrm{ext}}) > t_{\inf}(\alpha)$, then there exists $1 \leqslant q < n(n-1)/2$ such that each element of $[\alpha^q]^U$ has a standard reduction system.*
(vi) *If $t_{\sup}(\alpha_{\mathrm{ext}}) < t_{\sup}(\alpha)$, then there exists $1 \leqslant q < n(n-1)/2$ such that each element of $[\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.*

Roughly speaking, the first statement of the above theorem says that if the outermost component α_{ext} is simpler than the whole braid α up to conjugacy from a Garside-theoretic point of view, then every element of $[\alpha]^U$ has a standard reduction system. In this case, finding a reduction system is as easy as finding one element in the ultra summit set, because it is easy to find a standard reduction system of a given braid if it exists by the results in [BGN93]. In Section 7, we present three examples showing that the conditions in Theorem 7.4 cannot be weakened.

In [BGN95], Benardete, Gutiérrez and Nitecki showed that *if a braid is reducible, then there exists an element in its super summit set which has a standard reduction system.* (The notion of ultra summit set appeared later than their work, and from their proof we can replace 'super summit set' in their statement with 'ultra summit set.') While their result concerns the *existence* of an ultra summit element with a standard reduction system, Theorem 7.4(i)–(iii) show that, under a certain condition, *every* ultra summit element has a standard reduction system.

We remark that the six types of braids in Theorem 7.4 cover most reducible braids. The braid α_{ext} can be obtained, up to conjugacy, by deleting some strands from α, hence α_{ext} cannot be more complicated than α. Indeed, the following inequalities always hold (see Lemma 5.3):

$$\inf_s(\alpha_{\text{ext}}) \geqslant \inf_s(\alpha); \qquad \sup_s(\alpha_{\text{ext}}) \leqslant \sup_s(\alpha);$$
$$t_{\inf}(\alpha_{\text{ext}}) \geqslant t_{\inf}(\alpha); \qquad t_{\sup}(\alpha_{\text{ext}}) \leqslant t_{\sup}(\alpha).$$

Theorem 7.4 shows the characteristics of the braid conjugacy classes for which at least one of the above inequalities is strict.

We briefly explain the idea of proof of Theorem 7.4.

- In Section 6, we show that if α is a split braid with the minimal word length in the conjugacy class, then the outermost component $\mathcal{R}_{\text{ext}}(\alpha)$ of the canonical reduction system of α is standard. Since a positive braid has the minimal word length in the conjugacy class, we have the following: *if P is a positive split braid, then $\mathcal{R}_{\text{ext}}(P)$ is standard.*
- If a braid α commutes with a non-periodic reducible braid β, then the canonical reduction system of β is a reduction system of α. Combining this with the previous observation, we have the following: *if $\alpha P = P\alpha$ for some positive split braid P, then $\mathcal{R}_{\text{ext}}(P)$ is a standard reduction system of α.*
- If α belongs to the ultra summit set, then there exists a finite sequence $\alpha = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_m = \alpha$ for some $m \geqslant 1$, where $\alpha_{i+1} = A_i \alpha_i A_i^{-1}$ for some permutation braid A_i for $i = 0, \ldots, m-1$. If we let $T = A_{m-1} \cdots A_1 A_0$, then $T\alpha = \alpha T$. Exploiting the $\leqslant_R$-minimal elements of the standardizers $\mathrm{St}(\mathcal{R}_{\text{ext}}(\alpha_i))$, we show that T is a positive split braid if $\inf_s(\alpha_{\text{ext}}) > \inf_s(\alpha)$, from which Theorem 7.4(i) follows. The other statements are proved using this.

1.3. Organization

In Section 2, we review the Garside theory in brad groups. In Section 3, we study the normal form of the braids that send a standard curve system to a standard curve system. In Section 4, we prove Theorems 4.2 and 4.9. In Section 5, we study the properties of the outermost component α_{ext} of a non-periodic reducible braid α. In Section 6, we show that if a split braid has the minimal word length in the conjugacy class, then the outermost component of its canonical reduction system is standard. In Sections 7 and 8, we prove Theorem 7.4, using the results of the previous sections.

2. Garside theory in braid groups

We give necessary definitions and results on Garside theory in braid groups. See [Gar69, Thu92,EM94,BKL98,DP99,Deh02,FG03,Geb05] for details. The n-braid group B_n has the group presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geqslant 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \end{array} \right\rangle,$$

where σ_i is the isotopy class of the positive half Dehn-twist along the straight line segment connecting the punctures i and $i+1$. An n-braid can be regarded as a collection of n strands $l = l_1 \cup \cdots \cup l_n$ in $[0,1] \times D^2$ such that $|l \cap (\{t\} \times D^2)| = n$ for $0 \leqslant t \leqslant 1$ and $l \cap (\{0,1\} \times D^2) = \{0,1\} \times \{1, \ldots, n\}$.

2.1. Positive braid monoid

Let B_n^+ be the monoid generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the defining relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$; $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i-j| = 1$. B_n^+ is a (left and right) cancellative monoid that embeds in B_n under the canonical homomorphism. B_n^+ is called the *positive braid monoid* and its elements are called *positive braids*.

Definition 2.1. The partial orders $\leqslant_L$ and $\leqslant_R$ on B_n^+ are defined as follows: for $P, Q \in B_n^+$, $P \leqslant_L Q$ if $Q = PS$ for some $S \in B_n^+$; $P \leqslant_R Q$ if $Q = SP$ for some $S \in B_n^+$.

It is known that the posets $(B_n^+, \leqslant_L)$ and $(B_n^+, \leqslant_R)$ are lattices. Let $\wedge_L$ and $\vee_L$ (respectively, $\wedge_R$ and $\vee_R$) denote the gcd and the lcm with respect to $\leqslant_L$ (respectively, $\leqslant_R$). For positive braids P_1 and P_2, the gcd $P_1 \wedge_R P_2$ and the lcm $P_1 \vee_R P_2$ are characterized by the following properties:

- $P_1 = Q_1(P_1 \wedge_R P_2)$ and $P_2 = Q_2(P_1 \wedge_R P_2)$ for some $Q_1, Q_2 \in B_n^+$ with $Q_1 \wedge_R Q_2 = 1$;
- $P_1 \vee_R P_2 = R_1P_1 = R_2P_2$ for some $R_1, R_2 \in B_n^+$ with $R_1 \wedge_L R_2 = 1$.

The partial orders $\leqslant_L$ and $\leqslant_R$, and thus the lattice structures in B_n^+ can be extended to B_n as follows: for $\alpha, \beta \in B_n$, $\alpha \leqslant_L \beta$ if $\beta = \alpha P$ for some $P \in B_n^+$; $\alpha \leqslant_R \beta$ if $\beta = P\alpha$ for some $P \in B_n^+$.

Definition 2.2. The braid $\Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1$ is called the *fundamental braid* (or the *Garside element*). Let $\mathcal{D} = \{A \in B_n^+ \colon\ A \leqslant_L \Delta\}$. The elements of $\mathcal{D}$ are called *permutation braids* (or *simple elements*).

The fundamental braid Δ has the following properties: $A \leqslant_L \Delta$ if and only if $A \leqslant_R \Delta$ for $A \in B_n^+$; $\Delta \leqslant_L P$ if and only if $\Delta \leqslant_R P$ for $P \in B_n^+$; $\sigma_i \leqslant_L \Delta$ and $\sigma_i\Delta = \Delta\sigma_{n-i}$ for $i = 1, \ldots, n-1$. Permutation n-braids are in one-to-one correspondence with n-permutations: for an n-permutation θ, the diagram (in $[0,1] \times \mathbb{R}$) of the corresponding braid is obtained by connecting $(1,i) \in \{1\} \times \mathbb{R}$ to $(0, \theta(i)) \in \{0\} \times \mathbb{R}$ by a straight line for each $i = 1, \ldots, n$ and then making the ith strand lie above the jth strand whenever $i < j$.

For $P \in B_n^+$, let $\mathrm{s}_L(P) = P \wedge_L \Delta$ and $\mathrm{s}_R(P) = P \wedge_R \Delta$. It is known that for $P, Q \in B_n^+$,

$$\mathrm{s}_L(PQ) = \mathrm{s}_L\big(P\,\mathrm{s}_L(Q)\big) \quad \text{and} \quad \mathrm{s}_R(PQ) = \mathrm{s}_R\big(\mathrm{s}_R(P)Q\big).$$

For $\alpha \in B_n$, there are integers $u \leqslant v$ such that $\Delta^u \leqslant_L \alpha \leqslant_L \Delta^v$. Let $\inf(\alpha) = \max\{u \in \mathbb{Z}\colon \Delta^u \leqslant_L \alpha\}$ and $\sup(\alpha) = \min\{v \in \mathbb{Z}\colon \alpha \leqslant_L \Delta^v\}$.

Definition 2.3. The expression $\Delta^u A_1 \cdots A_m$ is called the *left* (*respectively, right*) *normal form* of α if $u = \inf(\alpha)$, $A_i \in \mathcal{D} \setminus \{1, \Delta\}$ and $\mathrm{s}_L(A_i \cdots A_m) = A_i$ (respectively, $\mathrm{s}_R(A_1 \cdots A_i) = A_i$) for $i = 1, \ldots, m$.

Definition 2.4. For $P \in B_n^+$, the *starting set* $S(P)$ and the *finishing set* $F(P)$ of P are defined as

$$S(P) = \{i \mid \sigma_i \leqslant_L P\} \quad \text{and} \quad F(P) = \{i \mid \sigma_i \leqslant_R P\}.$$

The following properties are well known [Thu92,EM94].

Lemma 2.5.

(i) *For a positive braid P, $S(\mathrm{s}_L(P)) = S(P)$.*
(ii) *If A is a permutation braid with induced permutation θ,*

$$S(A) = \bigl\{i \bigm| \theta^{-1}(i) > \theta^{-1}(i+1)\bigr\} \quad \textit{and} \quad F(A) = \bigl\{i \bigm| \theta(i) > \theta(i+1)\bigr\}.$$

(iii) *For permutation braids A and B, the expression AB is in left* (*respectively, right*) *normal form if and only if $F(A) \supset S(B)$* (*respectively, $F(A) \subset S(B)$*).

By Thurston [Thu92], an n-braid α has a unique expression

$$\alpha = P^{-1}Q,$$

where $P, Q \in B_n^+$ and $P \wedge_L Q = 1$. We call it the *np*-form of α. Similarly, we define the *pn*-form of α as $\alpha = PQ^{-1}$, where $P, Q \in B_n^+$ and $P \wedge_R Q = 1$.

Let τ be the inner automorphism of B_n defined by $\tau(\sigma_i) = \sigma_{n-i}$. Then $\Delta^{-1}\alpha\Delta = \tau(\alpha)$ for $\alpha \in B_n$. The following is known [Cha95, Lemma 2.3].

Lemma 2.6. *Let $P, Q \in B_n^+$. For $A \in \mathcal{D}$, let $\bar{A} = \Delta A^{-1}$.*

(i) *Let $P = A_m A_{m-1} \cdots A_1$ and $Q = A_{m+1} A_{m+2} \cdots A_l$ be in left normal forms. If $P^{-1}Q$ is in np-form, then $\Delta^{-m}\tau^{1-m}(\bar{A}_1) \cdots \tau^{-1}(\bar{A}_{m-1})\bar{A}_m A_{m+1} \cdots A_l$ is the left normal form of $P^{-1}Q$.*
(ii) *Let $P = A_1 A_2 \cdots A_m$ and $Q = A_l A_{l-1} \cdots A_{m+1}$ be in right normal forms. If PQ^{-1} is in pn-form, then $\Delta^{m-l}\tau^{m-l}(A_1) \cdots \tau^{m-l}(A_m)\tau^{m-l+1}(\bar{A}_{m+1}) \cdots \tau^{-1}(\bar{A}_{l-1})\bar{A}_l$ is the right normal form of PQ^{-1}.*

2.2. Conjugacy problem in braid groups

Let $\Delta^u A_1 \cdots A_m$ be the left normal form of $\alpha \in B_n$. The *cycling* $\mathbf{c}(\alpha)$ and the *decycling* $\mathbf{d}(\alpha)$ are defined by

$$\mathbf{c}(\alpha)=\Delta^u A_2\cdots A_m\tau^{-u}(A_1);$$
$$\mathbf{d}(\alpha)=\Delta^u\tau^u(A_m)A_1\cdots A_{m-1}.$$

Let $[\alpha]$ denote the conjugacy class of α. Let $\inf_s(\alpha)=\max\{\inf(\beta)\colon\ \beta\in[\alpha]\}$ and $\sup_s(\alpha)=\min\{\sup(\beta)\colon\ \beta\in[\alpha]\}$.

Definition 2.7. For $\alpha\in B_n$, the *super summit set* $[\alpha]^S$, the *ultra summit set* $[\alpha]^U$ and the *stable super summit set* $[\alpha]^{St}$ of α are defined as follows:

$$[\alpha]^S=\bigl\{\beta\in[\alpha]\colon\ \inf(\beta)=\inf_s(\alpha),\ \sup(\beta)=\sup_s(\alpha)\bigr\};$$
$$[\alpha]^U=\bigl\{\beta\in[\alpha]^S\colon\ \mathbf{c}^m(\beta)=\beta \text{ for some } m\geqslant 1\bigr\};$$
$$[\alpha]^{St}=\bigl\{\beta\in[\alpha]^S\colon\ \beta^m\in\bigl[\alpha^m\bigr]^S \text{ for all } m\geqslant 1\bigr\}.$$

By definition, $[\alpha]^U$ and $[\alpha]^{St}$ are subsets of $[\alpha]^S$.

Theorem 2.8. *Let $\alpha\in B_n$.*

(i) *If $\mathbf{c}^m(\alpha)=\alpha$ for some $m\geqslant 1$, then* $\inf(\alpha)=\inf_s(\alpha)$.
(ii) *If $\mathbf{d}^m(\alpha)=\alpha$ for some $m\geqslant 1$, then* $\sup(\alpha)=\sup_s(\alpha)$.
(iii) $\mathbf{c}^m\mathbf{d}^l(\alpha)\in[\alpha]^U$ *for some $m,l\geqslant 0$.*
(iv) *Both $[\alpha]^S$ and $[\alpha]^U$ are finite and nonempty.*
(v) *If $\beta\in[\alpha]^S$, then* $\mathbf{c}(\beta),\mathbf{d}(\beta),\tau(\beta)\in[\alpha]^S$. *The same is true for $[\alpha]^U$.*
(vi) *If $\beta\in[\alpha]^S$, then* $\mathbf{c}(\mathbf{d}(\alpha))=\mathbf{d}(\mathbf{c}(\alpha))$, $\tau(\mathbf{c}(\beta))=\mathbf{c}(\tau(\beta))$ *and* $\tau(\mathbf{d}(\beta))=\mathbf{d}(\tau(\beta))$.
(vii) *For $\beta,\beta'\in[\alpha]^S$, there is a finite sequence*

$$\beta=\beta_0\to\beta_1\to\cdots\to\beta_m=\beta'$$

such that for $i=0,\dots,m-1$, $\beta_i\in[\alpha]^S$ and $\beta_{i+1}=A_i\beta_iA_i^{-1}$ for some $A_i\in\mathcal{D}$. The same is true for $[\alpha]^U$.

For the results on stable super summit sets, see [LL06a,LL06b]. For $\beta\in[\alpha]^S$, let

$$C^S(\beta)=\bigl\{P\in B_n^+\colon\ P^{-1}\beta P\in[\beta]^S\bigr\};$$
$$C^U(\beta)=\bigl\{P\in B_n^+\colon\ P^{-1}\beta P\in[\beta]^U\bigr\}.$$

Both $C^S(\beta)$ and $C^U(\beta)$ are closed under $\wedge_L$ by Franco and González-Meneses [FG03] and Gebhardt [Geb05], respectively. The closedness under $\wedge_L$ makes the conjugacy algorithm more efficient.

For a nonempty subset $\mathcal{V}$ of B_n^+, we call an element $P\in\mathcal{V}$ the $\leqslant_R$*-minimal element* of $\mathcal{V}$ if $P\leqslant_R Q$ for all $Q\in\mathcal{V}$. By definition, the $\leqslant_R$-minimal element is unique if it exists. If $\mathcal{V}$ is closed under $\wedge_R$, then $\mathcal{V}$ has the $\leqslant_R$-minimal element.

The following notions are useful in studying powers [LL07a,LL06b]. For $\alpha\in B_n$, let

$$t_{\inf}(\alpha)=\lim_{m\to\infty}\frac{\inf(\alpha^m)}{m}\quad\text{and}\quad t_{\sup}(\alpha)=\lim_{m\to\infty}\frac{\sup(\alpha^m)}{m}.$$

The following lists important properties of $t_{\inf}(\cdot)$ and $t_{\sup}(\cdot)$. See Lemmas 3.2, 3.3, Theorem 3.13 in [LL07a], and Corollary 3.5 in [LL06b].

Theorem 2.9. *Let $\alpha \in B_n$.*

(i) $t_{\inf}(\gamma\alpha\gamma^{-1}) = t_{\inf}(\alpha)$ *and* $t_{\sup}(\gamma\alpha\gamma^{-1}) = t_{\sup}(\alpha)$ *for all* $\gamma \in B_n$.
(ii) $t_{\inf}(\alpha^m) = mt_{\inf}(\alpha)$ *and* $t_{\sup}(\alpha^m) = mt_{\sup}(\alpha)$ *for all* $m \geqslant 1$.
(iii) $\inf_s(\alpha) \leqslant t_{\inf}(\alpha) < \inf_s(\alpha) + 1$ *and* $\sup_s(\alpha) - 1 < t_{\sup}(\alpha) \leqslant \sup_s(\alpha)$.
(iv) $t_{\inf}(\alpha)$ *and* $t_{\sup}(\alpha)$ *are rational of the form* p/q *for some integers* p, q *with* $1 \leqslant q \leqslant n(n-1)/2$.

2.3. Duality between cycling and decycling

In many aspects, the cycling and the decycling are dual to each other. We define a variant of the cycling as follows so that the duality is more clear. See Lemmas 2.11 and 2.13.

Definition 2.10. For $\alpha \in B_n$, define $\mathbf{c}_0(\alpha) = \tau^{-1}(\mathbf{c}(\alpha))$.

Since $\tau^2(\beta) = \beta$ and $\tau(\mathbf{c}(\beta)) = \mathbf{c}(\tau(\beta))$ for $\beta \in [\alpha]^S$, we can replace $\mathbf{c}$ with $\mathbf{c}_0$ in Theorem 2.8 and in the definition of $[\alpha]^U$. In particular, for an element $\beta \in [\alpha]^S$, β belongs to the ultra summit set $[\alpha]^U$ if and only if $\mathbf{c}_0^m(\beta) = \beta$ for some $m \geqslant 1$.

Lemma 2.11. *Let $\Delta^u A_1 \cdots A_m$ be the left normal form of $\alpha \in B_n$.*

(i) *The set $\{P \in B_n^+\colon \inf(P\alpha) > \inf(\alpha)\}$ is nonempty and closed under $\wedge_R$. The $\leqslant_R$-minimal element A of this set is the permutation braid $\tau^{-u}(\Delta A_1^{-1})$ and satisfies $\mathbf{c}_0(\alpha) = A\alpha A^{-1}$.*
(ii) *The set $\{P \in B_n^+\colon \sup(\alpha P^{-1}) < \sup(\alpha)\}$ is nonempty and closed under $\wedge_R$. The $\leqslant_R$-minimal element A of this set is the permutation braid A_m and satisfies $\mathbf{d}(\alpha) = A\alpha A^{-1}$.*

Proof. We prove only (i) since (ii) can be proved similarly. Nonemptiness of $\{P \in B_n^+\colon \inf(P\alpha) > \inf(\alpha)\}$ is clear. Note that

- $(\beta\alpha) \wedge_R (\gamma\alpha) = (\beta \wedge_R \gamma)\alpha$ for all $\alpha, \beta, \gamma \in B_n$;
- $\inf(\alpha \wedge_R \beta) = \min\{\inf(\alpha), \inf(\beta)\}$ for all $\alpha, \beta \in B_n$.

If $\inf(P\alpha) > \inf(\alpha)$ and $\inf(Q\alpha) > \inf(\alpha)$ for positive braids P and Q, then

$$\inf\bigl((P \wedge_R Q)\alpha\bigr) = \inf\bigl((P\alpha) \wedge_R (Q\alpha)\bigr) = \min\bigl\{\inf(P\alpha), \inf(Q\alpha)\bigr\} > \inf(\alpha).$$

Therefore, the set $\{P \in B_n^+\colon \inf(P\alpha) > \inf(\alpha)\}$ is closed under $\wedge_R$.

It is easy to see that the $\leqslant_R$-minimal element A is $\tau^{-u}(\Delta A_1^{-1})$ and, hence,

$$\begin{aligned} A\alpha A^{-1} &= \bigl(\Delta\tau^{-u}\bigl(A_1^{-1}\bigr)\bigr)\bigl(\Delta^u A_1 \cdots A_m\bigr)\bigl(\tau^{-u}(A_1)\Delta^{-1}\bigr) \\ &= \Delta\bigl(\Delta^u A_2 \cdots A_m \tau^{-u}(A_1)\bigr)\Delta^{-1} = \Delta\mathbf{c}(\alpha)\Delta^{-1} = \tau^{-1}\bigl(\mathbf{c}(\alpha)\bigr) \\ &= \mathbf{c}_0(\alpha). \qquad \square \end{aligned}$$

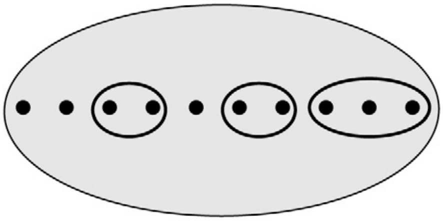


Fig. 2. The unnested standard curve system $\mathcal{C}_{\mathbf{n}}$ for $\mathbf{n} = (1, 1, 2, 1, 2, 3)$.

Definition 2.12. For $\alpha \in B_n$, the set

$$[\alpha]^U_{\mathbf{d}} = \{\beta \in [\alpha]^S \colon\ \mathbf{d}^m(\beta) = \beta \text{ for some } m \geqslant 1\}$$

is called the *ultra summit set of α with respect to decycling*.

The following lemma is easy to prove, so we omit the proof. It shows that there is a duality between $\mathbf{c}_0(\cdot) \leftrightarrow \mathbf{d}(\cdot)$, $\inf(\cdot) \leftrightarrow \sup(\cdot)$ and $[\cdot]^U \leftrightarrow [\cdot]^U_{\mathbf{d}}$.

Lemma 2.13. *Let $\alpha \in B_n$.*

(i) $\inf(\alpha) = -\sup(\alpha^{-1})$ *and* $\inf_s(\alpha) = -\sup_s(\alpha^{-1})$.
(ii) $\mathbf{c}_0(\alpha) = (\mathbf{d}(\alpha^{-1}))^{-1}$.
(iii) $\beta \in [\alpha]^S$ *if and only if* $\beta^{-1} \in [\alpha^{-1}]^S$.
(iv) $\beta \in [\alpha]^U$ *if and only if* $\beta^{-1} \in [\alpha^{-1}]^U_{\mathbf{d}}$.

3. Braids sending a standard curve to a standard curve

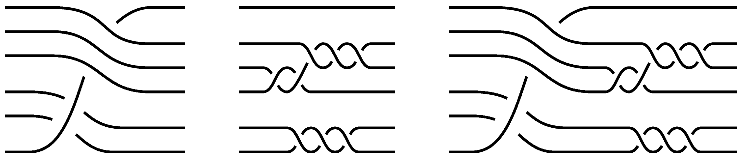
In this section we study the normal form of braids that send a standard curve system to a standard curve system. We collect basic properties of such braids in Lemma 3.5, from which the other results of this section follow easily.

We start by defining some notions. Throughout the paper, we do not distinguish the curves and the isotopy classes of curves.

Definition 3.1. A curve system means a finite collection of disjoint simple closed curves. A simple closed curve in D_n is said to be *essential* if it is homotopic neither to a point nor to a puncture nor to the boundary. An essential curve system in D_n is said to be *standard* if each component is isotopic to a round circle centered at the real axis as in Fig. 1. It is said to be *unnested* if none of its components encloses another component. See Fig. 2.

Definition 3.2. The n-braid group B_n acts on the set of curve systems in D_n. Let $\alpha * \mathcal{C}$ denote the left action of $\alpha \in B_n$ on the curve system $\mathcal{C}$ in D_n. An n-braid α is said to be *reducible* if $\alpha * \mathcal{C} = \mathcal{C}$ for some essential curve system $\mathcal{C}$ in D_n. Such a curve system $\mathcal{C}$ is called a *reduction system* of α.

The unnested standard curve systems in D_n are in one-to-one correspondence with the k-compositions of n for $2 \leqslant k \leqslant n-1$. Recall that an ordered k-tuple $\mathbf{n} = (n_1, \ldots, n_k)$ is a k-composition of n if $n_i \geqslant 1$ for each i and $n = n_1 + \cdots + n_k$.



(a) $\langle\sigma_1^{-1}\sigma_2\rangle_{\mathbf{n}}$. (b) $(\sigma_1^3 \oplus \sigma_1^{-2}\sigma_2^3 \oplus 1)$. (c) $\langle\sigma_1^{-1}\sigma_2\rangle_{\mathbf{n}}(\sigma_1^3 \oplus \sigma_1^{-2}\sigma_2^3 \oplus 1)$.

Fig. 3. $\mathbf{n} = (2, 3, 1)$.

Definition 3.3. For a composition $\mathbf{n} = (n_1, \dots, n_k)$ of n, let $\mathcal{C}_{\mathbf{n}}$ denote the curve system $\bigcup_{n_i \geqslant 2} C_i$, where C_i is the standard curve enclosing $\{m\colon \sum_{j=1}^{i-1} n_j < m \leqslant \sum_{j=1}^{i} n_j\}$. See Fig. 2.

The k-braid group B_k acts on the set of k-compositions of n via the induced permutations: for a k-composition $\mathbf{n} = (n_1, \dots, n_k)$ and $\alpha_0 \in B_k$ with induced permutation θ, $\alpha_0 * \mathbf{n} = (n_{\theta^{-1}(1)}, \dots, n_{\theta^{-1}(k)})$.

Definition 3.4. Let $\mathbf{n} = (n_1, \dots, n_k)$ be a composition of n.

- Let $\alpha_0 = l_1 \cup \dots \cup l_k$ be a k-braid with $l_i \cap (\{1\} \times D^2) = \{(1, i)\}$ for each i. Note that the strands of α_0 are numbered from bottom to top at its right end. We define $\langle\alpha_0\rangle_{\mathbf{n}}$ as the n-braid obtained from α_0 by taking n_i parallel copies of l_i for each i.
- Let $\alpha_i \in B_{n_i}$ for $i = 1, \dots, k$. We define $(\alpha_1 \oplus \dots \oplus \alpha_k)$ as the n-braid $\alpha_1'\alpha_2' \cdots \alpha_k'$, where α_i' is the image of α_i under the homomorphism $B_{n_i} \to B_n$ defined by $\sigma_j \mapsto \sigma_{n_1+\dots+n_{i-1}+j}$.

We will use the notation $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k)$ throughout the paper. See Fig. 3.

Lemma 3.5. *Let $\mathbf{n} = (n_1, \dots, n_k)$ be a composition of n.*

(i) *The expression $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k)$ is unique, i.e., if $\langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k) = \langle\beta_0\rangle_{\mathbf{n}}(\beta_1 \oplus \dots \oplus \beta_k)$, then $\alpha_i = \beta_i$ for $i = 0, \dots, k$.*

(ii) *If $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k)$, then $\alpha * \mathcal{C}_{\mathbf{n}}$ is standard and, further, $\alpha * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\alpha_0 * \mathbf{n}}$. Conversely, if $\alpha * \mathcal{C}_{\mathbf{n}}$ is standard, then α can be expressed as $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k)$.*

(iii) *Let $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k)$. If all α_i's are positive (respectively, permutation and fundamental) braids, then so is α.*

(iv) *$\langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \dots \oplus \alpha_k) = (\alpha_{\theta^{-1}(1)} \oplus \dots \oplus \alpha_{\theta^{-1}(k)})\langle\alpha_0\rangle_{\mathbf{n}}$, where θ is the induced permutation of α_0.*

(v) *$\langle\alpha_0\beta_0\rangle_{\mathbf{n}} = \langle\alpha_0\rangle_{\beta_0 * \mathbf{n}}\langle\beta_0\rangle_{\mathbf{n}}$.*

(vi) *$(\langle\alpha_0\rangle_{\mathbf{n}})^{-1} = \langle\alpha_0^{-1}\rangle_{\alpha_0 * \mathbf{n}}$.*

(vii) *$(\alpha_1\beta_1 \oplus \dots \oplus \alpha_k\beta_k) = (\alpha_1 \oplus \dots \oplus \alpha_k)(\beta_1 \oplus \dots \oplus \beta_k)$.*

(viii) *$(\alpha_1 \oplus \dots \oplus \alpha_k)^{-1} = (\alpha_1^{-1} \oplus \dots \oplus \alpha_k^{-1})$.*

(ix) *Let A_0 and B_0 be permutation k-braids. A_0B_0 is in left (respectively, right) normal form if and only if $\langle A_0\rangle_{B_0 * \mathbf{n}}\langle B_0\rangle_{\mathbf{n}}$ is in left (respectively, right) normal form.*

(x) *Let P_i, $i = 0, \dots, k$, be positive braids with appropriate braid indices. Let $A_i = \mathrm{s}_L(P_i)$ and $B_i = \mathrm{s}_R(P_i)$ for $i = 0, \dots, k$. Then*

$$\mathrm{s}_L\big((P_1 \oplus \dots \oplus P_k)\langle P_0\rangle_{\mathbf{n}}\big) = (A_1 \oplus \dots \oplus A_k)\langle A_0\rangle_{(A_0^{-1}P_0) * \mathbf{n}};$$

$$\mathrm{s}_R\big(\langle P_0\rangle_{\mathbf{n}}(P_1 \oplus \dots \oplus P_k)\big) = \langle B_0\rangle_{\mathbf{n}}(B_1 \oplus \dots \oplus B_k).$$

Proof. The statements from (i) to (viii) are easy to prove. Let us prove (ix) and (x).

(ix) Let $B_0 * \mathbf{n} = (n'_1, \dots, n'_k)$ and $N_i = n'_1 + \cdots + n'_i$ for $i = 1, \dots, k$. Then,

$$F\big(\langle A_0\rangle_{B_0*\mathbf{n}}\big) = \big\{N_i\colon\ i \in F(A_0)\big\};$$
$$S\big(\langle B_0\rangle_{\mathbf{n}}\big) = \big\{N_i\colon\ i \in S(B_0)\big\}.$$

Hence, $F(A_0) \supset S(B_0)$ if and only if $F(\langle A_0\rangle_{B_0*\mathbf{n}}) \supset S(\langle B_0\rangle_{\mathbf{n}})$, and $F(A_0) \subset S(B_0)$ if and only if $F(\langle A_0\rangle_{B_0*\mathbf{n}}) \subset S(\langle B_0\rangle_{\mathbf{n}})$.

(x) We prove only the second identity. The first one can be proved in a similar way. It is easy to see that $\mathrm{s}_R(\langle P_0\rangle_{\mathbf{n}}) = \langle B_0\rangle_{\mathbf{n}}$ by (ix) and that $\mathrm{s}_R(P_1 \oplus \cdots \oplus P_k) = (B_1 \oplus \cdots \oplus B_k)$. Let θ be the induced permutation of B_0. Then, by (iv)

$$\begin{aligned}
\mathrm{s}_R\big(\langle P_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k)\big) &= \mathrm{s}_R\big(\mathrm{s}_R\big(\langle P_0\rangle_{\mathbf{n}}\big)(P_1 \oplus \cdots \oplus P_k)\big)\\
&= \mathrm{s}_R\big(\langle B_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k)\big) = \mathrm{s}_R\big(\big(P_{\theta^{-1}(1)} \oplus \cdots \oplus P_{\theta^{-1}(k)}\big)\langle B_0\rangle_{\mathbf{n}}\big)\\
&= \mathrm{s}_R\big(\mathrm{s}_R\big(P_{\theta^{-1}(1)} \oplus \cdots \oplus P_{\theta^{-1}(k)}\big)\langle B_0\rangle_{\mathbf{n}}\big)\\
&= \mathrm{s}_R\big(\big(B_{\theta^{-1}(1)} \oplus \cdots \oplus B_{\theta^{-1}(k)}\big)\langle B_0\rangle_{\mathbf{n}}\big)\\
&= \mathrm{s}_R\big(\langle B_0\rangle_{\mathbf{n}}(B_1 \oplus \cdots \oplus B_k)\big) = \langle B_0\rangle_{\mathbf{n}}(B_1 \oplus \cdots \oplus B_k).
\end{aligned}$$

The last equality holds since $\langle B_0\rangle_{\mathbf{n}}(B_1 \oplus \cdots \oplus B_k)$ is a permutation braid by (iii). □

Let $\mathrm{br}(\alpha)$ denote the braid index of α.

Lemma 3.6. *Let $\alpha = \langle \alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k) \in B_n$.*

(i) $\inf(\alpha) = \min\{\inf(\alpha_i)\colon\ i = 0, \dots, k, \mathrm{br}(\alpha_i) \geqslant 2\}$.
(ii) $\sup(\alpha) = \max\{\sup(\alpha_i)\colon\ i = 0, \dots, k, \mathrm{br}(\alpha_i) \geqslant 2\}$.
(iii) *α is a positive (respectively, permutation and fundamental) braid if and only if each α_i is a positive (respectively, permutation and fundamental) braid for $i = 0, \dots, k$.*

Proof. (i) Let $r = \min\{\inf(\alpha_i)\colon\ i = 0, \dots, k, \mathrm{br}(\alpha_i) \geqslant 2\}$. Set $n_0 = k$. For $i = 0, \dots, k$, let $\alpha_i = \Delta_i^r P_i$, where Δ_i is the fundamental braid of B_{n_i} and $P_i \in B_{n_i}^+$. Let $P = \langle P_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k)$. By Lemma 3.5(iv), (v) and (vii),

$$\begin{aligned}
\alpha &= \big\langle \Delta_0^r P_0\big\rangle_{\mathbf{n}}\big(\Delta_1^r P_1 \oplus \cdots \oplus \Delta_k^r P_k\big)\\
&= \big\langle \Delta_0^r\big\rangle_{P_0*\mathbf{n}}\langle P_0\rangle_{\mathbf{n}}\big(\Delta_1^r \oplus \cdots \oplus \Delta_k^r\big)(P_1 \oplus \cdots \oplus P_k)\\
&= \big\langle \Delta_0^r\big\rangle_{P_0*\mathbf{n}}\big(\Delta_{\theta^{-1}(1)}^r \oplus \cdots \oplus \Delta_{\theta^{-1}(k)}^r\big)\langle P_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k)
\end{aligned}$$

where θ is the induced permutation of P_0. Since $P_0 * \mathbf{n} = (n_{\theta^{-1}(1)}, \dots, n_{\theta^{-1}(k)})$, we have $\langle \Delta_0^r\rangle_{P_0*\mathbf{n}}(\Delta_{\theta^{-1}(1)}^r \oplus \cdots \oplus \Delta_{\theta^{-1}(k)}^r) = \Delta^r$, and hence $\alpha = \Delta^r P$. Since $\inf(P_i) = 0$ for some P_i with $\mathrm{br}(P_i) \geqslant 2$, $\mathrm{s}_R(P) \neq \Delta$ by Lemma 3.5(x). Therefore $\inf(\alpha) = r$.

(ii) Since $\sup(\alpha) = -\inf(\alpha^{-1})$ by Lemma 2.13(i) and $\alpha^{-1} = (\alpha_1^{-1} \oplus \cdots \oplus \alpha_k^{-1})\langle \alpha_0^{-1}\rangle_{\alpha_0*\mathbf{n}}$ by Lemma 3.5(vi) and (viii), the assertion follows from (i).

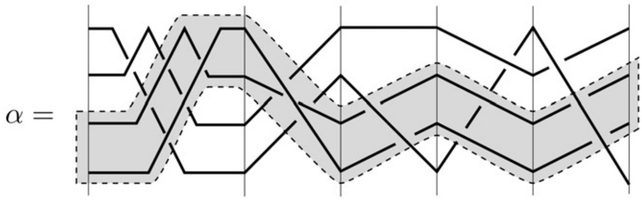


Fig. 4. The 4-braid α, whose normal form is of the form $\Delta^{-1}A_1A_2A_3A_4$, sends the standard curve system $\mathcal{C}_{(1,2,1)}$ to the standard curve system $\mathcal{C}_{(2,1,1)}$ as follows: $\mathcal{C}_{(2,1,1)} \xleftarrow{\Delta^{-1}} \mathcal{C}_{(1,1,2)} \xleftarrow{A_1} \mathcal{C}_{(2,1,1)} \xleftarrow{A_2} \mathcal{C}_{(1,2,1)} \xleftarrow{A_3} \mathcal{C}_{(2,1,1)} \xleftarrow{A_4} \mathcal{C}_{(1,2,1)}$.

(iii) Note that a braid β is a positive (respectively, permutation and fundamental) braid if and only if $\inf(\beta) \geqslant 0$ (respectively, $0 \leqslant \inf(\beta) \leqslant \sup(\beta) \leqslant 1$ and $\inf(\beta) = \sup(\beta) = 1$). Therefore, the assertion follows from (i) and (ii) and Lemma 3.5(iii). □

Lemma 3.7. *Let $\mathcal{C}$ be a standard curve system in D_n and $P \in B_n^+$ such that $P * \mathcal{C}$ is standard.*

(i) *If $P = QA$ and $A = \mathrm{s}_R(P)$, then $A * \mathcal{C}$ is standard.*
(ii) *If $P = AQ$ and $A = \mathrm{s}_L(P)$, then $Q * \mathcal{C}$ is standard.*

Proof. A curve system is standard if and only if each of its components is standard. Hence, we may assume that the given standard curve system $\mathcal{C}$ is unnested. Let $\mathcal{C} = \mathcal{C}_{\mathbf{n}}$ for a composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n.

(i) $P = \langle P_0\rangle_{\mathbf{n}}(P_1 \oplus \cdots \oplus P_k)$ for some positive braids P_i, $i = 0, \ldots, k$, by Lemmas 3.5(ii) and 3.6(iii). By Lemma 3.5(x), $A = \mathrm{s}_R(P) = \langle \mathrm{s}_R(P_0)\rangle_{\mathbf{n}}(\mathrm{s}_R(P_1) \oplus \cdots \oplus \mathrm{s}_R(P_k))$. By Lemma 3.5(ii), $A * \mathcal{C}$ is standard.

(ii) $P = (P_1 \oplus \cdots \oplus P_k)\langle P_0\rangle_{\mathbf{n}}$ for some positive braids P_i, $i = 0, \ldots, k$, by Lemmas 3.5(ii), (iv) and 3.6(iii). Let $A_i = \mathrm{s}_L(P_i)$ for $i = 0, \ldots, k$. Then $A = \mathrm{s}_L(P) = (A_1 \oplus \cdots \oplus A_k)\langle A_0\rangle_{(A_0^{-1}P_0)*\mathbf{n}}$ by Lemma 3.5(x). By Lemma 3.5(vi) and (viii), $A^{-1} = \langle A_0^{-1}\rangle_{P_0*\mathbf{n}}(A_1^{-1} \oplus \cdots \oplus A_k^{-1})$. By Lemma 3.5(ii),

$$\begin{aligned} Q * \mathcal{C}_{\mathbf{n}} &= \left(A^{-1}P\right) * \mathcal{C}_{\mathbf{n}} = A^{-1} * (P * \mathcal{C}_{\mathbf{n}}) = A^{-1} * \mathcal{C}_{P_0*\mathbf{n}} \\ &= \left(\left\langle A_0^{-1}\right\rangle_{P_0*\mathbf{n}}\left(A_1^{-1} \oplus \cdots \oplus A_k^{-1}\right)\right) * \mathcal{C}_{P_0*\mathbf{n}} = \mathcal{C}_{(A_0^{-1}P_0)*\mathbf{n}}. \end{aligned}$$

Hence $Q * \mathcal{C}$ is standard. □

Theorem 3.8. *Let $\mathcal{C}$ be a standard curve system in D_n and $\Delta^u A_1 \cdots A_m$ be the* (*left or right*) *normal form of $\alpha \in B_n$. If $\alpha * \mathcal{C}$ is standard, then so is $(A_i \cdots A_m) * \mathcal{C}$ for $i = 1, \ldots, m$.*

Proof. It is an immediate consequence of Lemma 3.7, because $(A_1 \cdots A_m) * \mathcal{C} = \Delta^{-u} * (\alpha * \mathcal{C})$ is standard. □

Roughly speaking, Theorem 3.8 says that if a braid α sends a standard curve system to a standard curve system, then so does each permutation braid in the normal form of α as in Fig. 4.

Corollary 3.9. *Let $\Delta^u A_1 \cdots A_m$ be the left normal form of an n-braid α. If α has a standard reduction system $\mathcal{C}$, then $\mathbf{c}_0(\alpha)$, $\mathbf{d}(\alpha)$ and $\tau(\alpha)$ have standard reduction systems $\tau^{-u}(\Delta A_1^{-1}) * \mathcal{C}$, $A_m * \mathcal{C}$ and $\Delta^{-1} * \mathcal{C}$, respectively.*

Proof. $A_m * \mathcal{C}$ is standard by Theorem 3.8. By Lemma 2.11,

$$\mathbf{d}(\alpha) * (A_m * \mathcal{C}) = \left(A_m \alpha A_m^{-1}\right) * (A_m * \mathcal{C}) = A_m * (\alpha * \mathcal{C}) = A_m * \mathcal{C}.$$

Therefore $\mathbf{d}(\alpha)$ has a standard reduction system $A_m * \mathcal{C}$. In the same way, $\tau(\alpha)$ and $\mathbf{c}_0(\alpha)$ have standard reduction systems $\Delta^{-1} * \mathcal{C}$ and $\tau^{-u}(\Delta A_1^{-1}) * \mathcal{C}$, respectively. $\square$

Corollary 3.10. *Let α be a reducible n-braid with a reduction system $\mathcal{C}$. There exists an element β of the ultra summit set $[\alpha]^U$ which has a standard reduction system. Precisely, there exists a positive braid P such that $\beta = P\alpha P^{-1}$ belongs to $[\alpha]^U$ and $P * \mathcal{C}$ is a standard reduction system of β.*

Proof. Let P_1 be a positive n-braid such that $P_1 * \mathcal{C}$ is standard. Then $P_1 \alpha P_1^{-1}$ has the standard reduction system $P_1 * \mathcal{C}$. Take $l, m \geqslant 0$ such that $\beta = \mathbf{c}_0^l \mathbf{d}^m (P_1 \alpha P_1^{-1})$ belongs to $[\alpha]^U$. Lemma 2.11 and Corollary 3.9 say that if $\gamma \in B_n$ has a standard reduction system $\mathcal{C}'$, then there are permutation braids A_1 and A_2 such that $\mathbf{c}_0(\gamma) = A_1 \gamma A_1^{-1}$ and $\mathbf{d}(\gamma) = A_2 \gamma A_2^{-1}$ have standard reduction systems $A_1 * \mathcal{C}'$ and $A_2 * \mathcal{C}'$, respectively. Hence, we can find a positive n-braid P_2 such that $\beta = P_2(P_1 \alpha P_1^{-1}) P_2^{-1}$ and $P_2 * (P_1 * \mathcal{C}) = (P_2 P_1) * \mathcal{C}$ is standard. Let $P_2 P_1 = P$. Then, $\beta = P\alpha P^{-1}$ and β has the standard reduction system $(P_2 P_1) * \mathcal{C} = P * \mathcal{C}$. $\square$

Corollary 3.11. *Let $\mathcal{C}$ be a standard curve system in D_n, and let $\alpha * \mathcal{C}$ be standard for an n-braid α.*

(i) *If $P^{-1} Q$ is the np-form of α, then $Q * \mathcal{C}$ is standard.*
(ii) *If $P Q^{-1}$ is the pn-form of α, then $Q^{-1} * \mathcal{C}$ is standard.*

Proof. By Lemma 2.6 and Theorem 3.8, $Q * \mathcal{C}$ and $Q^{-1} * \mathcal{C}$ are standard. $\square$

We remark that Theorem 3.8 and Corollary 3.10 were obtained also by Benardete, Gutiérrez and Nitecki [BGN95, Theorems 5.7 and 5.8], and that these two are enough to solve the reducibility problem because there is an efficient algorithm that decides whether a given braid has a standard reduction system or not and finds one if it has [BGN93]. However, Corollary 3.10 guarantees only the *existence* of an element (in the ultra summit set of a reducible braid) that has a standard reduction system. To solve the reducibility problem using only Corollary 3.10, we have to compute all the elements in the ultra summit set.

4. Standardizers of curve systems

Definition 4.1. For an essential curve system $\mathcal{C}$ in D_n, we define the *standardizer* of $\mathcal{C}$ as the set

$$\mathrm{St}(\mathcal{C}) = \left\{P \in B_n^+ \colon\ P * \mathcal{C} \text{ is standard}\right\}.$$

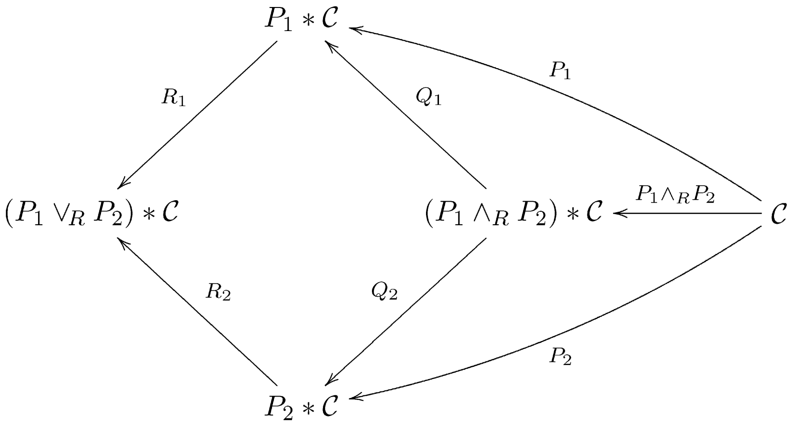


Fig. 5. $\mathcal{C}_1 \xrightarrow{P} \mathcal{C}_2$ means that $\mathcal{C}_2 = P * \mathcal{C}_1$.

This section is devoted to the study of properties of standardizers. Clearly, $\mathrm{St}(\mathcal{C})$ is nonempty for any essential curve system $\mathcal{C}$. Theorem 4.2 shows that standardizers are sublattices of B_n^+, hence they have unique $\leqslant_R$-minimal elements. The main result of this section is Theorem 4.9 that for any reduction system $\mathcal{C}$ of a reducible braid α, conjugating α by the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$ preserves the membership of the super summit set, ultra summit set and stable super summit set. Proposition 4.4 and Corollary 4.5 show that the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$ does not entangle any standard curve disjoint from $\mathcal{C}$. Proposition 4.8 is a characterization of the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$ in terms of normal form and lattice operations.

Theorem 4.2. *For an essential curve system $\mathcal{C}$ in D_n, its standardizer* $\mathrm{St}(\mathcal{C})$ *is closed under $\wedge_R$ and $\vee_R$, and hence a sublattice of B_n^+. Therefore* $\mathrm{St}(\mathcal{C})$ *contains a unique $\leqslant_R$-minimal element.*

Proof. (See Fig. 5.) Let $P_1, P_2 \in \mathrm{St}(\mathcal{C})$. Let $P_1 = Q_1(P_1 \wedge_R P_2)$ and $P_2 = Q_2(P_1 \wedge_R P_2)$ for $Q_1, Q_2 \in B_n^+$ with $Q_1 \wedge_R Q_2 = 1$. Then $P_2 = Q_2(P_1 \wedge_R P_2) = Q_2 Q_1^{-1} P_1$, and $Q_2 Q_1^{-1}$ is in *pn*-form. Since $P_1 * \mathcal{C}$ and $P_2 * \mathcal{C}$ are standard and

$$P_2 * \mathcal{C} = \left(Q_2 Q_1^{-1}\right) * (P_1 * \mathcal{C}),$$

$Q_1^{-1} * (P_1 * \mathcal{C}) = (P_1 \wedge_R P_2) * \mathcal{C}$ is standard by Corollary 3.11(ii).

Let $P_1 \vee_R P_2 = R_1 P_1 = R_2 P_2$ for $R_1, R_2 \in B_n^+$ with $R_1 \wedge_L R_2 = 1$. Then $R_2^{-1} R_1 P_1 = P_2$, and $R_2^{-1} R_1$ is the *np*-form. Since $P_1 * \mathcal{C}$ and $P_2 * \mathcal{C}$ are standard and

$$P_2 * \mathcal{C} = \left(R_2^{-1} R_1\right) * (P_1 * \mathcal{C}),$$

$R_1 * (P_1 * \mathcal{C}) = (P_1 \vee_R P_2) * \mathcal{C}$ is standard by Corollary 3.11(i). □

Let $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be essential curve systems such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then $\mathrm{St}(\mathcal{C}) \subset \mathrm{St}(\mathcal{C}_i)$ for $i = 1, 2$. Let P, P_1 and P_2 be the $\leqslant_R$-minimal elements of $\mathrm{St}(\mathcal{C})$, $\mathrm{St}(\mathcal{C}_1)$ and $\mathrm{St}(\mathcal{C}_2)$, respectively. By Theorem 4.2, $P_1 \leqslant_R P$ and $P_2 \leqslant_R P$, hence $(P_1 \vee_R P_2) \leqslant_R P$. One may expect that $P = P_1 \vee_R P_2$. However, the following example shows that it is not true in general.

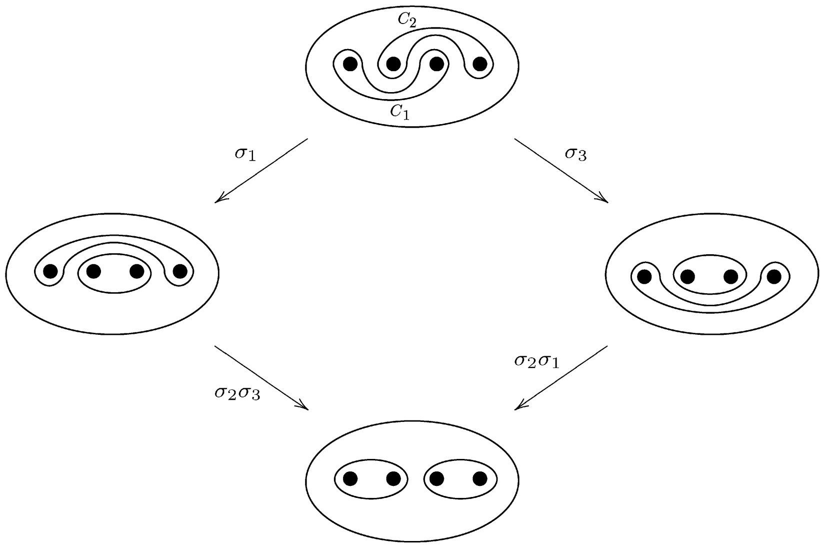


Fig. 6. Standardization of a curve system.

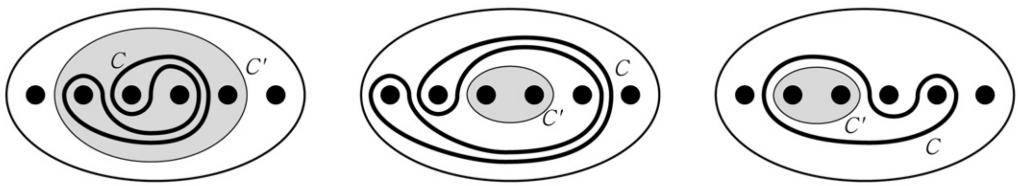


Fig. 7. C is inside C' in the first figure, and outside C' in the other two figures.

Example 4.3. Let C_1 and C_2 be the curves in D_4 as in Fig. 6. The $\leqslant_R$-minimal elements of $\mathrm{St}(C_1)$, $\mathrm{St}(C_2)$ and $\mathrm{St}(C_1 \cup C_2)$ are σ_1, σ_3 and $\sigma_2\sigma_1\sigma_3$, respectively. Note that $\sigma_2\sigma_1\sigma_3$ is not equal to $\sigma_1 \vee_R \sigma_3 = \sigma_1\sigma_3$.

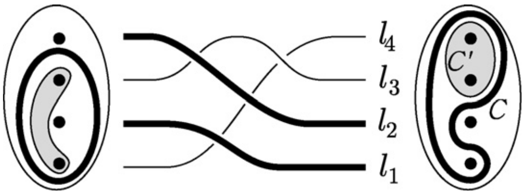
The following proposition shows that, when an essential curve C in D_n is standardized by the action of the $\leqslant_R$-minimal element of $\mathrm{St}(C)$, any other standard curve disjoint from C remains standard.

Proposition 4.4. *Let C be an essential simple closed curve in D_n and let P be the $\leqslant_R$-minimal element of* $\mathrm{St}(C)$. *For any standard curve C' in D_n with $C \cap C' = \emptyset$, the curve $P * C'$ is standard.*

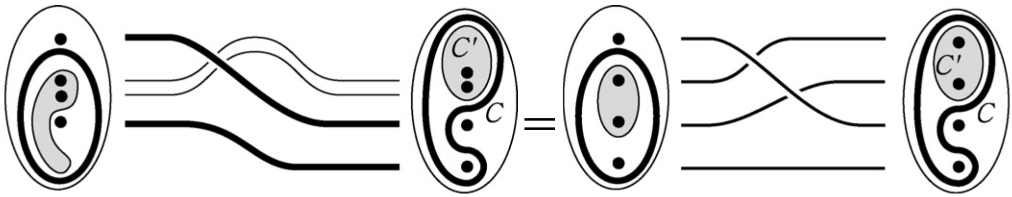
Proof. Let C' be a standard curve which is disjoint from C and encloses the punctures $\{r, r+1, \ldots, r+s\}$. Because C and C' are disjoint, C is either inside C' or outside C' as Fig. 7.

Case 1. C is inside C'.

There exists a positive braid Q written as a positive word on $\sigma_r, \ldots, \sigma_{r+s-1}$ such that $Q * C$ is standard. Since $Q \in \mathrm{St}(C)$ and P is the $\leqslant_R$-minimal element of $\mathrm{St}(C)$, we have $P \leqslant_R Q$,



(a) The braid diagram K of P.



(b) The braid diagram L of Q.

Fig. 8. The positive braid $P = \sigma_1\sigma_3\sigma_2\sigma_3$ standardizes the thick curve C in (a). The strands in $K'' = l_1 \cup l_2$ cross l_3 once and l_4 twice. The braid diagram L is the union of K'' and two parallel copies of l_3. It represents a positive braid Q which standardizes both C and C'.

hence $Q = RP$ for some positive braid R. In particular, P is written as a positive word on $\sigma_r, \dots, \sigma_{r+s-1}$, and hence $P * C' = C'$ is standard.

Case 2. C is outside C'.

For a braid diagram K, let $c(K)$ denote the number of crossings in K. Note that if all the crossings in K are positive, then K represents a positive braid Q with $|Q| = c(K)$, where $|Q|$ denotes the word length of Q with respect to σ_i's.

Claim. *Let C and C' be essential simple closed curves in D_n such that C' is standard and C is outside C'. Let P be an element* (*not necessarily the $\leqslant_R$-minimal element*) *of* $\mathrm{St}(C)$. *Then there is a positive braid Q such that $|Q| \leqslant |P|$ and both $Q * C$ and $Q * C'$ are standard.*

Proof of Claim. See Fig. 8 which illustrates this proof with a simple example. Let $K = l_1 \cup \cdots \cup l_n$ be a braid diagram of P in $[0, 1] \times \mathbb{R}$ such that the number of crossings in K is exactly $|P|$. Here we assume that the right end of l_i is $(1, i)$ for $i = 1, \dots, n$. Let $\{r, r+1, \dots, r+s\}$ be the set of punctures inside C'. Let $K' = l_r \cup l_{r+1} \cup \cdots \cup l_{r+s}$ and $K'' = K \setminus K'$. For $i = r, \dots, r+s$, let e_i be the number of crossings between l_i and K''. Let e_{i_0} be the minimum of $\{e_r, e_{r+1}, \dots, e_{r+s}\}$. Then

$$|P| = c(K) = c(K') + c(K'') + (e_r + \cdots + e_{r+s}) \geqslant c(K'') + (s+1)e_{i_0}.$$

Let L be the braid diagram which is the union of K'' and $(s+1)$ parallel copies of l_{i_0}, and let Q be the positive braid represented by L. Since all the crossings in L are positive,

$$|Q| = c(L) = c(K'') + (s+1)e_{i_0} \leqslant |P|.$$

By the construction of Q, both the curves $Q * C$ and $Q * C'$ are standard. $\square$

By the above claim, there exists a positive braid Q such that $|Q| \leqslant |P|$ and both $Q * C$ and $Q * C'$ are standard. Because P is the $\leqslant_R$-minimal element of $\mathrm{St}(C)$ and $Q * C$ is standard, we have $P \leqslant_R Q$. Since $|Q| \leqslant |P|$, we obtain $P = Q$, hence $P * C'$ is standard. $\square$

Proposition 4.4 says that if we standardize the components of a curve system $\mathcal{C} = C_1 \cup \cdots \cup C_k$ one after another by the $\leqslant_R$-minimal element of the standardizers as follows, then the product of the $\leqslant_R$-minimal elements used in this process is exactly the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$.

(i) Standardize the first component C_1 of $\mathcal{C}$ using the $\leqslant_R$-minimal element P_1 of $\mathrm{St}(C_1)$. Then $P_1 * \mathcal{C} = P_1 * C_1 \cup \cdots \cup P_1 * C_k$ and $P_1 * C_1$ is standard.
(ii) Standardize the second component $P_1 * C_2$ of $P_1 * \mathcal{C}$ by the $\leqslant_R$-minimal element P_2 of $\mathrm{St}(P_1 * C_2)$. Then the first two components $(P_2 P_1) * (C_1 \cup C_2)$ of $(P_2 P_1) * \mathcal{C}$ are standard.
(iii) Continue the above process. Then $(P_k \cdots P_1) * \mathcal{C}$ is standard. Corollary 4.5 shows that in fact $P_k \cdots P_1$ is the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$.

Corollary 4.5. *Let $\mathcal{C}, \mathcal{C}_1, \ldots, \mathcal{C}_k$ be essential curve systems in D_n such that $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$. Let P be the $\leqslant_R$-minimal element of* $\mathrm{St}(\mathcal{C})$.

(i) *If P_i is the $\leqslant_R$-minimal element of* $\mathrm{St}((P_{i-1} \cdots P_1) * \mathcal{C}_i)$*, then $P = P_k P_{k-1} \cdots P_1$.*
(ii) *For any standard curve C' disjoint from $\mathcal{C}$, the curve $P * C'$ is standard.*

Proof. We prove the corollary only for the case when each curve system $\mathcal{C}_i$ has only one component. The general case can be proved easily from this. Suppose that each curve system $\mathcal{C}_i$ has only one component.

Claim. *The following hold for each $i = 0, 1, \ldots, k$.*

(a) *$P_i P_{i-1} \cdots P_1 \leqslant_R P$.*
(b) *The curve $(P_i P_{i-1} \cdots P_1) * \mathcal{C}_j$ is standard for $j = 1, \ldots, i$.*
(c) *For any standard curve C' disjoint from $\mathcal{C}$, the curve $(P_i P_{i-1} \cdots P_1) * C'$ is standard.*

Proof of Claim. The statement is obvious for $i = 0$ since $P_i \cdots P_1$ is the identity. Using induction on i, assume that the statement is true for some i with $0 \leqslant i < k$. Since $P_i \cdots P_1 \leqslant_R P$,

$$P = Q(P_i \cdots P_1)$$

for some $Q \in B_n^+$. Since $Q * ((P_i \cdots P_1) * \mathcal{C}_{i+1}) = P * \mathcal{C}_{i+1}$ is standard and P_{i+1} is the $\leqslant_R$-minimal element of $\mathrm{St}((P_i \cdots P_1) * \mathcal{C}_{i+1})$, we have $P_{i+1} \leqslant_R Q$, hence

$$P_{i+1} P_i \cdots P_1 \leqslant_R Q(P_i \cdots P_1) = P.$$

By the induction hypothesis, $(P_i \cdots P_1) * C'$ and $(P_i \cdots P_1) * \mathcal{C}_j$ are standard curves disjoint from $(P_i \cdots P_1) * \mathcal{C}_{i+1}$ for $j = 1, \ldots, i$. Since P_{i+1} is the $\leqslant_R$-minimal element of $\mathrm{St}((P_i \cdots P_1) * \mathcal{C}_{i+1})$, $(P_{i+1} P_i \cdots P_1) * C'$ and $(P_{i+1} P_i \cdots P_1) * \mathcal{C}_j$ for $j = 1, \ldots, i$ are standard by Proposition 4.4. By definition of P_{i+1}, $(P_{i+1} P_i \cdots P_1) * \mathcal{C}_{i+1}$ is standard. $\square$

By (b) of the above claim, $(P_k P_{k-1} \cdots P_1) * \mathcal{C}$ is standard. Since P is the $\leqslant_R$ minimal element of $\mathrm{St}(\mathcal{C})$, $P \leqslant_R (P_k P_{k-1} \cdots P_1)$. By (a) of the claim, $(P_k P_{k-1} \cdots P_1) \leqslant_R P$, hence

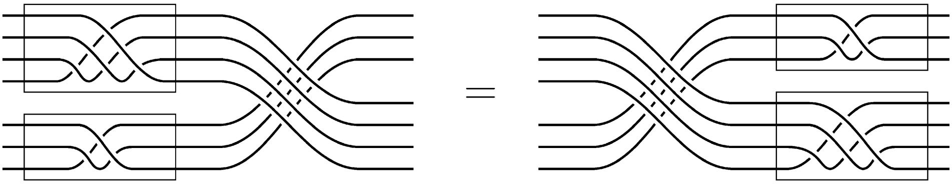


Fig. 9. The figure shows that $\delta_{(3,4)}\langle\sigma_1\rangle_{(4,3)} = \langle\sigma_1\rangle_{(4,3)}\delta_{(4,3)}$.

$P = P_k P_{k-1} \cdots P_1$. By (c) of the claim, $P * C'$ is standard for any standard curve C' disjoint from $\mathcal{C}$. $\square$

In the rest of this section, we use the following definition.

Definition 4.6. For a composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n, we define the symbol $\delta_{\mathbf{n}}$ and non-negative integers $N_0, N_1, \ldots, N_k$ as follows:

- $\delta_{\mathbf{n}} = \Delta_1 \oplus \cdots \oplus \Delta_k$, where Δ_i is the fundamental braid of B_{n_i} for $i = 1, \ldots, k$;
- $N_0 = 0$ and $N_i = n_1 + n_2 + \cdots + n_i$ for $i = 1, \ldots, k$.

Then, for a composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n and $\sigma_i \in B_k$, the following hold.

- If $A \leqslant_L \delta_{\mathbf{n}}$, then $A * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$.
- $S(\delta_{\mathbf{n}}) = F(\delta_{\mathbf{n}}) = \{1, \ldots, n-1\} \setminus \{N_1, \ldots, N_{k-1}\}$.
- $\sigma_i * \mathbf{n} = \sigma_i^{-1} * \mathbf{n} = (n_1, \ldots, n_{i-1}, n_{i+1}, n_i, n_{i+2}, \ldots, n_k)$.
- $\delta_{\mathbf{n}}\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}} = \langle\sigma_i\rangle_{\sigma_i*\mathbf{n}}\delta_{\sigma_i*\mathbf{n}}$. See Fig. 9.

Lemma 4.7. *Let $\mathbf{n} = (n_1, \ldots, n_k)$ be a composition of n.*

(i) *Let A be a permutation n-braid with induced permutation θ. Then $\delta_{\mathbf{n}}A$ is a permutation braid if and only if θ^{-1} is order-preserving on the set $\{N_{i-1}+1, \ldots, N_i\}$ for each $i = 1, \ldots, k$, that is,*

$$\theta^{-1}(N_{i-1}+1) < \theta^{-1}(N_{i-1}+2) < \cdots < \theta^{-1}(N_i).$$

(ii) *For a positive n-braid P, the starting set $S(\delta_{\mathbf{n}}P)$ is strictly greater than the starting set $S(\delta_{\mathbf{n}})$ if and only if $\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}} \leqslant_L P$ for some $i \in \{1, \ldots, k-1\}$.*

Proof. (i) It is an easy consequence of the fact that a positive braid P is a permutation braid if and only if any two of its strands cross at most once [Thu92, Lemma 9.1.10] or [EM94, Lemma 2.3]. See Fig. 10.

(ii) See Fig. 11. Suppose $\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}} \leqslant_L P$ for some $i \in \{1, \ldots, k-1\}$. Then $N_i \in S(\delta_{\mathbf{n}}P)$, hence $S(\delta_{\mathbf{n}}P)$ is strictly greater than $S(\delta_{\mathbf{n}})$. Conversely, suppose that $S(\delta_{\mathbf{n}}P)$ is strictly greater than $S(\delta_{\mathbf{n}})$. Let A be the permutation n-braid such that $\mathrm{s}_L(\delta_{\mathbf{n}}P) = \delta_{\mathbf{n}}A$, that is, $\delta_{\mathbf{n}}A$ is the first permutation braid in the left normal form of $\delta_{\mathbf{n}}P$. Then $N_i \in S(\delta_{\mathbf{n}}A)$ for some $i \in \{1, \ldots, k-1\}$. Let ω and θ be the induced permutations of $\delta_{\mathbf{n}}$ and A respectively. Then

$$\omega^{-1}(N_i) = N_{i-1}+1 \quad \text{and} \quad \omega^{-1}(N_i+1) = N_{i+1}.$$

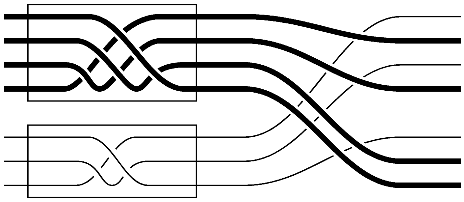


Fig. 10. The figure shows a permutation braid of the form $\delta_{(3,4)}A$ for a permutation braid A. If θ is the induced permutation of A, then θ^{-1} is order-preserving on each of the sets $\{1,2,3\}$ and $\{4,5,6,7\}$.

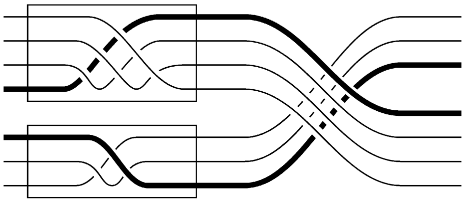


Fig. 11. The figure shows a permutation braid of the form $\delta_{(3,4)}A$ for a permutation braid A. If $3 \in S(\delta_{(3,4)}A)$, then two thick strands cross each other and, hence, $\langle\sigma_1\rangle_{(4,3)} \leqslant_L A$.

Since $N_i \in S(\delta_{\mathbf{n}}A)$, we have $(\omega\theta)^{-1}(N_i) > (\omega\theta)^{-1}(N_i+1)$ and, hence,

$$\theta^{-1}(N_{i-1}+1) > \theta^{-1}(N_{i+1}). \tag{1}$$

Because θ^{-1} is order-preserving on each of the sets $\{N_{i-1}+1, N_{i-1}+2, \ldots, N_i\}$ and $\{N_i+1, N_i+2, \ldots, N_{i+1}\}$, we have the following:

$$\theta^{-1}(N_{i-1}+1) < \cdots < \theta^{-1}(N_i-1) < \theta^{-1}(N_i); \tag{2}$$

$$\theta^{-1}(N_i+1) < \theta^{-1}(N_i+2) < \cdots < \theta^{-1}(N_{i+1}). \tag{3}$$

From (1), (2) and (3), we obtain $\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}} \leqslant_L A \leqslant_L P$. □

The following proposition characterizes the minimal element of the standardizer $\mathrm{St}(\mathcal{C})$ of a curve system $\mathcal{C}$.

Proposition 4.8. *Let $\mathcal{C}$ be an unnested curve system in D_n. Let P be a positive braid such that $P*\mathcal{C}$ is standard and, hence, $P*\mathcal{C} = \mathcal{C}_{\mathbf{n}}$ for some composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n. Then the following conditions are equivalent.*

(i) *P is the $\leqslant_R$-minimal element of the standardizer* $\mathrm{St}(\mathcal{C})$.
(ii) *$P \wedge_L \delta_{\mathbf{n}} = 1$ and $S(\delta_{\mathbf{n}}P) = S(\delta_{\mathbf{n}})$.*
(iii) *$P^{-1}(\delta_{\mathbf{n}}P)$ is in np-form.*
(iv) *$P^{-1}(\delta_{\mathbf{n}}^l P)$ is in np-form for some $l \geqslant 1$.*
(v) *$P^{-1}(\delta_{\mathbf{n}}^l P)$ is in np-form for all $l \geqslant 1$.*

Proof. We prove the equivalence by showing that (i) ⇔ (ii) ⇒ (v) ⇒ (iii) ⇒ (iv) ⇒ (ii). The implications (v) ⇒ (iii) and (iii) ⇒ (iv) are obvious.

(i) $\Rightarrow$ (ii) Let $A = P \wedge_L \delta_{\mathbf{n}}$ and let $P = AQ$ for some positive braid Q. Since $A \leqslant_L \delta_{\mathbf{n}}$, $A * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$, and hence

$$Q * \mathcal{C} = A^{-1} * (P * \mathcal{C}) = A^{-1} * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}.$$

Therefore $Q \in \mathrm{St}(\mathcal{C})$. By the $\leqslant_R$-minimality of P, we have $P = Q$ and, hence, $P \wedge_L \delta_{\mathbf{n}} = A = 1$.

Assume that $S(\delta_{\mathbf{n}} P)$ is strictly greater than $S(\delta_{\mathbf{n}})$. Then, by Lemma 4.7(ii), $P = \langle \sigma_i \rangle_{\sigma_i * \mathbf{n}} Q$ for some $i \in \{1, \dots, k-1\}$ and some positive braid Q. Since

$$Q * \mathcal{C} = \bigl(\langle \sigma_i \rangle_{\sigma_i * \mathbf{n}}\bigr)^{-1} * (P * \mathcal{C}) = \bigl\langle \sigma_i^{-1} \bigr\rangle_{\mathbf{n}} * \mathcal{C}_{\mathbf{n}},$$

$Q * \mathcal{C}$ is standard. This contradicts the $\leqslant_R$-minimality of P. Consequently, $S(\delta_{\mathbf{n}} P) = S(\delta_{\mathbf{n}})$.

(ii) $\Rightarrow$ (i) Let Q be the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$. Let $Q * \mathcal{C} = \mathcal{C}_{\mathbf{n}'}$ for some composition $\mathbf{n}'$ of n. Since $P * \mathcal{C}$ is standard, $P = RQ$ for some positive braid R. Since

$$R * \mathcal{C}_{\mathbf{n}'} = R * (Q * \mathcal{C}) = P * \mathcal{C} = \mathcal{C}_{\mathbf{n}},$$

the positive braid R sends the standard curve system $\mathcal{C}_{\mathbf{n}'}$ to the standard curve system $\mathcal{C}_{\mathbf{n}}$. Therefore, by Lemmas 3.5(ii) and 3.6(iii), $R = \langle R_0 \rangle_{\mathbf{n}'} (R_1 \oplus \cdots \oplus R_k)$ for some positive braids R_i with appropriate braid indices, and $R_0 * \mathbf{n}' = \mathbf{n}$.

If $(R_1 \oplus \cdots \oplus R_k) \neq 1$, then $P \wedge_L \delta_{\mathbf{n}} \neq 1$. This contradicts the hypothesis. Therefore $(R_1 \oplus \cdots \oplus R_k) = 1$.

If $R_0 \neq 1$, then $R_0 = \sigma_i R_0'$ for some $i \in \{1, \dots, k-1\}$ and a positive k-braid R_0'. Since $R_0' * \mathbf{n}' = \sigma_i^{-1} * (R_0 * \mathbf{n}') = \sigma_i^{-1} * \mathbf{n} = \sigma_i * \mathbf{n}$,

$$\langle R_0 \rangle_{\mathbf{n}'} = \langle \sigma_i \rangle_{R_0' * \mathbf{n}'} \bigl\langle R_0' \bigr\rangle_{\mathbf{n}'} = \langle \sigma_i \rangle_{\sigma_i * \mathbf{n}} \bigl\langle R_0' \bigr\rangle_{\mathbf{n}'}.$$

Since $\langle \sigma_i \rangle_{\sigma_i * \mathbf{n}} \leqslant_L \langle R_0 \rangle_{\mathbf{n}'} \leqslant_L P$, $S(\delta_{\mathbf{n}} P)$ is strictly greater than $S(\delta_{\mathbf{n}})$ by Lemmas 3.5(ii). This contradicts the hypothesis $S(\delta_{\mathbf{n}} P) = S(\delta_{\mathbf{n}})$. Therefore $R = 1$ and, hence, P is the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$.

(ii) $\Rightarrow$ (v) We first claim that $S(\delta_{\mathbf{n}}^l P) = S(\delta_{\mathbf{n}})$ for all $l \geqslant 1$. Let $\delta_{\mathbf{n}} A = \mathrm{s}_L(\delta_{\mathbf{n}} P)$. Since $S(\delta_{\mathbf{n}} A) = S(\delta_{\mathbf{n}} P)$ by Lemma 2.5(i) and $S(\delta_{\mathbf{n}} P) = S(\delta_{\mathbf{n}})$ by the hypothesis,

$$S(\delta_{\mathbf{n}} A) = S(\delta_{\mathbf{n}} P) = S(\delta_{\mathbf{n}}) = F(\delta_{\mathbf{n}}).$$

In particular, $F(\delta_{\mathbf{n}}) \supset S(\delta_{\mathbf{n}} A)$, and hence $\delta_{\mathbf{n}}(\delta_{\mathbf{n}} A)$ is in left normal form by Lemma 2.5(iii). Since $F(\delta_{\mathbf{n}}) = S(\delta_{\mathbf{n}})$, $\underbrace{\delta_{\mathbf{n}} \cdots \delta_{\mathbf{n}}}_{l-1}(\delta_{\mathbf{n}} A)$ is the left normal form of $\delta_{\mathbf{n}}^l A$ for all $l \geqslant 1$, and hence $S(\delta_{\mathbf{n}}^l P) = S(\delta_{\mathbf{n}}^l A) = S(\delta_{\mathbf{n}})$.

Now we have $S(\delta_{\mathbf{n}}^l P) = S(\delta_{\mathbf{n}})$ for all $l \geqslant 1$. By the hypothesis $P \wedge_L \delta_{\mathbf{n}} = 1$,

$$S(P) \cap S\bigl(\delta_{\mathbf{n}}^l P\bigr) = S(P) \cap S(\delta_{\mathbf{n}}) = \emptyset \quad \text{for all } l \geqslant 1.$$

Consequently, $P \wedge_L \delta_{\mathbf{n}}^l P = 1$ and $P^{-1}(\delta_{\mathbf{n}}^l P)$ is in *np*-form for all $l \geqslant 1$.

(iv) $\Rightarrow$ (ii) Let $P^{-1}(\delta_{\mathbf{n}}^l P)$ is in *np*-form for some $l \geqslant 1$, that is, $P \wedge_L (\delta_{\mathbf{n}}^l P) = 1$. Since $P \wedge_L \delta_{\mathbf{n}} \leqslant_L P \wedge_L (\delta_{\mathbf{n}}^l P)$, we have $P \wedge_L \delta_{\mathbf{n}} = 1$.

Assume that $S(\delta_{\mathbf{n}}P)$ is strictly greater than $S(\delta_{\mathbf{n}})$. By Lemma 4.7(ii), we have

$$P = \langle\sigma_i\rangle_{\sigma_i*\mathbf{n}}Q \tag{4}$$

for some $i \in \{1, \ldots, k-1\}$ and some positive braid Q. Since $\delta_{\mathbf{n}}\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}} = \langle\sigma_i\rangle_{\sigma_i*\mathbf{n}}\delta_{\sigma_i*\mathbf{n}}$,

$$\delta_{\mathbf{n}}^l P = \delta_{\mathbf{n}}^l\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}}Q = \langle\sigma_i\rangle_{\sigma_i*\mathbf{n}}\delta_{\sigma_i*\mathbf{n}}^l Q. \tag{5}$$

By (4) and (5), we obtain $\langle\sigma_i\rangle_{\sigma_i*\mathbf{n}} \leqslant_L P \wedge_L (\delta_{\mathbf{n}}^l P)$, which contracts the hypothesis that $P^{-1}(\delta_{\mathbf{n}}^l P)$ is in *np*-form. As a result, $S(\delta_{\mathbf{n}}P) = S(\delta_{\mathbf{n}})$. $\square$

Now we are ready to show that standardizing a reduction system $\mathcal{C}$ of a braid by the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{C})$ preserves the membership of the super summit set, ultra summit set and stable super summit set. The anonymous referee of this journal pointed out that our initial proof of the following theorem contains a mistake. The proof is corrected as suggested by the referee.

Theorem 4.9. *Let α be a reducible n-braid with a reduction system $\mathcal{C}$. Let P be the $\leqslant_R$-minimal element of* $\mathrm{St}(\mathcal{C})$. *Then the following hold.*

(i) $\inf(\alpha) \leqslant \inf(P\alpha P^{-1}) \leqslant \sup(P\alpha P^{-1}) \leqslant \sup(\alpha)$.
(ii) *If $\alpha \in [\alpha]^S$, then $P\alpha P^{-1} \in [\alpha]^S$.*
(iii) *If $\alpha \in [\alpha]^U$, then $P\alpha P^{-1} \in [\alpha]^U$.*
(iv) *If $\alpha \in [\alpha]^{St}$, then $P\alpha P^{-1} \in [\alpha]^{St}$.*

Proof. First, suppose that $\mathcal{C}$ is an unnested curve system. Let $P * \mathcal{C} = \mathcal{C}_{\mathbf{n}}$ for a composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n. Let $u = \sup(P)$. Define $\bar{P} = \Delta^u P^{-1}$ and $Q = \bar{P}\delta_{\mathbf{n}}^2 P$. By Proposition 4.8, $P^{-1}(\delta_{\mathbf{n}}^2 P)$ is in *np*-form, hence, by Lemma 2.6(i),

$$\bar{P} = Q \wedge_L \Delta^{\sup(\bar{P})}.$$

Since $(P\alpha P^{-1}) * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$, $P\alpha P^{-1} = \langle\beta_0\rangle_{\mathbf{n}}(\beta_1 \oplus \cdots \oplus \beta_k)$ for some β_i's with appropriate braid indices, and $\beta_0 * \mathbf{n} = \mathbf{n}$. Thus $P\alpha P^{-1}$ commutes with $\delta_{\mathbf{n}}^2$, and it follows that α commutes with $P^{-1}\delta_{\mathbf{n}}^2 P$. Therefore $Q\alpha Q^{-1} = (\Delta^u P^{-1}\delta_{\mathbf{n}}^2 P)\alpha(\Delta^u P^{-1}\delta_{\mathbf{n}}^2 P)^{-1} = \tau^{-u}(\alpha)$. That is,

$$Q^{-1}\tau^{-u}(\alpha)Q = \alpha. \tag{6}$$

Consider the following sets:

$$C(\alpha) = \{R \in B_n^+ \colon \inf(\alpha) \leqslant \inf(R^{-1}\alpha R) \leqslant \sup(R^{-1}\alpha R) \leqslant \sup(\alpha)\};$$
$$C^S(\alpha) = \{R \in B_n^+ \colon R^{-1}\alpha R \in [\alpha]^S\};$$
$$C^U(\alpha) = \{R \in B_n^+ \colon R^{-1}\alpha R \in [\alpha]^U\};$$
$$C^{St}(\alpha) = \{R \in B_n^+ \colon R^{-1}\alpha R \in [\alpha]^{St}\}.$$

By Franco and González-Meneses [FG03], Gebhardt [Geb05] and Lee and Lee [LL06a], all the sets $C(\alpha)$, $C^S(\alpha)$, $C^U(\alpha)$ and $C^{St}(\alpha)$ are closed under $\wedge_L$.

Suppose $\alpha \in [\alpha]^S$. Since $\tau^m(\alpha) \in [\alpha]^S$ for all $m \in \mathbb{Z}$, $\Delta^{\sup(\bar{P})} \in C^S(\tau^{-u}(\alpha))$. Since $Q \in C^S(\tau^{-u}(\alpha))$ by (6), we have $\bar{P} = Q \wedge_L \Delta^{\sup(\bar{P})} \in C^S(\tau^{-u}(\alpha))$. That is,

$$P\alpha P^{-1} = \bar{P}^{-1}\Delta^u\alpha\Delta^{-u}\bar{P} = \bar{P}^{-1}\tau^{-u}(\alpha)\bar{P} \in \left[\tau^{-u}(\alpha)\right]^S = [\alpha]^S.$$

Hence (ii) is proved. The other statements can be proved similarly.

Now we consider general case. For a reduction system $\mathcal{C}$ of α, we decompose $\mathcal{C}$ into $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$, where $\mathcal{C}_i$'s are inductively defined as the outermost component of $\mathcal{C} \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{i-1})$. By the construction, $\mathcal{C}_1, \ldots, \mathcal{C}_l$ are unnested reduction systems of α. For $i = 1, \ldots, l$, define positive braids P_i and conjugates α_i of α inductively as follows. Let $P_0 = 1$ and $\alpha_0 = \alpha$.

- P_i is the $\leqslant_R$-minimal element of $\mathrm{St}((P_{i-1} \cdots P_1) * \mathcal{C}_i)$;
- $\alpha_i = P_i\alpha_{i-1}P_i^{-1} = (P_i \cdots P_1)\alpha(P_i \cdots P_1)^{-1}$.

Note that each α_i is a reducible braid with a reduction system $(P_i \cdots P_1) * \mathcal{C}_{i+1}$ and that $P = P_l \cdots P_1$ by Corollary 4.5(i).

Suppose $\alpha \in [\alpha]^S$. By the previous discussion on the unnested case, $P_{i+1}\alpha_i P_{i+1}^{-1} \in [\alpha]^S$ for $i = 0, \ldots, l-1$, hence $P\alpha P^{-1} \in [\alpha]^S$. Therefore (ii) is proved. The other statements can be proved similarly. $\square$

5. Outermost components of non-periodic reducible braids

In this section we define the outermost component α_{ext} of a non-periodic reducible braid α using the $\leqslant_R$-minimal element of the standardizer of the canonical reduction system of α, and study its properties.

Recall the canonical reduction system of mapping classes. For a reduction system $\mathcal{C} \subset D_n$ of an n-braid α, let $D_{\mathcal{C}}$ be the closure of $D_n \setminus N(\mathcal{C})$ in D_n, where $N(\mathcal{C})$ is a regular neighborhood of $\mathcal{C}$. The restriction of α induces an automorphism on $D_{\mathcal{C}}$ that is well-defined up to isotopy. Due to Birman, Lubotzky and McCarthy [BLM83] and Ivanov [Iva92], for any n-braid α, there is a unique *canonical reduction system* $\mathcal{R}(\alpha)$ with the following properties.

(i) $\mathcal{R}(\alpha^m) = \mathcal{R}(\alpha)$ for all $m \neq 0$.
(ii) $\mathcal{R}(\beta\alpha\beta^{-1}) = \beta * \mathcal{R}(\alpha)$ for all $\beta \in B_n$.
(iii) The restriction of α to each component of $D_{\mathcal{R}(\alpha)}$ is either periodic or pseudo-Anosov. A reduction system with this property is said to be *adequate*.
(iv) If $\mathcal{C}$ is an adequate reduction system of α, then $\mathcal{R}(\alpha) \subset \mathcal{C}$.

By the properties of canonical reduction systems, a braid α is non-periodic reducible if and only if $\mathcal{R}(\alpha) \neq \emptyset$. Let $\mathcal{R}_{\mathrm{ext}}(\alpha)$ denote the collection of the outermost components of $\mathcal{R}(\alpha)$. Then, $\mathcal{R}_{\mathrm{ext}}(\alpha)$ is an unnested curve system satisfying the properties (i) and (ii). We remark that, while the canonical reduction systems are defined for the mapping classes of surfaces with genus, we have to restrict ourselves to the mapping classes of punctured disks in order to define the outermost component $\mathcal{R}_{\mathrm{ext}}(\alpha)$.

Lemma 5.1. *Let $\alpha, \beta \in B_n$ with $\mathcal{R}(\alpha) \neq \emptyset$. If $\alpha\beta = \beta\alpha$, then $\mathcal{R}(\alpha)$ and $\mathcal{R}_{\mathrm{ext}}(\alpha)$ are reduction systems of β.*

Proof. Since $\mathcal{R}(\alpha)=\mathcal{R}(\beta\alpha\beta^{-1})=\beta*\mathcal{R}(\alpha)$ and $\mathcal{R}_{\text{ext}}(\alpha)=\mathcal{R}_{\text{ext}}(\beta\alpha\beta^{-1})=\beta*\mathcal{R}_{\text{ext}}(\alpha)$, both $\mathcal{R}(\alpha)$ and $\mathcal{R}_{\text{ext}}(\alpha)$ are reduction systems of β. $\square$

Definition 5.2. Let $\alpha\in B_n$ with $\mathcal{R}(\alpha)\neq\emptyset$. Let P be the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{R}_{\text{ext}}(\alpha))$ and $\beta=P\alpha P^{-1}$. Since $\mathcal{R}_{\text{ext}}(\beta)$ is unnested and standard, $\mathcal{R}_{\text{ext}}(\beta)=\mathcal{C}_{\mathbf{n}}$ for a composition $\mathbf{n}=(n_1,\dots,n_k)$ of n, and β has the unique expression $\beta=\langle\beta_0\rangle_{\mathbf{n}}(\beta_1\oplus\cdots\oplus\beta_k)$ by Lemma 3.5(ii). We define the *outermost component* α_{ext} of α by $\alpha_{\text{ext}}=\beta_0$.

In other words, α_{ext} is the restriction of α to the outermost component of $D_n\setminus\mathcal{R}_{\text{ext}}(\alpha)$. This element is a priori defined up to conjugacy, but the use of the $\leqslant_R$-minimal element P determines the particular element β_0 to be chosen in the conjugacy class.

Lemma 5.3. *Let α be an n-braid with $\mathcal{R}(\alpha)\neq\emptyset$.*

(i) *If β is conjugate to α, then β_{ext} is conjugate to α_{ext}.*
(ii) $(\alpha^m)_{\text{ext}}=(\alpha_{\text{ext}})^m$ *for all $m\neq 0$.*
(iii) $\inf(\alpha)\leqslant\inf(\alpha_{\text{ext}})\leqslant\sup(\alpha_{\text{ext}})\leqslant\sup(\alpha)$.
(iv) $\inf_s(\alpha)\leqslant\inf_s(\alpha_{\text{ext}})\leqslant\sup_s(\alpha_{\text{ext}})\leqslant\sup_s(\alpha)$.
(v) $t_{\inf}(\alpha)\leqslant t_{\inf}(\alpha_{\text{ext}})\leqslant t_{\sup}(\alpha_{\text{ext}})\leqslant t_{\sup}(\alpha)$.

Proof. (i) is obvious. (ii) follows from $\mathcal{R}(\alpha^m)=\mathcal{R}(\alpha)$. (iii) follows from Lemma 3.6 and Theorem 4.9.

(iv) Choose any $\beta\in[\alpha]^S$. By (iii), we have

$$\inf_s(\alpha)=\inf(\beta)\leqslant\inf(\beta_{\text{ext}})\leqslant\sup(\beta_{\text{ext}})\leqslant\sup(\beta)=\sup_s(\alpha).$$

Since α_{ext} and β_{ext} are conjugate by (i),

$$\inf(\beta_{\text{ext}})\leqslant\inf_s(\alpha_{\text{ext}})\leqslant\sup_s(\alpha_{\text{ext}})\leqslant\sup(\beta_{\text{ext}}).$$

Combining the above two, we obtain $\inf_s(\alpha)\leqslant\inf_s(\alpha_{\text{ext}})\leqslant\sup_s(\alpha_{\text{ext}})\leqslant\sup_s(\alpha)$.

(v) By (ii) and (iii), for all $m\geqslant 1$,

$$\inf\bigl(\alpha^m\bigr)\leqslant\inf\bigl(\bigl(\alpha^m\bigr)_{\text{ext}}\bigr)=\inf\bigl((\alpha_{\text{ext}})^m\bigr)\leqslant\sup\bigl((\alpha_{\text{ext}})^m\bigr)=\sup\bigl(\bigl(\alpha^m\bigr)_{\text{ext}}\bigr)\leqslant\sup\bigl(\alpha^m\bigr).$$

Therefore,

$$\frac{\inf(\alpha^m)}{m}\leqslant\frac{\inf((\alpha_{\text{ext}})^m)}{m}\leqslant\frac{\sup((\alpha_{\text{ext}})^m)}{m}\leqslant\frac{\sup(\alpha^m)}{m}.$$

By taking $m\to\infty$, we obtain the desired inequalities for $t_{\inf}(\cdot)$ and $t_{\sup}(\cdot)$. $\square$

Lemma 5.4. *Let $\alpha\in B_n$ with $\mathcal{R}_{\text{ext}}(\alpha)$ standard. Then $\mathcal{R}_{\text{ext}}(\tau(\alpha))$, $\mathcal{R}_{\text{ext}}(\mathbf{c}_0(\alpha))$ and $\mathcal{R}_{\text{ext}}(\mathbf{d}(\alpha))$ are standard. Moreover,*

(i) $\tau(\alpha)_{\mathrm{ext}} = \tau(\alpha_{\mathrm{ext}})$;

(ii) $$\mathbf{c}_0(\alpha)_{\mathrm{ext}} = \begin{cases} \alpha_{\mathrm{ext}} & \textit{if } \inf(\alpha_{\mathrm{ext}}) > \inf(\alpha); \\ \mathbf{c}_0(\alpha_{\mathrm{ext}}) & \textit{if } \inf(\alpha_{\mathrm{ext}}) = \inf(\alpha); \end{cases}$$

(iii) $$\mathbf{d}(\alpha)_{\mathrm{ext}} = \begin{cases} \alpha_{\mathrm{ext}} & \textit{if } \sup(\alpha_{\mathrm{ext}}) < \sup(\alpha); \\ \mathbf{d}(\alpha_{\mathrm{ext}}) & \textit{if } \sup(\alpha_{\mathrm{ext}}) = \sup(\alpha). \end{cases}$$

Proof. $\mathcal{R}_{\mathrm{ext}}(\tau(\alpha)) = \mathcal{R}_{\mathrm{ext}}(\Delta^{-1}\alpha\Delta) = \Delta^{-1} * \mathcal{R}_{\mathrm{ext}}(\alpha)$ is obviously standard. $\mathcal{R}_{\mathrm{ext}}(\mathbf{c}_0(\alpha))$ and $\mathcal{R}_{\mathrm{ext}}(\mathbf{d}(\alpha))$ are standard by Corollary 3.9. Let $\mathcal{R}_{\mathrm{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}}$ for a composition $\mathbf{n} = (n_1, \dots, n_k)$ of n and $\alpha = \langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)$. Let Δ_i be the fundamental braid of B_{n_i} for $i = 1, \dots, k$ and Δ_0 be the fundamental braid of B_k. Note that $\alpha_0 * \mathbf{n} = \mathbf{n}$ and

$$\Delta = (\Delta_1 \oplus \cdots \oplus \Delta_k)\langle\Delta_0\rangle_{\Delta_0^{-1}*\mathbf{n}} = \langle\Delta_0\rangle_{\Delta_0^{-1}*\mathbf{n}}(\Delta_k \oplus \cdots \oplus \Delta_1).$$

Therefore,

$$\begin{aligned} \tau(\alpha) &= \Delta^{-1}\alpha\Delta \\ &= \langle\Delta_0^{-1}\rangle_{\mathbf{n}}\big(\Delta_1^{-1} \oplus \cdots \oplus \Delta_k^{-1}\big)\langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)\langle\Delta_0\rangle_{\Delta_0^{-1}*\mathbf{n}}(\Delta_k \oplus \cdots \oplus \Delta_1) \\ &= \langle\Delta_0^{-1}\alpha_0\Delta_0\rangle_{\Delta_0^{-1}*\mathbf{n}}\big(\Delta_k^{-1}\alpha_k\Delta_k \oplus \cdots \oplus \Delta_1^{-1}\alpha_1\Delta_1\big) \\ &= \langle\tau(\alpha_0)\rangle_{\Delta_0^{-1}*\mathbf{n}}\big(\tau(\alpha_k) \oplus \cdots \oplus \tau(\alpha_1)\big). \end{aligned}$$

Since $\mathcal{R}_{\mathrm{ext}}(\tau(\alpha)) = \Delta^{-1} * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\Delta_0^{-1}*\mathbf{n}}$, $\tau(\alpha)_{\mathrm{ext}} = \tau(\alpha_0) = \tau(\alpha_{\mathrm{ext}})$.

Let $\alpha = \Delta^u A_1 \cdots A_l$ be the left normal form of α. Since $\alpha * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$ is standard, $A_l * \mathcal{C}_{\mathbf{n}}$ is standard by Theorem 3.8. By Lemmas 3.5(ii) and 3.6(iii), A_l is expressed as $A_l = \langle A_{l,0}\rangle_{\mathbf{n}}(A_{l,1} \oplus \cdots \oplus A_{l,k})$, where $A_{l,i}$'s are permutation n_i-braids. Let θ_1 and θ_2 be the induced permutations of $\alpha_0 A_{l,0}^{-1}$ and $A_{l,0}^{-1}$ respectively. Then

$$\begin{aligned} \mathbf{d}(\alpha) &= A_l\alpha A_l^{-1} \\ &= \langle A_{l,0}\rangle_{\mathbf{n}}(A_{l,1} \oplus \cdots \oplus A_{l,k})\langle\alpha_0\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)\big(A_{l,1}^{-1} \oplus \cdots \oplus A_{l,k}^{-1}\big)\langle A_{l,0}^{-1}\rangle_{A_{l,0}*\mathbf{n}} \\ &= \langle A_{l,0}\alpha_0 A_{l,0}^{-1}\rangle_{A_{l,0}*\mathbf{n}}\big(A_{l,\theta_1(1)}\alpha_{\theta_2(1)}A_{l,\theta_2(1)}^{-1} \oplus \cdots \oplus A_{l,\theta_1(k)}\alpha_{\theta_2(k)}A_{l,\theta_2(k)}^{-1}\big). \end{aligned}$$

Recall Lemma 3.6(ii) that $\sup(\alpha_{\mathrm{ext}}) \leqslant \sup(\alpha)$. If $\sup(\alpha_{\mathrm{ext}}) < \sup(\alpha)$, then $A_{l,0} = 1$ and, hence, $\mathbf{d}(\alpha)_{\mathrm{ext}} = \alpha_{\mathrm{ext}}$. If $\sup(\alpha_{\mathrm{ext}}) = \sup(\alpha)$, then $A_{l,0} \neq 1$ and, hence, $\mathbf{d}(\alpha)_{\mathrm{ext}} = A_{l,0}\alpha_0 A_{l,0}^{-1} = \mathbf{d}(\alpha_{\mathrm{ext}})$.

For $\mathbf{c}_0(\alpha)$, use the identity $\mathbf{c}_0(\alpha) = \mathbf{d}(\alpha^{-1})^{-1}$. $\square$

Recall Lemma 5.3 that $\inf(\alpha) \leqslant \inf(\alpha_{\mathrm{ext}})$ and $\inf_s(\alpha) \leqslant \inf_s(\alpha_{\mathrm{ext}})$ for any $\alpha \in B_n$ with $\mathcal{R}(\alpha) \neq \emptyset$.

Lemma 5.5. *Let α be an n-braid with $\mathcal{R}(\alpha) \neq \emptyset$. Let β be an element of $[\alpha]^U$ with $\mathcal{R}_{\mathrm{ext}}(\beta)$ standard.*

(i) *Let* $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$. *Then,* $\inf(\beta_{\mathrm{ext}}) > \inf(\beta)$.
(ii) *Let* $\inf_s(\alpha_{\mathrm{ext}}) = \inf_s(\alpha)$. *Then,* $\inf(\beta_{\mathrm{ext}}) = \inf(\beta)$, *and* $\mathbf{c}_0^m(\beta_{\mathrm{ext}}) = \beta_{\mathrm{ext}}$ *for some* $m \geqslant 1$.

Proof. By Lemma 5.3(i), β_{ext} and α_{ext} are conjugate, hence $\inf(\beta_{\mathrm{ext}}) \leqslant \inf_s(\alpha_{\mathrm{ext}})$.

We first prove the following claim.

Claim. *Let* $\inf(\beta_{\mathrm{ext}}) = \inf(\beta)$. *Then,* $\mathbf{c}_0^m(\beta_{\mathrm{ext}}) = \beta_{\mathrm{ext}}$ *for some* $m \geqslant 1$, *and* $\inf_s(\alpha_{\mathrm{ext}}) = \inf(\beta_{\mathrm{ext}}) = \inf(\beta) = \inf_s(\alpha)$.

Proof of Claim. By Lemma 5.4(ii), the sequence $\{\inf(\mathbf{c}_0^i(\beta)_{\mathrm{ext}})\}_{i=0}^{\infty}$ is non-decreasing. Since $\beta \in [\alpha]^U$, $\mathbf{c}_0^m(\beta) = \beta$ for some $m \geqslant 1$. Therefore,

$$\inf\bigl(\mathbf{c}_0^i(\beta)_{\mathrm{ext}}\bigr) = \inf(\beta_{\mathrm{ext}}) \quad \text{for all } i \geqslant 0.$$

Since $\mathbf{c}_0^i(\beta) \in [\alpha]^U$ for all $i \geqslant 0$, we have $\inf(\mathbf{c}_0^i(\beta)) = \inf_s(\alpha) = \inf(\beta)$ for all $i \geqslant 0$. Hence

$$\inf\bigl(\mathbf{c}_0^i(\beta)_{\mathrm{ext}}\bigr) = \inf(\beta_{\mathrm{ext}}) = \inf(\beta) = \inf\bigl(\mathbf{c}_0^i(\beta)\bigr) \quad \text{for all } i \geqslant 0.$$

By Lemma 5.4(ii),

$$\mathbf{c}_0^i(\beta)_{\mathrm{ext}} = \mathbf{c}_0^i(\beta_{\mathrm{ext}}) \quad \text{for all } i \geqslant 0.$$

Since $\mathbf{c}_0^m(\beta) = \beta$, we obtain $\mathbf{c}_0^m(\beta_{\mathrm{ext}}) = \mathbf{c}_0^m(\beta)_{\mathrm{ext}} = \beta_{\mathrm{ext}}$.

By Theorem 2.8(i), $\inf(\beta_{\mathrm{ext}}) = \inf_s(\beta_{\mathrm{ext}}) = \inf_s(\alpha_{\mathrm{ext}})$. Therefore, $\inf_s(\alpha_{\mathrm{ext}}) = \inf(\beta_{\mathrm{ext}}) = \inf(\beta) = \inf_s(\alpha)$. □

(i) Assume $\inf(\beta_{\mathrm{ext}}) = \inf(\beta)$. Then $\inf_s(\alpha_{\mathrm{ext}}) = \inf(\beta_{\mathrm{ext}}) = \inf(\beta) = \inf_s(\alpha)$ by the above claim. This contradicts the hypothesis that $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$, hence $\inf(\beta_{\mathrm{ext}}) > \inf(\beta)$.

(ii) Since $\inf(\beta) \leqslant \inf(\beta_{\mathrm{ext}}) \leqslant \inf_s(\alpha_{\mathrm{ext}})$,

$$\inf(\beta) \leqslant \inf(\beta_{\mathrm{ext}}) \leqslant \inf_s(\alpha_{\mathrm{ext}}) = \inf_s(\alpha) = \inf(\beta).$$

Therefore $\inf(\beta_{\mathrm{ext}}) = \inf(\beta)$. By the claim, $\mathbf{c}_0^m(\beta_{\mathrm{ext}}) = \beta_{\mathrm{ext}}$ for some $m \geqslant 1$. □

The following proposition show that the property $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$ is preserved by taking powers.

Proposition 5.6. *If* $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$, *then* $\inf_s((\alpha^m)_{\mathrm{ext}}) > \inf_s(\alpha^m)$ *for all* $m \geqslant 1$.

Proof. By Theorem 6.1 in [Lee07], for any $\beta \in B_n$ and any $m \geqslant 1$,

$$\inf_s(\beta) \leqslant \frac{\inf_s(\beta^m)}{m} < \inf_s(\beta) + 1.$$

By Lemma 5.3(ii), $(\alpha^m)_{\mathrm{ext}} = (\alpha_{\mathrm{ext}})^m$ for all $m \neq 0$. Since $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$,

$$\frac{\inf_s(\alpha^m)}{m} < \inf_s(\alpha) + 1 \leqslant \inf_s(\alpha_{\mathrm{ext}}) \leqslant \frac{\inf_s((\alpha_{\mathrm{ext}})^m)}{m} = \frac{\inf_s((\alpha^m)_{\mathrm{ext}})}{m}$$

for all $m \geqslant 1$. Therefore $\inf_s((\alpha^m)_{\mathrm{ext}}) > \inf_s(\alpha^m)$ for all $m \geqslant 1$. □

6. Split braids

An n-braid α is called a *split braid* if it is conjugate to an element in the subgroup of B_n generated by $\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{n-1}$ for some $1 \leqslant i \leqslant n-1$ [Hum91]. In our terminology, $\alpha \in B_n$ is a split braid if it is conjugate to a braid β of the form $\beta = \langle 1\rangle_{\mathbf{n}}(\beta_1 \oplus \beta_2)$, where $\mathbf{n} = (i, n-i)$ for some $1 \leqslant i \leqslant n-1$, and $\beta_1 \in B_i$ and $\beta_2 \in B_{n-i}$.

The following lemma is easy to show, but we include a proof for completeness.

Lemma 6.1. *Let α be an n-braid.*

(i) *α is a split braid if and only if either α is the identity or α is non-periodic and reducible with $\alpha_{\mathrm{ext}} = 1$.*
(ii) *Let $\alpha = \langle 1\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)$ for a composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n such that $\mathcal{R}_{\mathrm{ext}}(\alpha) \neq \emptyset$. Then $\mathcal{R}_{\mathrm{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}}$ if and only if α_i is non-split for each $1 \leqslant i \leqslant k$.*

Proof. For unnested curve systems $\mathcal{C}$ and $\mathcal{C}'$ in D_n, let us write "$\mathcal{C}' \preccurlyeq \mathcal{C}$" if each component of $\mathcal{C}'$ is enclosed by (possibly parallel to) a component of $\mathcal{C}$, and write "$\mathcal{C}' \precneqq \mathcal{C}$" if $\mathcal{C}' \preccurlyeq \mathcal{C}$ and $\mathcal{C}' \neq \mathcal{C}$. Then $\preccurlyeq$ is a partial order over the set of unnested curve systems in D_n. For compositions $\mathbf{n} = (n_1, \ldots, n_k)$ and $\mathbf{n}'$ of n, $\mathcal{C}_{\mathbf{n}'} \preccurlyeq \mathcal{C}_{\mathbf{n}}$ if and only if $\mathbf{n}'$ is a refinement of $\mathbf{n}$, that is, for each $1 \leqslant i \leqslant k$, there exists a composition $(n'_{i,1}, \ldots, n'_{i,r_i})$ of n_i such that $\mathbf{n}' = (n'_{1,1}, \ldots, n'_{1,r_1}, \ldots, n'_{k,1}, \ldots, n'_{k,r_k})$.

Claim. *Let $\beta * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$ for a composition $\mathbf{n}$ of n, then β is written as $\beta = \langle \beta_0\rangle_{\mathbf{n}}(\beta_1 \oplus \cdots \oplus \beta_k)$. If β_0 is periodic or pseudo-Anosov, then $\mathcal{R}_{\mathrm{ext}}(\beta) \preccurlyeq \mathcal{C}_{\mathbf{n}}$, and there exists $P \in B_n^+$ such that both $P * \mathcal{R}_{\mathrm{ext}}(\beta)$ and $P * \mathcal{C}_{\mathbf{n}}$ are standard.*

Proof of Claim. Because β_0 is periodic or pseudo-Anosov, we can make an adequate reduction system of β from $\mathcal{C}_{\mathbf{n}}$ by adding some curves each of which is enclosed by a component of $\mathcal{C}_{\mathbf{n}}$. Because any adequate reduction system of β contains $\mathcal{R}_{\mathrm{ext}}(\beta)$ as a subset, we have $\mathcal{R}_{\mathrm{ext}}(\beta) \preccurlyeq \mathcal{C}_{\mathbf{n}}$. Let P be the $\leqslant_R$-minimal element of $\mathrm{St}(\mathcal{R}_{\mathrm{ext}}(\beta))$. Then $P * \mathcal{R}_{\mathrm{ext}}(\beta)$ is standard by the construction. Apply Corollary 4.5 to $\mathcal{C}_{\mathbf{n}} \backslash \mathcal{R}_{\mathrm{ext}}(\beta)$. Then $P * (\mathcal{C}_{\mathbf{n}} \backslash \mathcal{R}_{\mathrm{ext}}(\beta))$ and hence $P * \mathcal{C}_{\mathbf{n}}$ are standard. □

(i) It is obvious that if α is the identity or α is non-periodic and reducible with $\alpha_{\mathrm{ext}} = 1$ then α is a split braid. Conversely, suppose that α is a split braid. Taking a conjugate of α if necessary, we may assume that

$$\alpha = \langle 1\rangle_{\mathbf{n}}(\alpha_1 \oplus \alpha_2),$$

where $\mathbf{n} = (\ell, n-\ell)$ for some $1 \leqslant \ell \leqslant n-1$, and $\alpha_1 \in B_\ell$ and $\alpha_2 \in B_{n-\ell}$.

First, assume that $\mathcal{R}_{\mathrm{ext}}(\alpha) = \emptyset$, that is, α is periodic or pseudo-Anosov. Since split braids are a special type of reducible braids and since pseudo-Anosov braids cannot be reducible [FLP79], α is periodic. Therefore $\alpha^p = \Delta^{2m}$ for some $p \neq 0$ and $m \in \mathbb{Z}$, hence

$$\langle 1\rangle_{\mathbf{n}}\bigl(\alpha_1^p \oplus \alpha_2^p\bigr) = \alpha^p = \Delta^{2m} = \bigl\langle \Delta_0^{2m}\bigr\rangle_{\mathbf{n}}\bigl(\Delta_1^{2m} \oplus \Delta_2^{2m}\bigr),$$

where Δ_0, Δ_1 and Δ_2 are the fundamental braids of B_2, B_ℓ and $B_{n-\ell}$, respectively. By Lemma 3.5(i), we have $\Delta_0^{2m} = 1$, hence $m = 0$, and it follows that $\alpha^p = 1$. Because braid groups are torsion-free [Deh98], α is the identity.

Now, assume that $\mathcal{R}_{\text{ext}}(\alpha) \neq \emptyset$, that is, α is non-periodic and reducible. For a curve system $\mathcal{C}$ in a punctured disk D_m, let $\text{Out}(D_m \setminus \mathcal{C})$ denote the outermost component of $D_m \setminus \mathcal{C}$. By the above claim, we have $\mathcal{R}_{\text{ext}}(\alpha) \preccurlyeq \mathcal{C}_{\mathbf{n}}$ and hence $\mathcal{C}_{\mathbf{n}} \subset \text{Out}(D_n \setminus \mathcal{R}_{\text{ext}}(\alpha))$, and we may assume that $\mathcal{R}_{\text{ext}}(\alpha)$ is standard. Let α_{ext} be a k-braid. Because $\mathcal{R}_{\text{ext}}(\alpha)$ is standard, $\text{Out}(D_n \setminus \mathcal{R}_{\text{ext}}(\alpha))$ is canonically diffeomorphic to D_k. Let $\mathcal{C}'$ be the image of $\mathcal{C}_{\mathbf{n}}$ under this diffeomorphism. Then $\mathcal{C}'$ is a reduction system of α_{ext} such that the restriction of α_{ext} to $\text{Out}(D_k \setminus \mathcal{C}')$ is the same as the restriction of α to $\text{Out}(D_n \setminus \mathcal{C}_{\mathbf{n}})$ which is the identity. This means that α_{ext} is a split braid. Because α_{ext} is either periodic or pseudo-Anosov, the discussion in the above paragraph shows that α_{ext} is the identity.

(ii) Assume that α_ℓ is a split braid for some $1 \leqslant \ell \leqslant k$, hence α_ℓ is conjugate to $\langle 1\rangle_{\mathbf{n}_\ell}(\alpha'_\ell \oplus \alpha''_\ell)$, where $\mathbf{n}_\ell = (n'_\ell, n''_\ell)$ is a composition of n_ℓ, and $\alpha'_\ell \in B_{n'_\ell}$ and $\alpha''_\ell \in B_{n''_\ell}$. By taking a conjugate of α if necessary, we may assume that

$$\alpha = \langle 1\rangle_{\mathbf{n}'}\big(\alpha_1 \oplus \cdots \oplus \alpha_{\ell-1} \oplus \alpha'_\ell \oplus \alpha''_\ell \oplus \alpha_{\ell+1} \oplus \cdots \oplus \alpha_k\big),$$

where $\mathbf{n}' = (n_1, \dots, n_{\ell-1}, n'_\ell, n''_\ell, n_{\ell+1}, \dots, n_k)$. Note that $\mathcal{C}_{\mathbf{n}'} \precneqq \mathcal{C}_{\mathbf{n}}$. By the claim, $\mathcal{R}_{\text{ext}}(\alpha) \preccurlyeq \mathcal{C}_{\mathbf{n}'} \precneqq \mathcal{C}_{\mathbf{n}}$, hence $\mathcal{C}_{\mathbf{n}} \neq \mathcal{R}_{\text{ext}}(\alpha)$.

Conversely, assume that $\mathcal{R}_{\text{ext}}(\alpha) \neq \mathcal{C}_{\mathbf{n}}$. By the claim, we may assume that $\mathcal{R}_{\text{ext}}(\alpha) \precneqq \mathcal{C}_{\mathbf{n}}$ and $\mathcal{R}_{\text{ext}}(\alpha)$ is standard. Let $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}'}$ for a composition $\mathbf{n}'$ of n. Then $\mathbf{n}'$ is a refinement of $\mathbf{n}$, hence, for each i, there exists a composition $(n'_{i,1}, \dots, n'_{i,r_i})$ of n_i such that $\mathbf{n}' = (n'_{1,1}, \dots, n'_{1,r_1}, \dots, n'_{k,1}, \dots, n'_{k,r_k})$. Because α_{ext} is the identity by (i), α is written as

$$\alpha = \langle 1\rangle_{\mathbf{n}'}(\underbrace{\alpha_{1,1} \oplus \cdots \oplus \alpha_{1,r_1}}_{r_1} \oplus \cdots \oplus \underbrace{\alpha_{k,1} \oplus \cdots \oplus \alpha_{k,r_k}}_{r_k}).$$

Since $\mathcal{C}_{\mathbf{n}'} = \mathcal{R}_{\text{ext}}(\alpha) \neq \mathcal{C}_{\mathbf{n}}$ by the assumption, we have $r_\ell \geqslant 2$ for some $1 \leqslant \ell \leqslant k$. Comparing the above expression with $\alpha = \langle 1\rangle_{\mathbf{n}}(\alpha_1 \oplus \cdots \oplus \alpha_k)$, we have $\alpha_\ell = \langle 1\rangle_{\mathbf{n}'_\ell}(\alpha_{\ell,1} \oplus \cdots \oplus \alpha_{\ell,r_\ell})$ where $\mathbf{n}'_\ell = (n'_{\ell,1}, \dots, n'_{\ell,r_\ell})$. Since $r_\ell \geqslant 2$, α_ℓ is a split braid. $\square$

For $\alpha \in B_n$, let $|\alpha|$ denote the minimal word length of α with respect to $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}\}$. Then $|\alpha|$ is the minimum number of crossings in the braid diagram of α.

Proposition 6.2. *If $\alpha \neq 1$ is a split braid and $|\alpha|$ is minimal in the conjugacy class of α, then $\mathcal{R}_{\text{ext}}(\alpha)$ is standard.*

Proof. There exists a braid β in the conjugacy class of α such that $\mathcal{R}_{\text{ext}}(\beta)$ is standard. Therefore $\mathcal{R}_{\text{ext}}(\beta) = \mathcal{C}_{\mathbf{n}}$ for some composition $\mathbf{n} = (n_1, \dots, n_k)$ of n. Then by Lemma 6.1

$$\beta = \langle 1\rangle_{\mathbf{n}}(\beta_1 \oplus \cdots \oplus \beta_k)$$

for some non-split n_i-braids β_i. We may choose β so that $|\beta_i|$ is minimal in the conjugacy class of β_i for each $i \in \{1, \dots, k\}$.

Since α and β are conjugate, $\alpha = \gamma\beta\gamma^{-1}$ for some $\gamma \in B_n$. Let θ be the induced permutation of γ. For $i = 1, \dots, k$, let $S_i = \{j\colon n_1 + \cdots + n_{i-1} < j \leqslant n_1 + \cdots + n_i\}$ and $T_i = \{\theta(j)\colon j \in S_i\}$.

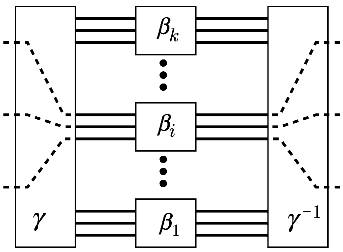


Fig. 12. The dotted strands indicate γ_i in the proof of Proposition 6.2.

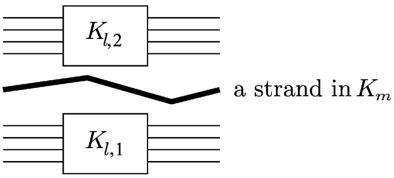


Fig. 13. Since there is no crossing between the strands in K_l and those in K_m, if a strand of K_m goes through K_l, then K_l is splitted into two parts $K_{l,1}$ and $K_{l,2}$.

Let γ_i be the result of forgetting the jth strand from γ for all $j \notin S_i$. (The strands of a braid are numbered from bottom to top at its right end.) See Fig. 12. Let α_i be the result of forgetting the jth strand from α for all $j \notin T_i$. Then $\alpha_i = \gamma_i \beta_i \gamma_i^{-1}$ for all $i = 1, \dots, k$.

Let K be a braid diagram of α such that the number of crossings in K is exactly $|\alpha|$. For $i = 1, \dots, k$, let K_i be the result of deleting the jth strand from K for all $j \notin T_i$. Then K_i is a braid diagram of α_i for all i. Let $c(K)$ and $c(K_i)$ denote the numbers of crossings in K and K_i, respectively. Then $|\alpha| = c(K)$, $|\alpha_i| \leqslant c(K_i)$ for each i and $\sum_{i=1}^{k} c(K_i) \leqslant c(K)$.

Since $|\alpha|$ is minimal in the conjugacy class, $|\alpha| \leqslant |\beta|$. Since $|\beta_i|$ is minimal in the conjugacy class, $|\beta_i| \leqslant |\alpha_i|$ for all $i = 1, \dots, k$. Hence

$$c(K) = |\alpha| \leqslant |\beta| = \sum_{i=1}^{k} |\beta_i| \leqslant \sum_{i=1}^{k} |\alpha_i| \leqslant \sum_{i=1}^{k} c(K_i) \leqslant c(K).$$

Therefore $c(K) = \sum_{i=1}^{k} c(K_i)$ and it follows that there is no crossing between the strands in K_i and those in K_j whenever $i \neq j$.

Now we claim that each T_l is a set of consecutive integers. On the contrary, assume that there exists $j \in T_m$ for some $m \neq l$ such that $i_1 < j < i_2$ for some $i_1, i_2 \in T_l$. Let $K_{l,1}$ be the result of deleting all ith strands from K_l with $i > j$ and let $K_{l,2} = K_l \setminus K_{l,1}$. See Fig. 13. Because there is no crossing between the strands in K_l and those in K_m, there is no crossing between $K_{l,1}$ and $K_{l,2}$. Therefore K_l is splitted into $K_{l,1}$ and $K_{l,2}$. This contradicts that α_l is non-split. Hence, each T_l is a set of consecutive integers.

Let $T_{i_1}, T_{i_2}, \dots, T_{i_k}$ be the rearrangement of T_j's such that the elements of the sets are increasing, and let $\mathbf{n}' = (n_{i_1}, \dots, n_{i_k})$. Then $\alpha = \langle 1 \rangle_{\mathbf{n}'}(\alpha_{i_1} \oplus \cdots \oplus \alpha_{i_k})$ and $\mathcal{R}_{\text{ext}}(\alpha) = \mathcal{C}_{\mathbf{n}'}$. Therefore $\mathcal{R}_{\text{ext}}(\alpha)$ is standard. $\square$

Corollary 6.3. *If $P \neq 1$ is a positive split braid, then $\mathcal{R}_{\text{ext}}(P)$ is standard.*

Proof. If P is a positive braid, then $|P|$ is minimal in the conjugacy class of P. $\square$

7. Ultra summit sets of reducible braids

In this section, we establish Theorem 7.4, the main result of this paper. Roughly speaking, it says that if the outermost component α_{ext} is simpler than the whole braid α from a Garside-theoretic point of view, then it is easy to find a reduction system of α.

Definition 7.1. Let $\alpha \in B_n$, $\beta \in [\alpha]^U$ and $m = \min\{l \geqslant 1\colon\ \mathbf{c}_0^l(\beta) = \beta\}$. For $i = 0, \ldots, m-1$, let A_i be the $\leqslant_R$-minimal element of $\{P \in B_n^+\colon\ \inf(P\mathbf{c}_0^i(\beta)) > \inf(\mathbf{c}_0^i(\beta))\}$. The product $A_{m-1}A_{m-2}\cdots A_0$ is called the *cycling commutator* of β and denoted T_β.

By definition, the cycling commutator T_β is a positive braid. By Lemma 2.11(i),

$$\begin{aligned} T_\beta \beta T_\beta^{-1} &= A_{m-1}\cdots A_2 A_1 A_0 \beta A_0^{-1} A_1^{-1} A_2^{-1}\cdots A_{m-1}^{-1} \\ &= A_{m-1}\cdots A_2 A_1 \mathbf{c}_0(\beta) A_1^{-1} A_2^{-1}\cdots A_{m-1}^{-1} \\ &= A_{m-1}\cdots A_2 \mathbf{c}_0^2(\beta) A_2^{-1}\cdots A_{m-1}^{-1} \\ &= \cdots = \mathbf{c}_0^m(\beta) = \beta. \end{aligned}$$

Lemma 7.2. *Let $\alpha \in B_n$ and $\beta \in [\alpha]^U$. Then the cycling commutator T_β is a non-identity positive braid with $T_\beta\beta = \beta T_\beta$.*

The following proposition is the key to Theorem 7.4. We prove it in Section 8.

Proposition 7.3. *Let α be a non-periodic reducible n-braid with $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$. For any element β of $[\alpha]^U$, the cycling commutator T_β is a split braid.*

Recall from Lemma 5.3 that $\inf_s(\alpha) \leqslant \inf_s(\alpha_{\mathrm{ext}}) \leqslant \sup_s(\alpha_{\mathrm{ext}}) \leqslant \sup_s(\alpha)$ and $t_{\inf}(\alpha) \leqslant t_{\inf}(\alpha_{\mathrm{ext}}) \leqslant t_{\sup}(\alpha_{\mathrm{ext}}) \leqslant t_{\sup}(\alpha)$ for any non-periodic reducible braid α.

Theorem 7.4. *Let α be a non-periodic reducible n-braid.*

(i) *If $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$, then each element of $[\alpha]^U$ has a standard reduction system.*
(ii) *If $\sup_s(\alpha_{\mathrm{ext}}) < \sup_s(\alpha)$, then each element of $[\alpha]^U_{\mathbf{d}}$ has a standard reduction system.*
(iii) *If α is a split braid, then each element of $[\alpha]^U \cup [\alpha]^U_{\mathbf{d}}$ has a standard reduction system.*
(iv) *If α_{ext} is periodic, then there exists $1 \leqslant q < n$ such that each element of $[\alpha^q]^U \cup [\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.*
(v) *If $t_{\inf}(\alpha_{\mathrm{ext}}) > t_{\inf}(\alpha)$, then there exists $1 \leqslant q < n(n-1)/2$ such that each element of $[\alpha^q]^U$ has a standard reduction system.*
(vi) *If $t_{\sup}(\alpha_{\mathrm{ext}}) < t_{\sup}(\alpha)$, then there exists $1 \leqslant q < n(n-1)/2$ such that each element of $[\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.*

Proof. (i) Let β be an element of $[\alpha]^U$. By Proposition 7.3, the cycling commutator T_β is a non-identity positive split braid. By Corollary 6.3, $\mathcal{R}_{\mathrm{ext}}(T_\beta)$ is standard. Since β commutes with T_β by Lemma 7.2, $\mathcal{R}_{\mathrm{ext}}(T_\beta)$ is a standard reduction system of β by Lemma 5.1.

(ii) Because $\inf_s((\alpha^{-1})_{\mathrm{ext}}) = \inf_s((\alpha_{\mathrm{ext}})^{-1}) = -\sup_s(\alpha_{\mathrm{ext}})$ and $\inf_s(\alpha^{-1}) = -\sup_s(\alpha)$, we have $\inf_s((\alpha^{-1})_{\mathrm{ext}}) > \inf_s(\alpha^{-1})$. By (i), each element of $[\alpha^{-1}]^U$ has a standard reduction system. Because $[\alpha]^U_{\mathbf{d}} = \{\beta^{-1}\colon \beta \in [\alpha^{-1}]^U\}$, we are done.

(iii) Let $\beta \in [\alpha]^U$. If $\inf_s(\alpha) < \inf_s(\alpha_{\mathrm{ext}})$, then β has a standard reduction system by (i). If $\inf_s(\alpha) = \inf_s(\alpha_{\mathrm{ext}})$, then $\inf(\beta) = \inf_s(\alpha) = \inf_s(\alpha_{\mathrm{ext}}) = 0$ and, hence, β is positive. Since β is split, $\mathcal{R}_{\mathrm{ext}}(\beta)$ is standard by Corollary 6.3.

Since α is a split braid, so is α^{-1}. Thus, every element of $[\alpha^{-1}]^U$ and, hence, $[\alpha]^U_{\mathbf{d}}$ has a standard reduction system.

(iv) Let k be the braid index of α_{ext}. Because α_{ext} is periodic, there exist $1 \leqslant q \leqslant k$ and $l \in \mathbb{Z}$ such that

$$(\alpha_{\mathrm{ext}})^q = \Delta_0^{2l},$$

where Δ_0 is the fundamental braid of B_k. Then $\Delta^{-2l}\alpha^q \neq 1$ is a split braid. By (iii), every element of $[\Delta^{-2l}\alpha^q]^U \cup [\Delta^{-2l}\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system. Since

$$\left[\alpha^q\right]^U = \left\{\Delta^{2l}\beta\colon \beta \in \left[\Delta^{-2l}\alpha^q\right]^U\right\} \quad \text{and} \quad \left[\alpha^q\right]^U_{\mathbf{d}} = \left\{\Delta^{2l}\beta\colon \beta \in \left[\Delta^{-2l}\alpha^q\right]^U_{\mathbf{d}}\right\},$$

each element of $[\alpha^q]^U \cup [\alpha^q]^U_{\mathbf{d}}$ has a standard reduction system.

(v) Recall from Theorem 2.9 that, for any $\gamma \in B_n$,

- $t_{\inf}(\gamma)$ is rational with denominator less than or equal to $|\Delta| = n(n-1)/2$;
- $\inf_s(\gamma) \leqslant t_{\inf}(\gamma) < \inf_s(\gamma) + 1$;
- $t_{\inf}(\gamma^m) = m t_{\inf}(\gamma)$ for all integers $m \geqslant 1$.

Let k be the braid index of α_{ext}. Then $t_{\inf}(\alpha_{\mathrm{ext}}) = p/q$ for some integers p, q with $1 \leqslant q \leqslant k(k-1)/2$. Since $t_{\inf}((\alpha_{\mathrm{ext}})^q) = q t_{\inf}(\alpha_{\mathrm{ext}})$ is an integer, we have $\inf_s((\alpha_{\mathrm{ext}})^q) = q t_{\inf}(\alpha_{\mathrm{ext}})$. Therefore,

$$\inf_s\left(\left(\alpha^q\right)_{\mathrm{ext}}\right) = \inf_s\left((\alpha_{\mathrm{ext}})^q\right) = q t_{\inf}(\alpha_{\mathrm{ext}}) > q t_{\inf}(\alpha) = t_{\inf}\left(\alpha^q\right) \geqslant \inf_s\left(\alpha^q\right).$$

By (i), every element of $[\alpha^q]^U$ has a standard reduction system.

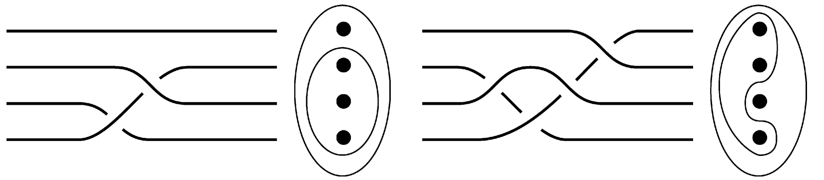
(vi) It can be proved in a way similar to (v). □

Now, let us consider the following algorithm. Let α be a given non-periodic n-braid.

Step 1. Applying cyclings and decyclings to α, obtain an element β of the set $[\alpha]^U \cap [\alpha]^U_{\mathbf{d}}$ together with an element γ such that $\alpha = \gamma\beta\gamma^{-1}$.
Step 2. Decide whether β has a standard reduction system or not.
Step 3. If β has no standard reduction system, then return "we cannot decide whether α is reducible," and halt.
Step 4. Find a standard reduction system, say $\mathcal{C}$, of β.
Step 5. Return "$\gamma * \mathcal{C}$ is a reduction system of α."

Note that, from definitions,

$$[\alpha]^U \cap [\alpha]^U_{\mathbf{d}} = \left\{\beta \in [\alpha]^S\colon \mathbf{c}^{\ell}(\beta) = \beta = \mathbf{d}^m(\beta) \text{ for some } \ell, m \geqslant 1\right\}.$$



(a) $\alpha = \sigma_1^{-1}\sigma_2 \in B_4$. (b) $\beta = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3$.

Fig. 14. α is a split braid. $\beta \in [\alpha]^S$ has no standard reduction system.

This set is called the *reduced super summit set*, and known to be nonempty [Lee00].

Theorem 7.4(i), (ii) and (iii) say that the above algorithm finds a reduction system of a non-periodic reducible braid α if either $\inf_s(\alpha_{\rm ext}) > \inf_s(\alpha)$, $\sup_s(\alpha_{\rm ext}) < \sup_s(\alpha)$, or α is a split braid. This implies that, roughly speaking, if the outermost component $\alpha_{\rm ext}$ is simpler than the whole braid α up to conjugacy, then we can find a reduction system of α from any element of $[\alpha]^U \cap [\alpha]^U_{\mathbf{d}}$.

In Theorem 7.4, the conditions in (v) and (vi) are weaker than those in (i) and (ii). Because $\inf_s(\cdot)$ and $\sup_s(\cdot)$ are integer-valued, Theorem 2.9(iii) implies the following.

- If $\inf_s(\alpha_{\rm ext}) > \inf_s(\alpha)$, then $\inf_s(\alpha_{\rm ext}) \geqslant \inf_s(\alpha) + 1$ and, hence,

$$t_{\rm inf}(\alpha_{\rm ext}) \geqslant \inf_s(\alpha_{\rm ext}) \geqslant \inf_s(\alpha) + 1 > t_{\rm inf}(\alpha).$$

- If $\sup_s(\alpha_{\rm ext}) < \sup_s(\alpha)$, then $\sup_s(\alpha_{\rm ext}) \leqslant \sup_s(\alpha) - 1$ and, hence,

$$t_{\rm sup}(\alpha_{\rm ext}) \leqslant \sup_s(\alpha_{\rm ext}) \leqslant \sup_s(\alpha) - 1 < t_{\rm sup}(\alpha).$$

Note that, for any $m \neq 0$, a braid α is reducible if and only if α^m is reducible. Therefore, in order to decide the reducibility of α, it suffices to decide the reducibility of α^m for an arbitrary $m \neq 0$. If $t_{\rm inf}(\alpha_{\rm ext}) > t_{\rm inf}(\alpha)$ or $t_{\rm sup}(\alpha_{\rm ext}) < t_{\rm sup}(\alpha)$, then the above algorithm, applied to α^m for $1 \leqslant m < n(n-1)/2$, finds a reduction system of α^m and, hence, decides the reducibility of α. Consequently, the non-periodic reducible braids whose reducibility are not decidable by Theorem 7.4 are those with $t_{\rm inf}(\alpha_{\rm ext}) = t_{\rm inf}(\alpha)$ and $t_{\rm sup}(\alpha_{\rm ext}) = t_{\rm sup}(\alpha)$.

We close this section with some examples. From the examples, we can see that, in each statement of Theorem 7.4, the assertion does not hold if one of the conditions is weakened.

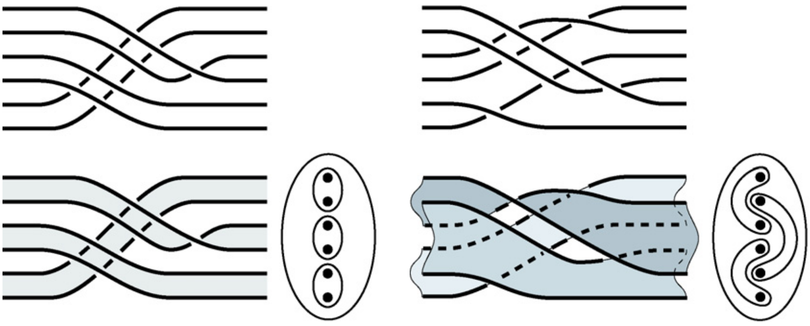
Example 7.5 shows that Theorem 7.4(i), (ii) and (iii) do not hold for super summit sets. Namely, there is a split braid who satisfies the conditions (i) and (ii) but whose super summit set contains an element without standard reduction system.

Example 7.5. Let $\alpha = \sigma_1^{-1}\sigma_2 \in B_4$ and $\beta = (\sigma_3\sigma_2)^{-1}\alpha(\sigma_3\sigma_2) = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3$. (See Fig. 14.) Then α is a split braid with

$$0 = \inf_s(\alpha_{\rm ext}) > \inf_s(\alpha) = -1 \quad \text{and} \quad 0 = \sup_s(\alpha_{\rm ext}) < \sup_s(\alpha) = 1$$

and $\beta \in [\alpha]^S$, but β has no standard reduction system.

Example 7.6 shows the following.



(a) $\alpha = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_3\sigma_4\sigma_3$. (b) $\beta = \sigma_4\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_4\sigma_3\sigma_2$.

Fig. 15. The 6-braid α is non-periodic reducible with $\inf_s(\alpha) = \inf_s(\alpha_{\text{ext}})$. The braid β belongs to $[\alpha]^U$, but β has no standard reduction system.

- Theorem 7.4(i) and (ii) do not hold for $\inf_s(\alpha_{\text{ext}}) = \inf_s(\alpha)$ and $\sup_s(\alpha_{\text{ext}}) = \sup_s(\alpha)$, respectively. Namely, there is a non-periodic reducible braid α with $\inf_s(\alpha_{\text{ext}}) = \inf_s(\alpha)$ and $\sup_s(\alpha_{\text{ext}}) = \sup_s(\alpha)$ such that the set $[\alpha]^U \cap [\alpha]^U_{\mathbf{d}}$ contains an element without standard reduction system.
- For a non-periodic reducible braid α with periodic α_{ext}, it is necessary to consider the ultra summit set $[\alpha^q]^U$ of some power of α in Theorem 7.4(iv). Namely, there is a non-periodic reducible α with periodic α_{ext} such that $[\alpha]^U$ contains an element without standard reduction system.

Example 7.6. Consider the following 6-braids in Fig. 15.

$$\alpha = \sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_3\sigma_4\sigma_3,$$
$$\beta = \left(\sigma_2\sigma_4^{-1}\right)^{-1}\alpha\left(\sigma_2\sigma_4^{-1}\right) = \sigma_4\sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_4\sigma_3\sigma_2.$$

Observe that α is a non-periodic reducible braid such that $\alpha_{\text{ext}} = \sigma_1\sigma_2$ is a periodic 3-braid. Since α_{ext}, α and β are all permutation braids, we have

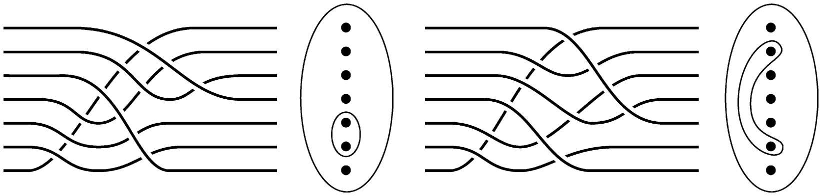
$$\inf_s(\alpha) = 0 = \inf_s(\alpha_{\text{ext}}); \qquad \sup_s(\alpha) = 1 = \sup_s(\alpha_{\text{ext}}); \quad \beta \in [\alpha]^U \cap [\alpha]^U_{\mathbf{d}}.$$

It is easy to see that β has no standard reduction system.

Example 7.7 is due to Juan González-Meneses and Bert Wiest. The authors are very grateful to them for providing it. It shows that Theorem 7.4(v) and (vi) do not hold for $t_{\inf}(\alpha_{\text{ext}}) = t_{\inf}(\alpha)$ and $t_{\sup}(\alpha_{\text{ext}}) = t_{\sup}(\alpha)$, respectively. More precisely, there exist a non-periodic reducible braid α with $t_{\inf}(\alpha_{\text{ext}}) = t_{\inf}(\alpha)$ and $t_{\sup}(\alpha_{\text{ext}}) = t_{\sup}(\alpha)$, and an element β such that, for each $q \geqslant 1$, the power β^q belongs to the set $[\alpha^q]^U \cap [\alpha^q]^U_{\mathbf{d}}$ but has no standard reduction system.

Example 7.7. Consider the following 7-braids in Fig. 16.

$$\alpha = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_6\sigma_5\sigma_4,$$
$$\beta = (\sigma_3\sigma_4\sigma_5)^{-1}\alpha(\sigma_3\sigma_4\sigma_5) = \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_6\sigma_5\sigma_4\sigma_3.$$



(a) $\alpha = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_3\sigma_2\sigma_1\sigma_5\sigma_4\sigma_6\sigma_5\sigma_4$. (b) $\beta = \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_3\sigma_5\sigma_6\sigma_5\sigma_4\sigma_3$.

Fig. 16. The 7-braid α is non-periodic reducible with $t_{\inf}(\alpha) = t_{\inf}(\alpha_{\rm ext}) = 0$ and $t_{\sup}(\alpha_{\rm ext}) = t_{\sup}(\alpha) = 1$. For all $q \geqslant 1$, the power β^q belongs to the set $[\alpha^q]^U \cap [\alpha^q]^U_{\mathbf{d}}$, but β^q has no standard reduction system.

Observe that

(i) both α and β are permutation braids;
(ii) α and β are non-periodic reducible braids with reduction systems as in Fig. 16;
(iii) because $\alpha_{\rm ext}$ is pseudo-Anosov, the curves in Fig. 16(a) and (b) are the only reduction systems of α^q and β^q, respectively, for all $q \neq 0$.

Let $B = \beta$. (Throughout the paper, we have used capital letters $A, B, \ldots$ to denote permutation braids.) The starting set and finishing set of B are

$$S(B) = \{1, 3, 6\} \quad \text{and} \quad F(B) = \{1, 3, 4, 6\}.$$

Since $S(B) \subset F(B)$, the left normal form of β^q is $\Delta^0 \underbrace{BB\cdots B}_{q}$ for all $q \geqslant 1$. In particular, for all $q \geqslant 1$,

$$\mathbf{c}(\beta^q) = \beta^q, \qquad \mathbf{d}(\beta^q) = \beta^q, \qquad \inf(\beta^q) = 0 \quad \text{and} \quad \sup(\beta^q) = q.$$

Therefore, for all $q \geqslant 1$, the power β^q belongs to the set $[\alpha^q]^U \cap [\alpha^q]^U_{\mathbf{d}}$ and

$$t_{\inf}(\alpha) = t_{\inf}(\beta) = \lim_{q\to\infty} \inf(\beta^q)/q = 0;$$
$$t_{\sup}(\alpha) = t_{\sup}(\beta) = \lim_{q\to\infty} \sup(\beta^q)/q = 1.$$

The outermost component $\alpha_{\rm ext}$ is obtained from α by deleting the second strand. Similarly to the above, we can see that $t_{\inf}(\alpha_{\rm ext}) = 0 = t_{\inf}(\alpha)$ and $t_{\sup}(\alpha_{\rm ext}) = 1 = t_{\sup}(\alpha)$.

8. Proof of Proposition 7.3

In this section, we prove Proposition 7.3 that if α is a non-periodic reducible n-braid with $\inf_s(\alpha_{\rm ext}) > \inf_s(\alpha)$, then for any element β of $[\alpha]^U$, the cycling commutator T_β is a split braid.

Throughout this section, the notation $\mathrm{St}^{\rm ext}(\gamma)$ is used as an abbreviation for $\mathrm{St}(\mathcal{R}_{\rm ext}(\gamma))$, the standardizer of the outermost component of the canonical reduction system of the braid γ. Therefore $\mathrm{St}^{\rm ext}(\gamma)$ consists of all positive braids P such that $P * \mathcal{R}_{\rm ext}(\gamma) = \mathcal{R}_{\rm ext}(P\gamma P^{-1})$ is standard. Recall that if $\gamma \in [\gamma]^U$ and P is the $\leqslant_R$-minimal element of $\mathrm{St}^{\rm ext}(\gamma)$, then $P\gamma P^{-1} \in [\gamma]^U$ by Theorem 4.9.

$$\begin{tikzcd}
\beta \arrow[r, "A_0"] \arrow[d, "P_0"] & \mathbf{c}_0(\beta) \arrow[r, "A_1"] \arrow[d, "P_1"] & \mathbf{c}_0^2(\beta) \arrow[r, "A_2"] \arrow[d, "P_2"] & \cdots \arrow[r, "A_{m-1}"] & \mathbf{c}_0^m(\beta) = \beta \arrow[d, "P_m = P_0"] \\
\gamma^{(0)} \arrow[r, "B_0"] & \gamma^{(1)} \arrow[r, "B_1"] & \gamma^{(2)} \arrow[r, "B_2"] & \cdots \arrow[r, "B_{m-1}"] & \gamma^{(m)} = \gamma^{(0)}
\end{tikzcd}$$

Fig. 17. "$\alpha \xrightarrow{A} \beta$" denotes $\beta = A\alpha A^{-1}$.

$$\begin{tikzcd}
\mathbf{c}_0^i(\beta) \arrow[r, "A_i"] \arrow[d, "P_i"] & \mathbf{c}_0^{i+1}(\beta) \arrow[d, "P_{i+1}"] \arrow[dr, bend left, "P'_{i+1}"] & \\
\gamma^{(i)} \arrow[r, "B_i"] \arrow[drr, bend right, "B'_i"] & \gamma^{(i+1)} \arrow[dr, "B''_i"] & \\
& & \mathbf{c}_0(\gamma^{(i)})
\end{tikzcd}$$

Fig. 18. "$\alpha \xrightarrow{A} \beta$" denotes $\beta = A\alpha A^{-1}$.

Let β be an element of the ultra summit set $[\alpha]^U$. Then $\mathbf{c}_0^m(\beta) = \beta$ for some $m \geqslant 1$. For each $i = 0, \dots, m$, we define n-braids A_i, P_i and $\gamma^{(i)}$ as follows (see Fig. 17):

- A_i is the $\leqslant_R$-minimal element of $\{P \in B_n^+ \colon \inf(P\mathbf{c}_0^i(\beta)) > \inf(\mathbf{c}_0^i(\beta))\}$;
- P_i is the $\leqslant_R$-minimal element of $\mathrm{St}^{\mathrm{ext}}(\mathbf{c}_0^i(\beta))$;
- $\gamma^{(i)} = P_i \mathbf{c}_0^i(\beta) P_i^{-1}$.

Then, for each $i = 0, \dots, m-1$,

- A_i is a permutation braid with $\mathbf{c}_0^{i+1}(\beta) = A_i \mathbf{c}_0^i(\beta) A_i^{-1}$ by Lemma 2.11(i);
- $\mathcal{R}_{\mathrm{ext}}(\gamma^{(i)})$ is standard because $\mathcal{R}_{\mathrm{ext}}(\gamma^{(i)}) = \mathcal{R}_{\mathrm{ext}}(P_i \mathbf{c}_0^i(\beta) P_i^{-1}) = P_i * \mathcal{R}_{\mathrm{ext}}(\mathbf{c}_0^i(\beta))$ and $P_i \in \mathrm{St}^{\mathrm{ext}}(\mathbf{c}_0^i(\beta))$;
- $\gamma^{(i)}$ belongs to $[\alpha]^U$ by Theorem 4.9.

Lemma 8.1. *For $i = 0, \dots, m-1$, there exists a permutation braid B_i such that $B_i P_i = P_{i+1} A_i$ and $\gamma^{(i+1)} = B_i \gamma^{(i)} B_i^{-1}$.*

Proof. (See Fig. 18.) Let B'_i be the $\leqslant_R$-minimal element of $\{P \in B_n^+ \colon \inf(P\gamma^{(i)}) > \inf(\gamma^{(i)})\}$. Then B'_i is a permutation braid by Lemma 2.11, and

$$\inf\bigl(B'_i \gamma^{(i)}\bigr) > \inf\bigl(\gamma^{(i)}\bigr) \quad \text{and} \quad \mathbf{c}_0\bigl(\gamma^{(i)}\bigr) = B'_i \gamma^{(i)} B_i'^{-1}.$$

Since both $\gamma^{(i)}$ and $\mathbf{c}_0^i(\beta)$ belong to $[\alpha]^U$, we have $\inf(\gamma^{(i)}) = \inf(\mathbf{c}_0^i(\beta)) = \inf_s(\alpha)$. Since

$$\inf\bigl(B'_i P_i \mathbf{c}_0^i(\beta)\bigr) = \inf\bigl(B'_i \gamma^{(i)} P_i\bigr) \geqslant \inf\bigl(B'_i \gamma^{(i)}\bigr) > \inf\bigl(\gamma^{(i)}\bigr) = \inf\bigl(\mathbf{c}_0^i(\beta)\bigr),$$

$B_i'P_i$ belongs to the set $\{P \in B_n^+\colon \inf(P\mathbf{c}_0^i(\beta)) > \inf(\mathbf{c}_0^i(\beta))\}$. Since A_i is the $\leqslant_R$-minimal element of this set, we have $A_i \leqslant_R B_i'P_i$, and hence

$$B_i'P_i = P_{i+1}'A_i \tag{7}$$

for some $P_{i+1}' \in B_n^+$. Note that

$$P_{i+1}'\mathbf{c}_0^{i+1}(\beta)P_{i+1}'^{-1} = P_{i+1}'A_i\mathbf{c}_0^i(\beta)A_i^{-1}P_{i+1}'^{-1} = B_i'P_i\mathbf{c}_0^i(\beta)P_i^{-1}B_i'^{-1} = B_i'\gamma^{(i)}B_i'^{-1} = \mathbf{c}_0\big(\gamma^{(i)}\big).$$

Since $\mathcal{R}_{\mathrm{ext}}(\mathbf{c}_0(\gamma^{(i)}))$ is standard by Lemma 5.4, P_{i+1}' belongs to $\mathrm{St}^{\mathrm{ext}}(\mathbf{c}_0^{i+1}(\beta))$. Since P_{i+1} is the $\leqslant_R$-minimal element of $\mathrm{St}^{\mathrm{ext}}(\mathbf{c}_0^{i+1}(\beta))$, we have $P_{i+1} \leqslant_R P_{i+1}'$. Therefore,

$$P_{i+1}' = B_i''P_{i+1} \tag{8}$$

for some $B_i'' \in B_n^+$. Observe that

$$P_{i+1}A_i\mathbf{c}_0^i(\beta)A_i^{-1}P_{i+1}^{-1} = P_{i+1}\mathbf{c}_0^{i+1}(\beta)P_{i+1}^{-1} = \gamma^{(i+1)}.$$

Since $\mathcal{R}_{\mathrm{ext}}(\gamma^{(i+1)})$ is standard, $P_{i+1}A_i$ belongs to $\mathrm{St}^{\mathrm{ext}}(\mathbf{c}_0^i(\beta))$. Since P_i is the $\leqslant_R$-minimal element of $\mathrm{St}^{\mathrm{ext}}(\mathbf{c}_0^i(\beta))$, we have $P_i \leqslant_R P_{i+1}A_i$. Therefore

$$P_{i+1}A_i = B_iP_i \tag{9}$$

for some $B_i \in B_n^+$. It is obvious that $\gamma^{(i+1)} = B_i\gamma^{(i)}B_i^{-1}$. From (7), (8) and (9),

$$B_i'P_i = P_{i+1}'A_i = B_i''P_{i+1}A_i = B_i''B_iP_i.$$

Therefore $B_i' = B_i''B_i$. Since B_i' is a permutation braid and $B_i \leqslant_R B_i'$, the positive braid B_i is a permutation braid as desired. $\square$

Let $\mathcal{R}_{\mathrm{ext}}(\gamma^{(0)}) = \mathcal{C}_{\mathbf{n}}$ for a composition $\mathbf{n} = (n_1, \ldots, n_k)$ of n. Let Δ_i be the fundamental braid of B_{n_i}.

Lemma 8.2. *For $i = 0, \ldots, m-1$, $\mathcal{R}_{\mathrm{ext}}(\gamma^{(i)}) = \mathcal{C}_{\mathbf{n}}$ and $B_i \leqslant_R (\Delta_1 \oplus \cdots \oplus \Delta_k)$.*

Proof. Using induction on i, it suffices to show the following:

$$\text{If } \mathcal{R}_{\mathrm{ext}}\big(\gamma^{(i)}\big) = \mathcal{C}_{\mathbf{n}}, \quad \text{then } B_i \leqslant_R (\Delta_1 \oplus \cdots \oplus \Delta_k) \quad \text{and} \quad \mathcal{R}_{\mathrm{ext}}\big(\gamma^{(i+1)}\big) = \mathcal{C}_{\mathbf{n}}.$$

Suppose $\mathcal{R}_{\mathrm{ext}}(\gamma^{(i)}) = \mathcal{C}_{\mathbf{n}}$. By Lemma 3.5(ii) and (iv),

$$\gamma^{(i)} = (\gamma_1 \oplus \cdots \oplus \gamma_k)\langle\gamma_0\rangle_{\mathbf{n}},$$

where $\gamma_0 = \gamma^{(i)}{}_{\mathrm{ext}} \in B_k$ and $\gamma_j \in B_{n_j}$ for $j = 1, \ldots, k$. Since $\inf_s(\alpha_{\mathrm{ext}}) > \inf_s(\alpha)$ (from the hypothesis) and $\gamma^{(i)} \in [\alpha]^U$, we have $\inf(\gamma_{\mathrm{ext}}^{(i)}) > \inf(\gamma^{(i)})$ by Lemma 5.5. By Lemma 3.6,

$$\inf(\gamma_0) = \inf\big(\gamma_{\mathrm{ext}}^{(i)}\big) > \inf\big(\gamma^{(i)}\big) = \min\big\{\inf(\gamma_i)\colon i = 0, \ldots, k, \mathrm{br}(\gamma_i) \geqslant 2\big\}.$$

Therefore $\inf(\gamma_0) > \inf(\gamma_j)$ for some $j \geqslant 1$ with $\mathrm{br}(\gamma_j) \geqslant 2$, and

$$\begin{aligned}
\inf\bigl((\Delta_1 \oplus \cdots \oplus \Delta_k)\gamma^{(i)}\bigr) &= \inf\bigl((\Delta_1\gamma_1 \oplus \cdots \oplus \Delta_k\gamma_k)\langle\gamma_0\rangle_{\mathbf{n}}\bigr) \\
&= \min\bigl(\bigl\{\inf(\Delta_j\gamma_j)\colon\ j = 1, \ldots, k, \mathrm{br}(\gamma_j) \geqslant 2\bigr\} \cup \bigl\{\inf(\gamma_0)\bigr\}\bigr) \\
&= \min\bigl(\bigl\{\inf(\gamma_j) + 1\colon\ j = 1, \ldots, k, \mathrm{br}(\gamma_j) \geqslant 2\bigr\} \cup \bigl\{\inf(\gamma_0)\bigr\}\bigr) \\
&> \inf\bigl(\gamma^{(i)}\bigr).
\end{aligned}$$

So $(\Delta_1 \oplus \cdots \oplus \Delta_k) \in \{P \in B_n^+\colon\ \inf(P\gamma^{(i)}) > \inf(\gamma^{(i)})\}$. Recall, from the proof of Lemma 8.1, that $B_i \leqslant_R B_i'$, where B_i' is the $\leqslant_R$-minimal element of $\{P \in B_n^+\colon\ \inf(P\gamma^{(i)}) > \inf(\gamma^{(i)})\}$. Therefore,

$$B_i \leqslant_R B_i' \leqslant_R (\Delta_1 \oplus \cdots \oplus \Delta_k)$$

as desired. This implies that B_i has the decomposition $B_i = (B_{i,1} \oplus \cdots \oplus B_{i,k})$ for some permutation n_j-braid $B_{i,j}$'s. By Lemma 3.5(ii), $B_i * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$. Therefore, $\mathcal{R}_{\mathrm{ext}}(\gamma^{(i+1)}) = \mathcal{R}_{\mathrm{ext}}(B_i\gamma^{(i)}B_i^{-1}) = B_i * \mathcal{R}_{\mathrm{ext}}(\gamma^{(i)}) = B_i * \mathcal{C}_{\mathbf{n}} = \mathcal{C}_{\mathbf{n}}$. $\square$

Let $S = B_{m-1} \cdots B_0$. Then S is a split braid by Lemma 8.2. Note that the cycling commutator of β is $T_\beta = A_{m-1} \cdots A_0$. Since $P_0^{-1}SP_0 = T_\beta$ by Lemma 8.1, T_β is a split braid and the proof is completed.

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