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Hermitian structures on the derived category of coherent sheaves

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Abstract

The main objective of the present paper is to set up the theoretical basis and the language needed to deal with the problem of direct images of Hermitian vector bundles for projective non-necessarily smooth morphisms. To this end, we first define Hermitian structures on the objects of the bounded derived category of coherent sheaves on a smooth complex variety. Secondly we extend the theory of Bott–Chern classes to these Hermitian structures. Finally we introduce the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ whose morphisms are projective morphisms with a Hermitian structure on the relative tangent complex.

Résumé

L'objectif de cet article est d'établir les bases et le langage nécessaires pour traiter la question des images directes de fibrés hermitiens par des morphismes projectifs non nécessairement lisses. À cette fin, on définit dans un premier temps des structures hermitiennes sur les objets de la catégorie dérivée bornée des faisceaux cohérents sur une variété complexe lisse. Ensuite on étend la théorie des classes de Bott–Chern à ces structures hermitiennes. Finalement on introduit la catégorie $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ dont les morphismes sont les morphismes projectifs munis d'une structure hermitienne sur le complexe tangent relatif.

Keywords: Derived category; Coherent sheaves; Meager complexes

1. Introduction

Derived categories were introduced in the 60's of the last century by Grothendieck and Verdier in order to study and generalize duality phenomenons in Algebraic Geometry (see [21,28]). Since then, derived categories had become a standard tool in Algebra and Geometry and the right framework to define derived functors and to study homological properties. A paradigmatic example is the definition of direct image of sheaves. Given a map $\pi : X \to Y$ between

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varieties and a sheaf $\mathcal{F}$ on X, there is a notion of direct image $\pi_*\mathcal{F}$. We are not specifying what kind of variety or sheaf we are talking about because the same circle of ideas can be used in many different settings. This direct image is not exact in the sense that if $f:\mathcal{F}\to\mathcal{G}$ is a surjective map of sheaves, the induced morphism $\pi_*f:\pi_*\mathcal{F}\to\pi_*\mathcal{G}$ is not necessarily surjective. One then can define a *derived* functor $R\pi_*$ that takes values in the derived category of sheaves on Y and that is exact in an appropriate sense. This functor encodes a lot of information about the topology of the fibres of the map π.

The interest for the derived category of coherent sheaves on a variety exploded with the celebrated 1994 lecture by Kontsevich [23], interpreting mirror symmetry as an equivalence between the derived category of the Fukaya category of certain symplectic manifold and the derived category of coherent sheaves of a dual complex manifold. In the last decades, many interesting results about the derived category of coherent sheaves have been obtained, like the Bondal–Orlov Theorem [8] that shows that a projective variety with ample canonical or anti-canonical bundle can be recovered from its derived category of coherent sheaves. Moreover, new tools for studying algebraic varieties have been developed in the context of derived categories like the Fourier–Mukai transform [25]. The interested reader is referred to books like [22] and [2] for a thorough exposition of recent developments in this area.

Hermitian vector bundles are ubiquitous in Mathematics. An interesting problem is to define the direct image of Hermitian vector bundles. More concretely, let $\pi:X\to Y$ be a proper holomorphic map of complex manifolds and let $\overline{E}=(E,h)$ be a Hermitian holomorphic vector bundle on X. We would like to define the direct image $\pi_*\overline{E}$ as something as close as possible to a Hermitian vector bundle on Y. The information that would be easier to extract from such a direct image is encoded in the determinant of the cohomology [15], that can be defined directly. Assume that π is a submersion and that we have chosen a Hermitian metric on the relative tangent bundle T_π of π satisfying certain technical conditions. Then the determinant line bundle $\lambda(E)=\det(R\pi_*E)$ can be equipped with the Quillen metric [26,4,5], that depends on the metrics on E and T_π and is constructed using the analytic torsion [27]. The Quillen metric has applications in Arithmetic Geometry [16,15,20] and also in String Theory [30,1]. Assume furthermore that the higher direct image sheaves $R^i\pi_*E$ are locally free. In general it is not possible to define an analogue of the Quillen metric as a Hermitian metric on each vector bundle $R^i\pi_*E$. But following Bismut and Köhler [7], one can do something almost as good. We can define the L^2-metric on $R^i\pi_*E$ and *correct* it using the higher analytic torsion forms. Although this *corrected metric* is not properly a Hermitian metric, it is enough for constructing characteristic forms and it appears in the Arithmetic Grothendieck–Riemann–Roch Theorem in higher degrees [18].

The main objective of the present paper is to set up the theoretical basis and the language needed to deal with the problem of direct images of Hermitian vector bundles for projective non-necessarily smooth morphisms. This program will be continued in the subsequent paper [11] where we give an axiomatic characterization of analytic torsion forms and we generalize them to projective morphisms. The ultimate goal of this program is to state and prove an Arithmetic Grothendieck–Riemann–Roch Theorem for general projective morphisms. This last result will be the topic of a forthcoming paper.

When dealing with direct images of Hermitian vector bundles for non-smooth morphisms, one is naturally led to consider Hermitian structures on objects of the bounded derived category of coherent sheaves $\mathbf{D}^b$. One reason for this is that, for a non-smooth projective morphism π, instead of the relative tangent bundle one should consider the relative tangent complex, that defines an object of $\mathbf{D}^b(X)$. Another reason is that, in general, the higher direct images $R^i\pi_*E$ are coherent sheaves and the derived direct image $R\pi_*E$ is an object of $\mathbf{D}^b(Y)$.

Thus the first goal of this paper is to define Hermitian structures on objects of the derived category. A possible starting point is to define a Hermitian metric on an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ as an isomorphism $E\dashrightarrow\mathcal{F}$ in $\mathbf{D}^b(X)$, with E a bounded complex of vector bundles, together with a choice of a Hermitian metric on each constituent vector bundle of E. Here we find a problem, because even being X smooth, in the bounded derived category of coherent sheaves of X, not every object can be represented by a bounded complex of locally free sheaves (see [29] and Remark 3.1). Thus the previous idea does not work for general complex manifolds. To avoid this problem we will restrict ourselves to the algebraic category. Thus, from now on the letters $X, Y, \dots$ will denote smooth algebraic varieties over $\mathbb{C}$, and all sheaves will be algebraic.

With the previous definition of Hermitian metric, for each object of $\mathbf{D}^b(X)$ we obtain a class of metrics that is too wide. Different constructions that ought to produce the same metric produce in fact different metrics. This indicates that we may define a Hermitian structure as an equivalence class of Hermitian metrics.

Let us be more precise. Being $\mathbf{D}^b(X)$ a triangulated category, to every morphism $\mathcal{F} \overset{f}{\dashrightarrow} \mathcal{G}$ in $\mathbf{D}^b(X)$ we can associate its cone, that is defined up to a (not unique) isomorphism by the fact that

$$\mathcal{F} \dashrightarrow \mathcal{G} \dashrightarrow \operatorname{cone}(f) \dashrightarrow \mathcal{F}[1]$$

is a distinguished triangle. If now $\mathcal{F}$ and $\mathcal{G}$ are provided with Hermitian metrics, we want that $\operatorname{cone}(f)$ has an induced Hermitian structure that is well defined up to *isometry*. By choosing a representative of the map f by means of morphisms of complexes of vector bundles, we can induce a Hermitian metric on $\operatorname{cone}(f)$, but this Hermitian metric depends on the choices. The idea behind the definition of Hermitian structures is to introduce the finest equivalence relation between metrics such that all possible induced Hermitian metrics on $\operatorname{cone}(f)$ are equivalent.

Once we have defined Hermitian structures a new invariant of X can be naturally defined. Namely, the set of Hermitian structures on a zero object of $\mathbf{D}^b(X)$ is an abelian group that we denote by $\overline{\mathbf{KA}}(X)$ (Definition 2.34). In the same way that $K_0(X)$ is the universal abelian group for additive characteristic classes of vector bundles, $\overline{\mathbf{KA}}(X)$ is the universal abelian group for secondary characteristic classes of acyclic complexes of Hermitian vector bundles (Theorem 2.35).

Secondary characteristic classes constitute other of the central topics of this paper. Recall that to each vector bundle we can associate its Chern character, that is an additive characteristic class. If the vector bundle is provided with a Hermitian metric, we can use Chern–Weil theory to construct a concrete representative of the Chern character, that is a differential form. This characteristic form is additive only for orthogonally split short exact sequences and not for general short exact sequences. Bott–Chern classes were introduced in [9] and are secondary classes that measure the lack of additivity of the characteristic forms.

The Bott–Chern classes have been extensively used in Arakelov Geometry [19,6] and they can be used to construct characteristic classes in higher K-theory [14]. The second goal of this paper is to extend the definition of additive Bott–Chern classes to the derived category. This is the most general definition of additive Bott–Chern classes and encompasses both, the Bott–Chern classes defined in [6] and the ones defined in [24] (Example 4.16).

Finally, recall that the Hermitian structure on the direct image of a Hermitian vector bundle should also depend on a Hermitian structure on the relative tangent complex. Thus the last goal of this paper is to introduce the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ (Definition 5.7, Theorem 5.11). The objects of this category are smooth algebraic varieties over $\mathbb{C}$ and the morphisms are pairs $\overline{f} = (f, \overline{T}_f)$ formed by a projective morphism of smooth complex varieties f, together with a Hermitian structure on the relative tangent complex T_f. The main difficulty here is to define the composition of two such morphisms. The remarkable fact is that the Hermitian cone construction enables us to define a composition rule for these morphisms.

We describe with more details the contents of each section.

In Section 2 we define and characterize the notion of *meager complex* (Definition 2.9 and Theorem 2.13). Roughly speaking, meager complexes are bounded acyclic complexes of Hermitian vector bundles whose Bott–Chern classes vanish for structural reasons. We then introduce the concept of tight morphism (Definition 2.19) and tight equivalence relation (Definition 2.27) between bounded complexes of Hermitian vector bundles. We explain a series of useful computational rules on the monoid of Hermitian vector bundles modulo tight equivalence relation, that we call *acyclic calculus* (Theorem 2.30). We prove that the submonoid of acyclic complexes modulo meager complexes has a structure of abelian group, this is the group $\overline{\mathbf{KA}}(X)$ mentioned previously.

With these tools at hand, in Section 3 we define Hermitian structures on objects of $\mathbf{D}^b(X)$ and we introduce the category $\overline{\mathbf{D}}^b(X)$. The objects of the category $\overline{\mathbf{D}}^b(X)$ are objects of $\mathbf{D}^b(X)$ together with a Hermitian structure, and the morphisms are just morphisms in $\mathbf{D}^b(X)$. Theorem 3.13 is devoted to describe the structure of the forgetful functor $\overline{\mathbf{D}}^b(X) \to \mathbf{D}^b(X)$. In particular, we show that the group $\overline{\mathbf{KA}}(X)$ acts on the fibers of this functor, freely and transitively.

An important example of use of Hermitian structures is the construction of the *Hermitian cone* of a morphism in $\overline{\mathbf{D}}^b(X)$ (Definition 3.14), which is well defined only up to tight isomorphism. We also study several elementary constructions in $\overline{\mathbf{D}}^b(X)$. Here we mention the classes of isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^b(X)$. These classes lie in the group $\overline{\mathbf{KA}}(X)$ and their properties are listed in Theorem 3.33. As an application we show that $\overline{\mathbf{KA}}(X)$ receives classes from $K_1(X)$ (Proposition 3.35).

Section 4 is devoted to the extension of Bott–Chern classes to the derived category. For every additive genus, we associate to each isomorphism or distinguished triangle in $\overline{\mathbf{D}}^b(X)$ a Bott–Chern class satisfying properties analogous to the classical ones.

We conclude the paper with Section 5, where we extend the definition of Bott–Chern classes to multiplicative genera and in particular to the Todd genus. In this section we also define the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$.

2. Meager complexes and acyclic calculus

The aim of this section is to construct a universal group for additive Bott–Chern classes of acyclic complexes of Hermitian vector bundles. To this end we first introduce and study the class of meager complexes. Any Bott–Chern class that is additive for certain short exact sequences of acyclic complexes (see Theorem 2.35) and that vanishes on orthogonally split complexes, necessarily vanishes on meager complexes. Then we develop an acyclic calculus that will ease the task to check if a particular complex is meager. Finally we introduce the group $\overline{\mathbf{KA}}$, which is the universal group for additive Bott–Chern classes.

Let X be a complex algebraic variety over $\mathbb{C}$, namely a reduced and separated scheme of finite type over $\mathbb{C}$. We denote by $\mathbf{V}^b(X)$ the exact category of bounded complexes of algebraic vector bundles on X. Assume in addition that X is smooth over $\mathbb{C}$. Then $\overline{\mathbf{V}}^b(X)$ is defined as the category of pairs $\overline{E} = (E, h)$, where $E \in \operatorname{Ob} \mathbf{V}^b(X)$ and h is a smooth Hermitian metric on the complex of analytic vector bundle E^{an}. From now on we shall make no distinction between E and E^{an}. The complex E will be called *the underlying complex of* $\overline{E}$. We will denote by the symbol $\sim$ the quasi-isomorphisms in any of the above categories.

A basic construction in $\mathbf{V}^b(X)$ is the cone of a morphism of complexes. Recall that, if $f: E \to F$ is such a morphism, then, as a graded vector bundle $\operatorname{cone}(f) = E[1] \oplus F$, the differential is given by $\mathrm{d}(x, y) = (-\mathrm{d}x, f(x) + \mathrm{d}y)$. The construction of a cone allow us to detect quasi-isomorphisms. In fact, we have the following result whose proof is easy.

Proposition 2.1. *A morphism complexes f is a quasi-isomorphism if and only if* $\operatorname{cone}(f)$ *is acyclic.*

We can extend the cone construction easily to $\overline{\mathbf{V}}^b(X)$ as follows.

Definition 2.2. If $f: \overline{E} \to \overline{F}$ is a morphism in $\overline{\mathbf{V}}^b(X)$, *the Hermitian cone* of f, denoted by $\overline{\operatorname{cone}}(f)$, is defined as the cone of f provided with the orthogonal sum Hermitian metric.

When the morphism is clear from the context we will sometimes denote $\overline{\operatorname{cone}}(f)$ by $\overline{\operatorname{cone}}(\overline{E}, \overline{F})$.

Remark 2.3. Let $f: \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{V}}^b(X)$. Then there is an exact sequence of complexes

$$0 \to \overline{F} \to \overline{\operatorname{cone}}(f) \to \overline{E}[1] \to 0,$$

whose constituent short exact sequences are orthogonally split. Conversely, let

$$0 \to \overline{F} \to \overline{G} \to \overline{E}[1] \to 0$$

be a short exact sequence all whose constituent exact sequences are orthogonally split, and $s: E[1] \to G$ the section given by the orthogonal splitting. Then the image of $\mathrm{d}s - s\mathrm{d}$ belongs to F and, in fact, determines a morphism of complexes

$$f_s := \mathrm{d}s - s\mathrm{d}: \overline{E} \to \overline{F}.$$

Moreover, there is a natural isometry $\overline{G} \cong \overline{\operatorname{cone}}(f_s)$.

The Hermitian cone has the following useful property.

Lemma 2.4. *Consider a diagram in* $\overline{\mathbf{V}}^b(X)$

$$\begin{array}{ccc} \overline{E}' & \xrightarrow{f'} & \overline{F}' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{E} & \xrightarrow{f} & \overline{F}. \end{array}$$

Assume that the diagram is commutative up to homotopy and fix a homotopy h. The homotopy h induces morphisms of complexes

$$\psi : \overline{\mathrm{cone}}(f') \to \overline{\mathrm{cone}}(f),$$
$$\phi : \overline{\mathrm{cone}}(-g') \to \overline{\mathrm{cone}}(g)$$

and there is a natural isometry of complexes

$$\overline{\mathrm{cone}}(\phi) \xrightarrow{\sim} \overline{\mathrm{cone}}(\psi).$$

Moreover, let h' be a second homotopy between $g \circ f'$ and $f \circ g'$ and let ψ' be the induced morphism. If there exists a higher homotopy between h and h', then ψ and ψ' are homotopically equivalent.

Proof. Since $h : E' \to F[-1]$ is a homotopy between gf' and fg', we have

$$gf' - fg' = \mathrm{d}h + h\mathrm{d}. \tag{2.5}$$

First of all, define the arrow $\psi : \overline{\mathrm{cone}}(f') \to \overline{\mathrm{cone}}(f)$ by the following rule:

$$\psi(x', y') = \big(g'(x'), g(y') + h(x')\big).$$

From the definition of the differential of a cone and the homotopy relation (2.5), one easily checks that ψ is a morphism of complexes. Now apply the same construction to the diagram

$$\begin{array}{ccc} \overline{E}' & \xrightarrow{-g'} & \overline{E} \\ {\scriptstyle -f'}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{F}' & \xrightarrow{g} & \overline{F}. \end{array} \tag{2.6}$$

The diagram (2.6) is still commutative up to homotopy and h provides such a homotopy. We obtain a morphism of complexes $\phi : \overline{\mathrm{cone}}(-g') \to \overline{\mathrm{cone}}(g)$, defined by the rule

$$\phi(x', x) = \big(-f'(x'), f(x) + h(x')\big).$$

One easily checks that a suitable reordering of factors sets an isometry of complexes between $\overline{\mathrm{cone}}(\phi)$ and $\overline{\mathrm{cone}}(\psi)$. Assume now that h' is a second homotopy and that there is a higher homotopy $s : \overline{E}' \to \overline{F}[-2]$ such that

$$h' - h = \mathrm{d}s - s\mathrm{d}.$$

Let $H : \overline{\mathrm{cone}}(f') \to \overline{\mathrm{cone}}(f)[-1]$ be given by $H(x', y') = (0, s(x'))$. Then

$$\psi' - \psi = \mathrm{d}H + H\mathrm{d}.$$

Hence ψ and ψ' are homotopically equivalent. □

Recall that, given a morphism of complexes $f : \overline{E} \to \overline{F}$, we use the abuse of notation $\overline{\mathrm{cone}}(f) = \overline{\mathrm{cone}}(\overline{E}, \overline{F})$. As seen in the previous lemma, sometimes it is natural to consider $\overline{\mathrm{cone}}(-f)$. With the notation above it will be denoted also by $\overline{\mathrm{cone}}(\overline{E}, \overline{F})$. Note that this ambiguity is harmless because there is a natural isometry between $\overline{\mathrm{cone}}(f)$ and $\overline{\mathrm{cone}}(-f)$. Of course, when more than one morphism between $\overline{E}$ and $\overline{F}$ is considered, the above notation should be avoided.

With this convention, Lemma 2.4 can be written as

$$\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(\overline{E}', \overline{E}), \overline{\mathrm{cone}}(\overline{F}', \overline{F})\big) \cong \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(\overline{E}', \overline{F}'), \overline{\mathrm{cone}}(\overline{E}, \overline{F})\big). \tag{2.7}$$

Definition 2.8. We will denote by $\mathscr{M}_0 = \mathscr{M}_0(X)$ the subclass of $\overline{\mathbf{V}}^b(X)$ consisting of

(i) the orthogonally split complexes;
(ii) all objects $\overline{E}$ such that there is an acyclic complex $\overline{F}$ of $\overline{\mathbf{V}}^b(X)$, and an isometry $\overline{E} \to \overline{F} \oplus \overline{F}[1]$.

We want to stabilize $\mathscr{M}_0$ with respect to Hermitian cones.

Definition 2.9. We will denote by $\mathscr{M} = \mathscr{M}(X)$ the smallest subclass of $\overline{\mathbf{V}}^b(X)$ that satisfies the following properties:

(i) it contains $\mathscr{M}_0$;
(ii) if $f : \overline{E} \to \overline{F}$ is a morphism and two of $\overline{E}$, $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ belong to $\mathscr{M}$, then so does the third.

The elements of $\mathscr{M}(X)$ will be called *meager complexes*.

We next give a characterization of meager complexes. For this, we introduce two auxiliary classes.

Definition 2.10.

(i) Let $\mathscr{M}_F$ be the subclass of $\overline{\mathbf{V}}^b(X)$ that contains all complexes $\overline{E}$ that have a finite filtration Fil such that
  (A) for every $p, n \in \mathbb{Z}$, the exact sequences

$$0 \to \operatorname{Fil}^{p+1} \overline{E}^n \to \operatorname{Fil}^p \overline{E}^n \to \operatorname{Gr}^p_{\operatorname{Fil}} \overline{E}^n \to 0,$$

  with the induced metrics, are orthogonally split short exact sequences of vector bundles;
  (B) the complexes $\operatorname{Gr}^\bullet_{\operatorname{Fil}} \overline{E}$ belong to $\mathscr{M}_0$.

(ii) Let $\mathscr{M}_S$ be the subclass of $\overline{\mathbf{V}}^b(X)$ that contains all complexes $\overline{E}$ such that there is a morphism of complexes $f : \overline{E} \to \overline{F}$ and both $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ belong to $\mathscr{M}_F$.

Lemma 2.11. *Let $0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$ be an exact sequence in $\overline{\mathbf{V}}^b(X)$ whose constituent rows are orthogonally split. Assume $\overline{E}$ and $\overline{G}$ are in $\mathscr{M}_F$. Then $\overline{F} \in \mathscr{M}_F$. In particular, $\mathscr{M}_F$ is closed under cone formation.*

Proof. For the first claim, notice that the filtrations of $\overline{E}$ and $\overline{G}$ induce a filtration on $\overline{F}$ satisfying conditions 2.10(A) and 2.10(B). The second claim then follows by Remark 2.3. □

Example 2.12. Given any complex $\overline{E} \in \operatorname{Ob} \overline{\mathbf{V}}^b(X)$, the complex $\overline{\operatorname{cone}}(\operatorname{id}_{\overline{E}})$ belongs to $\mathscr{M}_F$. This can be seen by induction on the length of $\overline{E}$ using Lemma 2.11 and the bête filtration of $\overline{E}$. For the starting point of the induction one takes into account that, if $\overline{E}$ has only one non-zero degree, then $\overline{\operatorname{cone}}(\operatorname{id}_{\overline{E}})$ is orthogonally split. In fact, this argument shows something slightly stronger. Namely, the complex $\overline{\operatorname{cone}}(\operatorname{id}_{\overline{E}})$ admits a finite filtration Fil satisfying 2.10(A) and such that the complexes $\operatorname{Gr}^\bullet_{\operatorname{Fil}} \overline{\operatorname{cone}}(\operatorname{id}_{\overline{E}})$ are orthogonally split.

Theorem 2.13. *The equality $\mathscr{M} = \mathscr{M}_S$ holds.*

Proof. We start by proving that $\mathscr{M}_F \subset \mathscr{M}$. Let $\overline{E} \in \mathscr{M}_F$ and let Fil be any filtration that satisfies conditions 2.10(A) and 2.10(B). We show that $\overline{E} \in \mathscr{M}$ by induction on the length of Fil. If Fil has length one, then $\overline{E}$ belongs to $\mathscr{M}_0 \subset \mathscr{M}$. If the length of Fil is $k > 1$, let p be such that $\operatorname{Fil}^p \overline{E} = \overline{E}$ and $\operatorname{Fil}^{p+1} \overline{E} \neq \overline{E}$. On the one hand, $\operatorname{Gr}^p_{\operatorname{Fil}} \overline{E}[-1] \in \mathscr{M}_0 \subset \mathscr{M}$ and, on the other hand, the filtration Fil induces a filtration on $\operatorname{Fil}^{p+1} \overline{E}$ fulfilling conditions 2.10(A) and 2.10(B) and has length $k - 1$. Thus, by induction hypothesis, $\operatorname{Fil}^{p+1} \overline{E} \in \mathscr{M}$. Then, by condition 2.10(A) and Remark 2.3, we can write $\overline{E}$ as the cone of a morphism between two elements of $\mathscr{M}$. By the condition 2.9(ii), we deduce that $\overline{E} \in \mathscr{M}$.

Clearly, the fact that $\mathscr{M}_F \subset \mathscr{M}$ implies that $\mathscr{M}_S \subset \mathscr{M}$. Thus, to prove the theorem, it only remains to show that $\mathscr{M}_S$ satisfies the condition 2.9(ii).

The content of the next result is that the apparent asymmetry in the definition of $\mathscr{M}_S$ is not real.

Lemma 2.14. *Let $\overline{E} \in \operatorname{Ob} \overline{\mathbf{V}}^b(X)$. Then there is a morphism $f : \overline{E} \to \overline{F}$ with $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ in $\mathscr{M}_F$ if and only if there is a morphism $g : \overline{G} \to \overline{E}$ with $\overline{G}$ and $\overline{\operatorname{cone}}(g)$ in $\mathscr{M}_F$.*

Proof. Assume that there is a morphism $f : \overline{E} \to \overline{F}$ with $\overline{F}$ and $\overline{\operatorname{cone}}(f)$ in $\mathscr{M}_F$. Then, write $\overline{G} = \overline{\operatorname{cone}}(f)[-1]$ and let $g : \overline{G} \to \overline{E}$ be the natural map. By hypothesis, $\overline{G} \in \mathscr{M}_F$. Moreover, since there is a natural isometry

$$\overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}(\overline{E}, \overline{F})[-1], \overline{E}\big) \cong \overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}(\operatorname{id}_{\overline{E}})[-1], \overline{F}\big),$$

by Example 2.12 and Lemma 2.11 we obtain that $\overline{\operatorname{cone}}(g) \in \mathscr{M}_F$. Thus we have proved one implication. The proof of the other implication is analogous. □

Let now $f : \overline{E} \to \overline{F}$ be a morphism of complexes with $\overline{E}, \overline{F} \in \mathscr{M}_S$. We want to show that $\overline{\operatorname{cone}}(f) \in \mathscr{M}_S$. By Lemma 2.14, there are morphisms of complexes $g : \overline{G} \to \overline{E}$ and $h : \overline{H} \to \overline{F}$ with $\overline{G}, \overline{H}, \overline{\operatorname{cone}}(g), \overline{\operatorname{cone}}(h) \in \mathscr{M}_F$. We consider the map $\overline{G} \to \overline{\operatorname{cone}}(h)$ induced by $f \circ g$. Then we write

$$\overline{G'} = \overline{\operatorname{cone}}\big(\overline{G}, \overline{\operatorname{cone}}(h)\big)[-1].$$

By Lemma 2.11, we have that $\overline{G'} \in \mathscr{M}_F$. We denote by $g' : G' \to E$ and $k : G' \to H$ the maps $g'(a,b,c) = g(a)$ and $k(a,b,c) = -b$.

There is an exact sequence

$$0 \to \overline{\operatorname{cone}}(h) \to \overline{\operatorname{cone}}\big(g'\big) \to \overline{\operatorname{cone}}(g) \to 0$$

whose constituent short exact sequences are orthogonally split. Since $\overline{\operatorname{cone}}(h)$ and $\overline{\operatorname{cone}}(g)$ belong to $\mathscr{M}_F$, Lemma 2.11 implies that $\overline{\operatorname{cone}}(g')$ belongs to $\mathscr{M}_F$ as well.

There is a diagram

$$\begin{array}{ccc} \overline{G'} & \xrightarrow{\;k\;} & \overline{H} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle h} \\ \overline{E} & \xrightarrow{\;f\;} & \overline{F} \end{array} \tag{2.15}$$

that commutes up to homotopy. We fix the homotopy $s : \overline{G}' \to \overline{F}$ given by $s(a,b,c) = c$. By Lemma 2.4 there is a natural isometry

$$\overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}\big(g'\big), \overline{\operatorname{cone}}(h)\big) \cong \overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}(-k), \overline{\operatorname{cone}}(f)\big).$$

Applying Lemma 2.11 again, we have that $\overline{\operatorname{cone}}(-k)$ and $\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(g'), \overline{\operatorname{cone}}(h))$ belong to $\mathscr{M}_F$. Therefore $\overline{\operatorname{cone}}(f)$ belongs to $\mathscr{M}_S$.

Lemma 2.16. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{V}}^b(X)$.*

(i) *If $\overline{E} \in \mathscr{M}_S$ and $\overline{\operatorname{cone}}(f) \in \mathscr{M}_F$ then $\overline{F} \in \mathscr{M}_S$.*
(ii) *If $\overline{F} \in \mathscr{M}_S$ and $\overline{\operatorname{cone}}(f) \in \mathscr{M}_F$ then $\overline{E} \in \mathscr{M}_S$.*

Proof. Assume that $\overline{E} \in \mathscr{M}_S$ and $\overline{\operatorname{cone}}(f) \in \mathscr{M}_F$. Let $g : \overline{G} \to \overline{E}$ with $\overline{G} \in \mathscr{M}_F$ and $\overline{\operatorname{cone}}(g) \in \mathscr{M}_F$. By Lemma 2.11 and Example 2.12, $\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(\mathrm{id}_{\overline{G}}), \overline{\operatorname{cone}}(f)) \in \mathscr{M}_F$. But there is a natural isometry of complexes

$$\overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}(\mathrm{id}_{\overline{G}}), \overline{\operatorname{cone}}(f)\big) \cong \overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}(g)[-1], \overline{G}\big), \overline{F}\big).$$

Since, by Lemma 2.11, $\overline{\operatorname{cone}}(\overline{\operatorname{cone}}(g)[-1], \overline{G}) \in \mathscr{M}_F$, then $\overline{F} \in \mathscr{M}_S$.

The second statement of the lemma is proved using the dual argument. □

Lemma 2.17. *Let $f : \overline{E} \to \overline{F}$ be a morphism in $\overline{\mathbf{V}}^b(X)$.*

(i) *If $\overline{E} \in \mathscr{M}_F$ and $\overline{\operatorname{cone}}(f) \in \mathscr{M}_S$ then $\overline{F} \in \mathscr{M}_S$.*
(ii) *If $\overline{F} \in \mathscr{M}_F$ and $\overline{\operatorname{cone}}(f) \in \mathscr{M}_S$ then $\overline{E} \in \mathscr{M}_S$.*

Proof. Assume that $\overline{E} \in \mathscr{M}_F$ and $\overline{\operatorname{cone}}(f) \in \mathscr{M}_S$. Let $g : \overline{G} \to \overline{\operatorname{cone}}(f)$ with $\overline{G}$ and $\overline{\operatorname{cone}}(\overline{G}, \overline{\operatorname{cone}}(f))$ in $\mathscr{M}_F$. There is a natural isometry of complexes

$$\overline{\operatorname{cone}}\big(\overline{G}, \overline{\operatorname{cone}}(f)\big) \cong \overline{\operatorname{cone}}\big(\overline{\operatorname{cone}}\big(\overline{G}[-1], \overline{E}\big), \overline{F}\big)$$

that shows $\overline{F} \in \mathscr{M}_S$.

The second statement of the lemma is proved by a dual argument. □

Assume now that $f:\overline{E}\to\overline{F}$ is a morphism in $\overline{\mathbf{V}}^b(X)$ and $\overline{E},\overline{\mathrm{cone}}(f)\in\mathscr{M}_S$. Let $g:\overline{G}\to\overline{E}$ with $\overline{G}$, $\overline{\mathrm{cone}}(g)\in\mathscr{M}_F$. There is a natural isometry

$$\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(\overline{G},\overline{E}),\overline{\mathrm{cone}}(\mathrm{id}_{\overline{F}})\big)\cong\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(\overline{G},\overline{F}),\overline{\mathrm{cone}}(\overline{E},\overline{F})\big),$$

that implies $\overline{\mathrm{cone}}(\overline{\mathrm{cone}}(\overline{G},\overline{F}),\overline{\mathrm{cone}}(\overline{E},\overline{F}))\in\mathscr{M}_F$. By Lemma 2.16, we deduce that $\overline{\mathrm{cone}}(\overline{G},\overline{F})\in\mathscr{M}_S$. By Lemma 2.17, $\overline{F}\in\mathscr{M}_S$.

With f as above, the fact that, if $\overline{F}$ and $\overline{\mathrm{cone}}(f)$ belong to $\mathscr{M}_S$ so does $\overline{E}$, is proved by a similar argument. In conclusion, $\mathscr{M}_S$ satisfies the condition 2.9(ii), hence $\mathscr{M}\subset\mathscr{M}_S$, which completes the proof of the theorem. □

The class of meager complexes satisfies the next list of properties, that follow almost directly from Theorem 2.13.

Theorem 2.18.

(i) *If $\overline{E}$ is a meager complex and $\overline{F}$ is a Hermitian vector bundle, then the complexes $\overline{F}\otimes\overline{E}$, $\mathrm{Hom}(\overline{F},\overline{E})$ and $\mathrm{Hom}(\overline{E},\overline{F})$, with the induced metrics, are meager. In particular, the dual of a meager complex with its induced metric is meager.*
(ii) *If $\overline{E}^{*,*}$ is a bounded double complex of Hermitian vector bundles and all rows (or columns) are meager complexes, then the complex $\mathrm{Tot}(\overline{E}^{*,*})$ is meager.*
(iii) *If $\overline{E}$ is a meager complex and $\overline{F}$ is another complex of Hermitian vector bundles, then the complexes*

$$\overline{E}\otimes\overline{F}=\mathrm{Tot}\big(\big(\overline{F}^i\otimes\overline{E}^j\big)_{i,j}\big),$$
$$\underline{\mathrm{Hom}}(\overline{E},\overline{F})=\mathrm{Tot}\big(\mathrm{Hom}\big(\big(\overline{E}^{-i},\overline{F}^j\big)_{i,j}\big)\big)\quad\textit{and}$$
$$\underline{\mathrm{Hom}}(\overline{F},\overline{E})=\mathrm{Tot}\big(\mathrm{Hom}\big(\big(\overline{F}^{-i},\overline{E}^j\big)_{i,j}\big)\big),$$

are meager.
(iv) *If $f:X\to Y$ is a morphism of smooth complex varieties and $\overline{E}$ is a meager complex on Y, then $f^*\overline{E}$ is a meager complex on X.*

We now introduce the notion of tight morphism.

Definition 2.19. A morphism $f:\overline{E}\to\overline{F}$ in $\overline{\mathbf{V}}^b(X)$ is said to be *tight* if $\overline{\mathrm{cone}}(f)$ is a meager complex.

Proposition 2.20.

(i) *Every meager complex is acyclic.*
(ii) *Every tight morphism is a quasi-isomorphism.*

Proof. Let $\overline{E}\in\mathscr{M}_F(X)$. Let Fil be any filtration that satisfies conditions 2.10(A) and 2.10(B). By definition, the complexes $\mathrm{Gr}^p_{\mathrm{Fil}}\overline{E}$ belong to $\mathscr{M}_0$, so they are acyclic. Hence $\overline{E}$ is acyclic.

If $\overline{E}\in\mathscr{M}_S(X)$, let $\overline{F}$ and $\overline{\mathrm{cone}}(f)$ be as in Definition 2.10(ii). Then, $\overline{F}$ and $\overline{\mathrm{cone}}(f)$ are acyclic, hence $\overline{E}$ is also acyclic. Thus we have proved the first statement. The second statement follows from the first and Proposition 2.1. □

Since every meager complex is acyclic, it is natural to ask if some kind of converse is true. Namely, given an acyclic complex of vector bundles E does there exist a choice of Hermitian metrics on the vector bundles such that the resulting complex of Hermitian vector bundles $\overline{E}$ is meager. At this point we do not know the answer, but we suspect that the answer is no. See Remark 2.38 for a more detailed discussion.

Proposition 2.21. *Let $f:\overline{E}\to\overline{F}$ and $g:\overline{F}\to\overline{G}$ be two tight morphisms. Then $g\circ f$ is tight.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \overline{E} & \xrightarrow{f} & \overline{F} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{E} & \xrightarrow{g\circ f} & \overline{G}. \end{array}$$

By Lemma 2.4, there are morphisms of complexes

$$\psi : \overline{\mathrm{cone}}(f) \to \overline{\mathrm{cone}}(g \circ f),$$
$$\phi : \overline{\mathrm{cone}}(-\mathrm{id}) \to \overline{\mathrm{cone}}(g).$$

Since g and $-\mathrm{id}$ are tight, then $\overline{\mathrm{cone}}(\phi)$ is meager. Again by Lemma 2.4, $\overline{\mathrm{cone}}(\psi)$ is meager. Since f is tight we deduce that $g \circ f$ is tight. □

Many arguments used for proving that a certain complex is meager or a certain morphism is tight involve cumbersome diagrams. In order to ease these arguments we will develop a calculus of acyclic complexes.

Before starting we need some preliminary lemmas.

Lemma 2.22. *Let $\overline{E}$, $\overline{F}$ be objects of $\overline{\mathbf{V}}^b(X)$. Then the following conditions are equivalent.*

(i) *There exist an object $\overline{G}$ and a diagram*

$$\begin{array}{ccccc} & & \overline{G} & & \\ & {\scriptstyle f}\swarrow{\scriptstyle \sim} & & \searrow{\scriptstyle g} & \\ \overline{E} & & & & \overline{F}, \end{array}$$

such that f is a quasi-isomorphism and $\overline{\mathrm{cone}}(g) \oplus \overline{\mathrm{cone}}(f)[1]$ is meager.

(ii) *There exist an object $\overline{G}$ and a diagram*

$$\begin{array}{ccccc} & & \overline{G} & & \\ & {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} & \\ \overline{E} & & & & \overline{F}, \end{array}$$

such that f and g are tight morphisms.

Proof. Clearly, since tight morphisms are quasi-isomorphisms, (ii) implies (i). To prove the converse implication, if $\overline{G}$ satisfies the conditions of (i), we put $G' = G \oplus \overline{\mathrm{cone}}(f)$ and consider the morphisms $f' : \overline{G}' \to E$ and $g' : G' \to F$ induced by the first projection $G' \to G$. Then

$$\overline{\mathrm{cone}}\big(f'\big) = \overline{\mathrm{cone}}(f) \oplus \overline{\mathrm{cone}}(f)[1],$$

that is meager because $\overline{\mathrm{cone}}(f)$ is acyclic, and

$$\overline{\mathrm{cone}}\big(g'\big) = \overline{\mathrm{cone}}(g) \oplus \overline{\mathrm{cone}}(f)[1],$$

that is meager by hypothesis. □

Lemma 2.23. *Any diagram of tight morphisms, of the following types*:

$$\begin{array}{ccccc} \overline{E} & & & & \overline{G} \\ & {\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} & \\ & & \overline{F} & & \\ & & \text{(i)} & & \end{array} \qquad \begin{array}{ccccc} & & \overline{H} & & \\ & {\scriptstyle f'}\swarrow & & \searrow{\scriptstyle g'} & \\ \overline{E} & & & & \overline{G} \\ & & \text{(ii)} & & \end{array} \tag{2.24}$$

can be completed into a diagram of tight morphisms

$$\begin{array}{ccccc} & & \overline{H} & & \\ & \swarrow^{f'} & & \searrow^{g'} & \\ \overline{E} & & & & \overline{G} \\ & \searrow_{f} & & \swarrow_{g} & \\ & & \overline{F}, & & \end{array} \tag{2.25}$$

which commutes up to homotopy.

Proof. We prove the statement only for the case (i), the other one being analogous. Note that there is a natural arrow $\overline{G} \to \overline{\text{cone}}(f)$. Define

$$\overline{H} = \overline{\text{cone}}\big(\overline{G}, \overline{\text{cone}}(f)\big)[-1].$$

With this choice, diagram (2.24)(i) becomes commutative up to homotopy, taking the projection $H \to F[-1]$ as homotopy. We first show that $\overline{\text{cone}}(\overline{H}, \overline{G})$ is meager. Indeed, there is a natural isometry

$$\overline{\text{cone}}(\overline{H}, \overline{G}) \cong \overline{\text{cone}}\big(\overline{\text{cone}}(\text{id}_{\overline{G}}), \overline{\text{cone}}(\overline{E}, \overline{F})[-1]\big)$$

and the right-hand side complex is meager. Now for $\overline{\text{cone}}(\overline{H}, \overline{E})$. By Lemma 2.4, there is an isometry

$$\overline{\text{cone}}\big(\overline{\text{cone}}(\overline{H}, \overline{E}), \overline{\text{cone}}(\overline{G}, \overline{F})\big) \cong \overline{\text{cone}}\big(\overline{\text{cone}}(\overline{H}, \overline{G}), \overline{\text{cone}}(\overline{E}, \overline{F})\big). \tag{2.26}$$

The right-hand side complex is meager, hence the left-hand side is meager as well. Since, by hypothesis, $\overline{\text{cone}}(\overline{G}, \overline{F})$ is meager, the same is true for $\overline{\text{cone}}(\overline{H}, \overline{E})$. □

Definition 2.27. We will say that two complexes $\overline{E}$ and $\overline{F}$ are *tightly related* if any of the equivalent conditions of Lemma 2.22 holds.

Proposition 2.28. *To be tightly related is an equivalence relation.*

Proof. The reflexivity and symmetry are obvious. The transitivity follows from Proposition 2.21 and Lemma 2.23. □

Definition 2.29. We denote by $\overline{\mathbf{V}}^b(X)/\mathscr{M}$ the set of classes of tightly related complexes. The class of a complex $\overline{E}$ will be denoted by $[\overline{E}]$.

Theorem 2.30 *(Acyclic calculus).*

(i) *For a complex $\overline{E} \in \operatorname{Ob} \overline{\mathbf{V}}^b(X)$, the class $[\overline{E}] = 0$ if and only if $\overline{E} \in \mathscr{M}$.*
(ii) *The operation $\oplus$ induces an operation, that we denote by $+$, in $\overline{\mathbf{V}}^b(X)/\mathscr{M}$. With this operation $\overline{\mathbf{V}}^b(X)/\mathscr{M}$ is an associative abelian semigroup.*
(iii) *For a complex $\overline{E}$, there exists a complex $\overline{F}$ such that $[\overline{F}] + [\overline{E}] = 0$, if and only if $\overline{E}$ is acyclic. In this case $[\overline{E}[1]] = -[\overline{E}]$.*
(iv) *For every morphism $f : \overline{E} \to \overline{F}$, if E is acyclic, then the equality*

$$\big[\overline{\text{cone}}(\overline{E}, \overline{F})\big] = [\overline{F}] - [\overline{E}]$$

holds.

(v) *For every morphism $f : \overline{E} \to \overline{F}$, if F is acyclic, then the equality*

$$\big[\overline{\text{cone}}(\overline{E}, \overline{F})\big] = [\overline{F}] + \big[\overline{E}[1]\big]$$

holds.

(vi) *Given a diagram*

$$\begin{array}{ccc} \overline{E}' & \xrightarrow{f'} & \overline{F}' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{E} & \xrightarrow{f} & \overline{F} \end{array}$$

in $\overline{\mathbf{V}}^b(X)$*, that commutes up to homotopy, then for every choice of homotopy we have*

$$\big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}\big(f'\big), \overline{\mathrm{cone}}(f)\big)\big] = \big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}\big(-g'\big), \overline{\mathrm{cone}}(g)\big)\big].$$

(vii) *Let* $f : \overline{E} \to \overline{F}$*,* $g : \overline{F} \to \overline{G}$ *be morphisms of complexes. Then*

$$\big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(g \circ f), \overline{\mathrm{cone}}(g)\big)\big] = \big[\overline{\mathrm{cone}}(f)[1]\big],$$
$$\big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(f), \overline{\mathrm{cone}}(g \circ f)\big)\big] = \big[\overline{\mathrm{cone}}(g)\big].$$

If one of f *or* g *is a quasi-isomorphism, then*

$$\big[\overline{\mathrm{cone}}(g \circ f)\big] = \big[\overline{\mathrm{cone}}(g)\big] + \big[\overline{\mathrm{cone}}(f)\big].$$

If $g \circ f$ *is a quasi-isomorphism, then*

$$\big[\overline{\mathrm{cone}}(g)\big] = \big[\overline{\mathrm{cone}}(f)[1]\big] + \big[\overline{\mathrm{cone}}(g \circ f)\big].$$

Proof. The statements (i) and (ii) are immediate. For assertion (iii), observe that, if $\overline{E}$ is acyclic, then $\overline{E} \oplus \overline{E}[1]$ is meager. Thus

$$[\overline{E}] + \big[\overline{E}[1]\big] = \big[\overline{E} \oplus \overline{E}[1]\big] = 0.$$

Conversely, if $[\overline{F}] + [\overline{E}] = 0$, then $\overline{F} \oplus \overline{E}$ is meager, hence acyclic. Thus $\overline{E}$ is acyclic.

For property (iv) we consider the map $\overline{F} \oplus \overline{E}[1] \to \overline{\mathrm{cone}}(f)$ defined by the map $\overline{F} \to \overline{\mathrm{cone}}(f)$. There is a natural isometry

$$\overline{\mathrm{cone}}\big(\overline{F} \oplus \overline{E}[1], \overline{\mathrm{cone}}(f)\big) \cong \overline{\mathrm{cone}}\big(\overline{E} \oplus \overline{E}[1], \overline{\mathrm{cone}}(\mathrm{id}_F)\big).$$

Since the right-hand complex is meager, so is the first. In consequence

$$\big[\overline{\mathrm{cone}}(f)\big] = \big[\overline{F} \oplus \overline{E}[1]\big] = [\overline{F}] + \big[\overline{E}[1]\big] = [\overline{F}] - [\overline{E}].$$

Statement (v) is proved analogously.

Statement (vi) is a direct consequence of Lemma 2.4.

Statement (vii) follows from the previous properties by using an argument similar to the one in Proposition 2.21. □

Remark 2.31. If $f : \overline{E} \to \overline{F}$ is a morphism and neither $\overline{E}$ nor $\overline{F}$ are acyclic, then $[\overline{\mathrm{cone}}(f)]$ depends on the homotopy class of f and not only on $\overline{E}$ and $\overline{F}$. For instance, let $\overline{E}$ be a non-acyclic complex of Hermitian bundles. Consider the zero map and the identity map $0, \mathrm{id} : \overline{E} \to \overline{E}$. Since, by Example 2.12, we know that $\overline{\mathrm{cone}}(\mathrm{id})$ is meager, then $[\overline{\mathrm{cone}}(\mathrm{id})] = 0$. By contrast,

$$\big[\overline{\mathrm{cone}}(0)\big] = [\overline{E}] + \big[\overline{E}[-1]\big] \neq 0$$

because $\overline{E}$ is not acyclic. This implies that we cannot extend Theorem 2.30(iv) or (v) to the case when none of the complexes are acyclic.

Corollary 2.32.

(i) *Let*

$$0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0$$

be a short exact sequence in $\overline{\mathbf{V}}^b(X)$ *all whose constituent short exact sequences are orthogonally split. If either* $\overline{E}$ *or* $\overline{G}$ *is acyclic, then*

$$[\overline{F}] = [\overline{E}] + [\overline{G}].$$

(ii) *Let* $\overline{E}^{*,*}$ *be a bounded double complex of Hermitian vector bundles. If the columns of* $\overline{E}^{*,*}$ *are acyclic, then*

$$\big[\mathrm{Tot}\big(\overline{E}^{*,*}\big)\big] = \sum_k (-1)^k \big[\overline{E}^{k,*}\big].$$

If the rows are acyclic, then

$$\big[\mathrm{Tot}\big(\overline{E}^{*,*}\big)\big] = \sum_k (-1)^k \big[\overline{E}^{*,k}\big].$$

In particular, if rows and columns are acyclic

$$\sum_k (-1)^k \big[\overline{E}^{k,*}\big] = \sum_k (-1)^k \big[\overline{E}^{*,k}\big].$$

Proof. The first item follows from Theorem 2.30(iv) and (v), by using Remark 2.3. The second assertion follows from the first by induction on the size of the complex, by using the usual filtration of $\mathrm{Tot}(E^{*,*})$. □

As an example of the use of the acyclic calculus we prove

Proposition 2.33. *Let* $f : \overline{E} \to \overline{F}$ *and* $g : \overline{F} \to \overline{G}$ *be morphisms of complexes. If two of* f, g, $g \circ f$ *are tight, then so is the third.*

Proof. We have already proved in Proposition 2.21 that, if f and g are tight, then $g \circ f$ is tight. We prove the remaining cases using acyclic calculus. Since tight morphisms are quasi-isomorphisms, by Theorem 2.30(vii)

$$\big[\overline{\mathrm{cone}}(g \circ f)\big] = \big[\overline{\mathrm{cone}}(f)\big] + \big[\overline{\mathrm{cone}}(g)\big].$$

Hence the result follows from Theorem 2.30(i). □

Definition 2.34. We will denote by $\overline{\mathbf{KA}}(X)$ the set of invertible elements of $\overline{\mathbf{V}}^b(X)/\mathscr{M}$. This is an abelian subgroup. By Theorem 2.30(iii) the group $\overline{\mathbf{KA}}(X)$ agrees with the image in $\overline{\mathbf{V}}^b(X)/\mathscr{M}$ of the class of acyclic complexes.

The group $\overline{\mathbf{KA}}(X)$ is a universal abelian group for additive Bott–Chern classes. More precisely, let us denote by $\overline{\mathbf{V}}^0(X)$ the full subcategory of $\overline{\mathbf{V}}^b(X)$ of acyclic complexes.

Theorem 2.35. *Let* $\mathscr{G}$ *be an abelian group and let* $\varphi : \mathrm{Ob}\,\overline{\mathbf{V}}^0(X) \to \mathscr{G}$ *be an assignment such that*

(i) (*Normalization*) *Every complex of the form*

$$\overline{E} : 0 \to \overline{A} \xrightarrow{\mathrm{id}} \overline{A} \to 0$$

satisfies $\varphi(\overline{E}) = 0$.

(ii) (*Additivity for exact sequences*) *For every short exact sequence in* $\overline{\mathbf{V}}^0(X)$

$$0 \to \overline{E} \to \overline{F} \to \overline{G} \to 0,$$

all whose constituent short exact sequences are orthogonally split, we have

$$\varphi(\overline{F}) = \varphi(\overline{E}) + \varphi(\overline{G}).$$

Then φ *factorizes through a group homomorphism* $\widetilde{\varphi} : \overline{\mathbf{KA}}(X) \to \mathscr{G}$.

Proof. The second condition tells us that φ is a morphism of semigroups. Thus we only need to show that it vanishes on meager complexes.

Again by the second condition, it is enough to prove that φ vanishes on the class $\mathscr{M}_0$. Both conditions together imply that φ vanishes on orthogonally split complexes. Therefore, by Example 2.12, it vanishes on complexes of the form $\overline{\mathrm{cone}}(\mathrm{id}_E)$. Once more by the second condition, if E is acyclic,

$$\varphi(E)+\varphi\big(E[1]\big)=\varphi\big(\overline{\mathrm{cone}}(\mathrm{id}_E)\big)=0.$$

Thus φ vanishes also on the complexes described in Definition 2.8(ii). Hence φ vanishes on the class $\mathscr{M}$. □

Remark 2.36. The considerations of this section carry over to the category of complex analytic varieties. If M is a complex analytic variety, one thus obtains for instance a group $\overline{\mathbf{KA}}^{\mathrm{an}}(M)$. Observe that, by GAGA principle, whenever X is a proper smooth algebraic variety over $\mathbb{C}$, the group $\overline{\mathbf{KA}}^{\mathrm{an}}(X^{\mathrm{an}})$ is canonically isomorphic to $\overline{\mathbf{KA}}(X)$.

As an example, we consider the simplest case $\operatorname{Spec}\mathbb{C}$ and we compute the group $\overline{\mathbf{KA}}(\operatorname{Spec}\mathbb{C})$. Given an acyclic complex E of $\mathbb{C}$-vector spaces, there is a canonical isomorphism

$$\alpha:\det E\to\mathbb{C}.$$

If we have an acyclic complex of Hermitian vector bundles $\overline{E}$, there is an induced metric on $\det E$. If we put on $\mathbb{C}$ the trivial Hermitian metric, then there is a well defined positive real number $\|\alpha\|$, namely the norm of the isomorphism α.

Theorem 2.37. *The assignment $\overline{E}\mapsto\log\|\alpha\|$ induces an isomorphism*

$$\widetilde{\tau}:\overline{\mathbf{KA}}(\operatorname{Spec}\mathbb{C})\xrightarrow{\simeq}\mathbb{R}.$$

Proof. First, we observe that the assignment in the theorem satisfies the hypothesis of Theorem 2.35. Thus, $\widetilde{\tau}$ exists and is a group morphism. Second, for every $a\in\mathbb{R}$ we consider the acyclic complex

$$e^a:=0\to\overline{\mathbb{C}}\xrightarrow{e^a}\overline{\mathbb{C}}\to 0,$$

where $\overline{\mathbb{C}}$ has the standard metric and the left copy of $\overline{\mathbb{C}}$ sits in degree 0. Since $\widetilde{\tau}([e^a])=a$ we deduce that $\widetilde{\tau}$ is surjective.

Next we prove that the complexes of the form $[e^a]$ form a set of generators of $\overline{\mathbf{KA}}(\operatorname{Spec}\mathbb{C})$. Let $\overline{E}=(\overline{E}^*,f^*)$ be an acyclic complex. Let $r=\sum_i\operatorname{rk}(E^i)$. We will show by induction on r that $[\overline{E}]=\sum_k(-1)^{i_k}[e^{a_k}]$ for certain integers i_k and real numbers a_k. Let n be the smallest integer such that $f^n:E^n\to E^{n+1}$ is non-zero. Let $v\in E^n\setminus\{0\}$. By acyclicity, f^n is injective, hence $\|f^n(v)\|\neq 0$. Set $i_1=n$ and $a_1=\log(\|f^n(v)\|/\|v\|)$ and consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \overline{\mathbb{C}} & \xrightarrow{e^a} & \overline{\mathbb{C}} & \to & 0 & \to & \cdots \\
 & & \downarrow{\scriptstyle\gamma^n} & & \downarrow{\scriptstyle\gamma^{n+1}} & & \downarrow & & \\
0 & \to & \overline{E}^n & \to & \overline{E}^{n+1} & \to & \overline{E}^{n+2} & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \overline{F}^n & \to & \overline{F}^{n+1} & \to & \overline{F}^{n+2} & \to & \cdots, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\gamma^n(1) = v$, $\gamma^{n+1}(1) = f^n(v)$ and all the columns are orthogonally split short exact sequences. By Corollary 2.32(i) and Theorem 2.30(iii), we have

$$[\overline{E}] = (-1)^{i_1}\left[e^{a_1}\right] + [\overline{F}].$$

Thus we deduce the claim.

Considering now the diagram

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{e^a} & \overline{\mathbb{C}} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle e^b} \\ \overline{\mathbb{C}} & \xrightarrow[e^{a+b}]{} & \overline{\mathbb{C}}, \end{array}$$

and using Corollary 2.32(ii) we deduce that $[e^a] + [e^b] = [e^{a+b}]$ and $[e^{-a}] = -[e^a]$. Therefore every element of $\overline{\mathbf{KA}}(\operatorname{Spec}\mathbb{C})$ is of the form $[e^a]$. Hence $\widetilde{\tau}$ is also injective. □

Remark 2.38. The Bott–Chern secondary characteristic classes depend on the relationship between the Hermitian metric and the algebraic or holomorphic structure. Therefore the group $\overline{\mathbf{KA}}(X)$ encodes algebraic as well as metric information. We can try to isolate the algebraic and metric information as follows.

On the one hand, we define the group $\mathbf{KA}(X)$ by mimicking the definition of $\overline{\mathbf{KA}}(X)$ but without metrics. Namely, we denote by $\mathscr{M}^{\mathrm{alg}}(X)$ the smallest subclass of $\mathbf{V}^b(X)$ that contains all the split acyclic complexes and all the complexes of the form $F \oplus F[1]$, with F an acyclic complex, and such that, if $f : E \to F$ is a morphism and two of E, F and $\operatorname{cone}(f)$ belong to $\mathscr{M}^{\mathrm{alg}}(X)$, then so does the third. Then $\mathbf{KA}(X)$ is the set of invertible elements in $\mathbf{V}^b(X)/\mathscr{M}^{\mathrm{alg}}(X)$. There is a natural map $\overline{\mathbf{KA}}(X) \to \mathbf{KA}(X)$. Clearly $\mathbf{KA}(\operatorname{Spec}(\mathbb{C})) = 0$, but we expect that, in general, $\mathbf{KA}(X) \neq 0$. If this is the case, this group would be a very interesting group were each extension of vector bundles will have a class. Moreover, any acyclic complex whose class in $\mathbf{KA}(X)$ is different from zero will not be meager for any choice of Hermitian metrics. It is clear that the definition of $\overline{\mathbf{KA}}$ can be generalized to any situation where there are exact complexes and split exact complexes.

On the other hand, we consider X^{dif} the differentiable manifold associated to X and $\mathbf{V}^b(X^{\mathrm{dif}})$ the exact category of bounded complexes of differentiable vector bundles on X^{dif}. Clearly, we can define $\overline{\mathbf{KA}}(X^{\mathrm{dif}})$ and there is a map $\overline{\mathbf{KA}}(X) \to \overline{\mathbf{KA}}(X^{\mathrm{dif}})$. Since every exact complex of differentiable vector bundles is split, then $\overline{\mathbf{KA}}(X^{\mathrm{dif}})$ only contains metric information.

3. Definition of $\overline{\mathbf{D}}^b(X)$ and basic constructions

Let X be a smooth algebraic variety over $\mathbb{C}$. We denote by $\mathbf{Coh}(X)$ the abelian category of coherent sheaves on X and by $\mathbf{D}^b(X)$ its bounded derived category. The objects of $\mathbf{D}^b(X)$ are complexes of quasi-coherent sheaves with bounded coherent cohomology. The reader is referred to [17] for an introduction to derived categories. For notational convenience, we also introduce $\mathbf{C}^b(X)$, the abelian category of bounded cochain complexes of coherent sheaves on X. Arrows in $\mathbf{D}^b(X)$ will be written as $\dashrightarrow$, while arrows in $\mathbf{C}^b(X)$ will be denoted by $\to$. The symbol $\sim$ will mean either quasi-isomorphism in $\mathbf{C}^b(X)$ or isomorphism in $\mathbf{D}^b(X)$. Every functor from $\mathbf{D}^b(X)$ to another category will tacitly be assumed to be the derived functor. Therefore we will denote just by f_*, f^*, $\otimes$ and $\underline{\mathrm{Hom}}$ the derived direct image, inverse image, tensor product and internal Hom. Finally, we will refer to (complexes of) locally free sheaves by normal upper case letters (such as F) whereas we reserve script upper case letters for (complexes of) quasi-coherent sheaves in general (for instance $\mathcal{F}$).

Remark 3.1. Because X is in particular a smooth Noetherian scheme over $\mathbb{C}$, every object $\mathcal{F}$ of $\mathbf{C}^b(X)$ admits a quasi-isomorphism $F \to \mathcal{F}$, with F an object of $\mathbf{V}^b(X)$. Hence, if $\mathcal{F}$ is an object in $\mathbf{D}^b(X)$, then there is an isomorphism $F \dashrightarrow \mathcal{F}'$ in $\mathbf{D}^b(X)$, for some object $F \in \mathbf{V}^b(X)$. In general, the analogous statement is no longer true if we work with complex manifolds, as shown by the counterexample [29, Appendix, Corollary A.5].

For the sake of completeness, we recall how morphisms in $\mathbf{D}^b(X)$ between bounded complexes of vector bundles can be represented.

Lemma 3.2.

(i) *Let F, G be bounded complexes of vector bundles on X. Every morphism $F \dashrightarrow G$ in $\mathbf{D}^b(X)$ may be represented by a diagram in $\mathbf{C}^b(X)$*

$$\begin{array}{ccc} & E & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ F & & G, \end{array}$$

where $E \in \operatorname{Ob} \mathbf{V}^b(X)$ and f is a quasi-isomorphism.

(ii) *Let E, E', F, G be bounded complexes of vector bundles on X. Let f, f', g, g' be morphisms in $\mathbf{C}^b(X)$ as in the diagram below, with f, f' quasi-isomorphisms. These data define the same morphism $F \dashrightarrow G$ in $\mathbf{D}^b(X)$ if, and only if, there exists a bounded complex of vector bundles E'' and a diagram*

$$\begin{array}{ccccc} & & E'' & & \\ & {\scriptstyle \alpha}\swarrow & & \searrow{\scriptstyle \beta} & \\ & E & & E' & \\ {\scriptstyle f}\swarrow & & {\scriptstyle g}\searrow\!\!\!\swarrow{\scriptstyle f'} & & \searrow{\scriptstyle g'} \\ F & & & & G, \end{array}$$

whose squares are commutative up to homotopy and where α and β are quasi-isomorphisms.

Proof. This follows from the equivalence of $\mathbf{D}^b(X)$ with the localization of the homotopy category of $\mathbf{C}^b(X)$ with respect to the class of quasi-isomorphisms and Remark 3.1. □

Proposition 3.3. *Let $f : \overline{E} \to \overline{E}$ be an endomorphism in $\overline{\mathbf{V}}^b(X)$ that represents id_E in $\mathbf{D}^b(X)$. Then $\overline{\mathrm{cone}}(f)$ is meager.*

Proof. By Lemma 3.2(ii), the fact that f represents the identity in $\mathbf{D}^b(X)$ means that there are diagrams

$$\begin{array}{ccc} \overline{E}' & \xrightarrow[\sim]{\alpha} & \overline{E} \\ {\scriptstyle \beta}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathrm{id}_E} \\ \overline{E} & \xrightarrow[\mathrm{id}_E]{} & \overline{E}, \end{array} \qquad \begin{array}{ccc} \overline{E}' & \xrightarrow[\sim]{\alpha} & \overline{E} \\ {\scriptstyle \beta}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle f} \\ \overline{E} & \xrightarrow[\mathrm{id}_E]{} & \overline{E}, \end{array}$$

that commute up to homotopy. By Theorem 2.30(iv) and (vi) the equalities

$$\big[\overline{\mathrm{cone}}(\alpha)\big] - \big[\overline{\mathrm{cone}}(\mathrm{id}_E)\big] = \big[\overline{\mathrm{cone}}(\beta)\big] - \big[\overline{\mathrm{cone}}(\mathrm{id}_E)\big],$$
$$\big[\overline{\mathrm{cone}}(\alpha)\big] - \big[\overline{\mathrm{cone}}(\mathrm{id}_E)\big] = \big[\overline{\mathrm{cone}}(\beta)\big] - \big[\overline{\mathrm{cone}}(f)\big]$$

hold in the group $\overline{\mathbf{KA}}(X)$ (observe that these relations do not depend on the choice of homotopies). Therefore

$$\big[\overline{\mathrm{cone}}(f)\big] = \big[\overline{\mathrm{cone}}(\mathrm{id}_E)\big] = 0.$$

Hence $\overline{\mathrm{cone}}(f)$ is meager. □

Definition 3.4. Let $\mathcal{F}$ be an object of $\mathbf{D}^b(X)$. A *Hermitian metric on $\mathcal{F}$* consists of the following data:

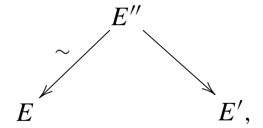
- an isomorphism $E \overset{\sim}{\dashrightarrow} \mathcal{F}$ in $\mathbf{D}^b(X)$, where $E \in \operatorname{Ob} \mathbf{V}^b(X)$;
- an object $\overline{E} \in \operatorname{Ob} \overline{\mathbf{V}}^b(X)$, whose underlying complex is E.

We write $\overline{E} \dashrightarrow \mathcal{F}$ to refer to the data above and we call it a *metrized object of* $\mathbf{D}^b(X)$.

Our next task is to define the category $\overline{\mathbf{D}}^b(X)$, whose objects are objects of $\mathbf{D}^b(X)$ provided with equivalence classes of metrics. We will show that in this category there is a Hermitian cone well defined up to isometries.

Lemma 3.5. *Let* $\overline{E}, \overline{E}' \in \mathrm{Ob}(\overline{\mathbf{V}}^b(X))$ *and consider an arrow* $E \dashrightarrow E'$ *in* $\mathbf{D}^b(X)$. *Then the following statements are equivalent*:

(i) *for any diagram*



(3.6)

that represents $E \dashrightarrow E'$, *and any choice of Hermitian metric on* E'', *the complex*

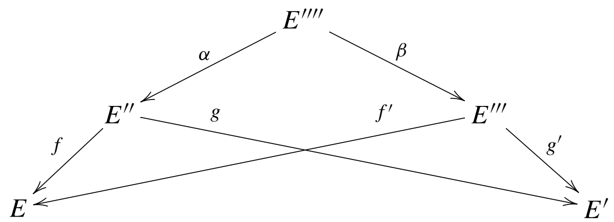
$$\overline{\mathrm{cone}}\big(\overline{E}'', \overline{E}\big)[1] \oplus \overline{\mathrm{cone}}\big(\overline{E}'', \overline{E}'\big) \tag{3.7}$$

is meager;

(ii) *there is a diagram* (3.6) *that represents* $E \dashrightarrow E'$, *and a choice of Hermitian metric on* E'', *such that the complex* (3.7) *is meager*;

(iii) *there is a diagram* (3.6) *that represents* $E \dashrightarrow E'$, *and a choice of Hermitian metric on* E'', *such that the arrows* $\overline{E}'' \to \overline{E}$ *and* $\overline{E}'' \to \overline{E}'$ *are tight morphisms.*

Proof. Clearly (i) implies (ii). To prove the converse we assume the existence of an $\overline{E}''$ such that the complex (3.7) is meager, and let $\overline{E}'''$ be any complex that satisfies the hypothesis of (i). Then there is a diagram



whose squares commute up to homotopy. Using acyclic calculus we have

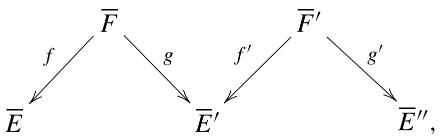
$$\begin{aligned}\big[\overline{\mathrm{cone}}\big(g'\big)\big] - \big[\overline{\mathrm{cone}}\big(f'\big)\big] &= \big[\overline{\mathrm{cone}}(\beta)\big] + \big[\overline{\mathrm{cone}}(g)\big] - \big[\overline{\mathrm{cone}}(\alpha)\big] - \big[\overline{\mathrm{cone}}(\beta)\big] - \big[\overline{\mathrm{cone}}(f)\big] + \big[\overline{\mathrm{cone}}(\alpha)\big] \\ &= \big[\overline{\mathrm{cone}}(g)\big] - \big[\overline{\mathrm{cone}}(f)\big] = 0.\end{aligned}$$

Now repeat the argument of Lemma 2.22 to prove that (ii) and (iii) are equivalent. The only point is to observe that the diagram constructed in Lemma 2.22 represents the same morphism in the derived category as the original diagram. □

Definition 3.8. Let $\mathcal{F} \in \mathrm{Ob}\,\mathbf{D}^b(X)$ and let $\overline{E} \dashrightarrow \mathcal{F}$ and $\overline{E}' \dashrightarrow \mathcal{F}$ be two Hermitian metrics on $\mathcal{F}$. We say that they *fit tightly* if the induced arrow $\overline{E} \dashrightarrow \overline{E}'$ satisfies any of the equivalent conditions of Lemma 3.5.

Theorem 3.9. *The relation "to fit tightly" is an equivalence relation.*

Proof. The reflexivity and the symmetry are obvious. To prove the transitivity, consider a diagram



where all the arrows are tight morphisms. By Lemma 2.23, this diagram can be completed into a diagram

$$\begin{array}{ccccc} & & \overline{F}'' & & \\ & \overset{\alpha}{\swarrow} & & \overset{\beta}{\searrow} & \\ & \overline{F} & & \overline{F}' & \\ \overset{f}{\swarrow} & & \overset{g}{\searrow}\quad \overset{f'}{\swarrow} & & \overset{g'}{\searrow} \\ \overline{E} & & \overline{E}' & & \overline{E}'', \end{array}$$

where all the arrows are tight morphisms and the square commutes up to homotopy. Now observe that $f \circ \alpha$ and $g' \circ \beta$ represent the morphism $E \dashrightarrow E''$ in $\mathbf{D}^b(X)$ and are both tight morphisms by Proposition 2.33. This finishes the proof. □

Definition 3.10. We denote by $\overline{\mathbf{D}}^b(X)$ the category whose objects are pairs $\overline{\mathcal{F}} = (\mathcal{F}, h)$ where $\mathcal{F}$ is an object of $\mathbf{D}^b(X)$ and h is an equivalence class of metrics that fit tightly, and with morphisms

$$\operatorname{Hom}_{\overline{\mathbf{D}}^b(X)}(\overline{\mathcal{F}}, \overline{\mathcal{G}}) = \operatorname{Hom}_{\mathbf{D}^b(X)}(\mathcal{F}, \mathcal{G}).$$

A class h of metrics will be called *a Hermitian structure*, and may be referenced by any representative $\overline{E} \dashrightarrow \mathcal{F}$ or, if the arrow is clear, by the complex $\overline{E}$. We will denote by $\bar{0} \in \operatorname{Ob} \overline{\mathbf{D}}^b(X)$ a zero object of $\mathbf{D}^b(X)$ provided with a trivial Hermitian structure given by any meager complex.

If the underlying complex to an object $\overline{\mathcal{F}}$ is acyclic, then its Hermitian structure has a well defined class in $\overline{\mathbf{KA}}(X)$. We will use the notation $[\overline{\mathcal{F}}]$ for this class.

Definition 3.11. A morphism in $\overline{\mathbf{D}}^b(X)$, $f : (\overline{E} \dashrightarrow \mathcal{F}) \dashrightarrow (\overline{F} \dashrightarrow \mathcal{G})$, is called *a tight isomorphism* whenever the underlying morphism $f : \mathcal{F} \dashrightarrow \mathcal{G}$ is an isomorphism and the metric on $\mathcal{G}$ induced by f and $\overline{E}$ fits tightly with $\overline{F}$. An object of $\overline{\mathbf{D}}^b(X)$ will be called *meager* if it is tightly isomorphic to the zero object with the trivial metric.

Remark 3.12. A word of warning should be said about the use of acyclic calculus to show that a particular map is a tight isomorphism. The subtle point comes from the difference between the equivalence relations "tightly related" and "to fit tightly". Let $\overline{E} \dashrightarrow \mathcal{F}$ and $\overline{E}' \dashrightarrow \mathcal{F}$ be two Hermitian metrics on $\mathcal{F}$ then $\overline{E}$ and $\overline{E}'$ are tightly related if there is a diagram like (3.6) such that both arrows are tight morphism, while they fit tightly if there is a diagram like (3.6) *that represents* id_F and such that both arrows are tight morphism. Thus the assignment $\operatorname{Ob} \overline{\mathbf{D}}^b(X) \to \overline{\mathbf{V}}^b(X)/\mathscr{M}$ that sends $\overline{E} \dashrightarrow \mathcal{F}$ to $[\overline{E}]$ is not injective. For a more concrete example, let $0 < r \neq 1$ be a real number and consider the trivial bundle $\mathcal{O}_X$ with the trivial metric $\|1\| = 1$ and with the metric $\|1\|' = 1/r$. Then the product by r induces an isometry between both bundles. Thus the complexes $(\mathcal{O}_X, \|\cdot\|)$ and $(\mathcal{O}_X, \|\cdot\|')$ are tightly related, but both metrics do not fit tightly because the product by r does not represent $\mathrm{id}_{\mathcal{O}_X}$.

Thus the right procedure to show that a morphism $f : (\overline{E} \dashrightarrow \mathcal{F}) \dashrightarrow (\overline{F} \dashrightarrow \mathcal{G})$ is a tight isomorphism, is not to show that $[\overline{E}] = [\overline{F}]$, but to construct a diagram

$$\begin{array}{ccc} & \overline{G} & \\ \overset{\alpha}{\swarrow}{\scriptstyle\sim} & & \overset{\beta}{\searrow} \\ \overline{E} & & \overline{F} \end{array}$$

that represents f and use the acyclic calculus to show that $[\overline{\operatorname{cone}}(\beta)] - [\overline{\operatorname{cone}}(\alpha)] = 0$. Observe that this subtle point is related with the fact that, if $\overline{E}$ is not acyclic, then $[\overline{E}]$ does not have an inverse.

By definition, the forgetful functor $\mathfrak{F} : \overline{\mathbf{D}}^b(X) \to \mathbf{D}^b(X)$ is fully faithful. The structure of this functor will be given in the next result that we suggestively summarize by saying that $\overline{\mathbf{D}}^b(X)$ is a principal fibered category over $\mathbf{D}^b(X)$ with structural group $\overline{\mathbf{KA}}(X)$ provided with a flat connection.

Theorem 3.13. *The functor $\mathfrak{F} : \overline{\mathbf{D}}^b(X) \to \mathbf{D}^b(X)$ defines a structure of category fibered in groupoids. Moreover*

(i) *The fiber $\mathfrak{F}^{-1}(0)$ is the grupoid associated to the abelian group $\overline{\mathbf{KA}}(X)$. The object $\bar{0}$ is the neutral element of $\overline{\mathbf{KA}}(X)$.*
(ii) *For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$, the fiber $\mathfrak{F}^{-1}(\mathcal{F})$ is the grupoid associated to a torsor over $\overline{\mathbf{KA}}(X)$. The action of $\overline{\mathbf{KA}}(X)$ over $\mathfrak{F}^{-1}(\mathcal{F})$ is given by orthogonal direct sum. We will denote this action by $+$.*
(iii) *Every isomorphism $f : \mathcal{F} \dashrightarrow \mathcal{G}$ in $\mathbf{D}^b(X)$ determines an isomorphism of $\overline{\mathbf{KA}}(X)$-torsors*

$$\mathfrak{t}_f : \mathfrak{F}^{-1}(\mathcal{F}) \to \mathfrak{F}^{-1}(\mathcal{G}),$$

that sends the Hermitian structure $\overline{E} \overset{\epsilon}{\dashrightarrow} \mathcal{F}$ to the Hermitian structure $\overline{E} \overset{f\circ\epsilon}{\dashrightarrow} \mathcal{G}$. This isomorphism will be called the parallel transport along f.
(iv) *Given two isomorphisms $f : \mathcal{F} \dashrightarrow \mathcal{G}$ and $g : \mathcal{G} \dashrightarrow \mathcal{H}$, the equality*

$$\mathfrak{t}_{g\circ f} = \mathfrak{t}_g \circ \mathfrak{t}_f$$

holds.

Proof. Recall that $\mathfrak{F}^{-1}(\mathcal{F})$ is the subcategory of $\overline{\mathbf{D}}^b(X)$ whose objects satisfy $\mathfrak{F}(A) = \mathcal{F}$ and whose morphisms satisfy $\mathfrak{F}(f) = \mathrm{id}_{\mathcal{F}}$. The first assertion is trivial. To prove that $\mathfrak{F}^{-1}(\mathcal{F})$ is a torsor under $\overline{\mathbf{KA}}(X)$, we need to show that $\overline{\mathbf{KA}}(X)$ acts freely and transitively on this fiber. For the freeness, it is enough to observe that if for $\overline{E} \in \overline{\mathbf{V}}^b(X)$ and $\overline{M} \in \overline{\mathbf{V}}^0(X)$, the complexes $\overline{E}$ and $\overline{E} \oplus \overline{M}$ represent the same Hermitian structure, then the inclusion $\overline{E} \hookrightarrow \overline{E} \oplus \overline{M}$ is tight. Hence $\overline{\mathrm{cone}}(\overline{E}, \overline{E} \oplus \overline{M})$ is meager. Since

$$\overline{\mathrm{cone}}(\overline{E}, \overline{E} \oplus \overline{M}) = \overline{\mathrm{cone}}(\overline{E}, \overline{E}) \oplus \overline{M}$$

and $\overline{\mathrm{cone}}(\overline{E}, \overline{E})$ is meager, we deduce that $\overline{M}$ is meager. For the transitivity, any two Hermitian structures on $\mathcal{F}$ are related by a diagram

$$\begin{array}{ccc} & \overline{E}'' & \\ {\scriptstyle f,\,\sim}\swarrow & & \searrow{\scriptstyle g,\,\sim} \\ \overline{E} & & \overline{E}'. \end{array}$$

After possibly replacing $\overline{E}''$ by $\overline{E}'' \oplus \overline{\mathrm{cone}}(f)$, we may assume that f is tight. We consider the natural arrow $\overline{E}'' \to \overline{E}' \oplus \overline{\mathrm{cone}}(g)[1]$ induced by g. Observe that $\overline{\mathrm{cone}}(g)[1]$ is acyclic. Finally, we find

$$\overline{\mathrm{cone}}\big(\overline{E}'', \overline{E}' \oplus \overline{\mathrm{cone}}(g)[1]\big) = \overline{\mathrm{cone}}(g) \oplus \overline{\mathrm{cone}}(g)[1],$$

that is meager. Thus the Hermitian structure represented by $\overline{E}''$ agrees with the Hermitian structure represented by $\overline{E}' \oplus \overline{\mathrm{cone}}(g)[1]$.

The remaining properties are straightforward. □

Our next objective is to define the cone of a morphism in $\overline{\mathbf{D}}^b(X)$. This will be an object of $\overline{\mathbf{D}}^b(X)$ uniquely defined up to tight isomorphism. Let $f : (\overline{E} \dashrightarrow \mathcal{F}) \dashrightarrow (\overline{E}' \dashrightarrow \mathcal{G})$ be a morphism in $\overline{\mathbf{D}}^b(X)$, where $\overline{E}$ and $\overline{E}'$ are representatives of the Hermitian structures.

Definition 3.14. A *Hermitian cone* of f, denoted by $\overline{\mathrm{cone}}(f)$, is an object $(\mathrm{cone}(f), h_f)$ of $\overline{\mathbf{D}}^b(X)$ where:

- $\mathrm{cone}(f) \in \mathrm{Ob}\,\mathbf{D}^b(X)$ is a choice of cone of f. Namely an object of $\mathbf{D}^b(X)$ completing f into a distinguished triangle;
- h_f is a Hermitian structure on $\mathrm{cone}(f)$ constructed as follows. The morphism f induces an arrow $E \dashrightarrow E'$. Choose a diagram of complexes of Hermitian vector bundles

$$\begin{array}{ccc} & \overline{E}'' & \\ {\scriptstyle \alpha,\,\sim}\swarrow & & \searrow{\scriptstyle \beta} \\ \overline{E} & & \overline{E}' \end{array} \qquad (3.15)$$

that represents $E \dashrightarrow E'$, with α tight. Put

$$\overline{C}(f) = \overline{\mathrm{cone}}(\beta).$$

Since $\alpha^{-1} : E \dashrightarrow E''$ is an isomorphism, there is a distinguished triangle

$$\overline{E} \dashrightarrow \overline{E}' \dashrightarrow \overline{C}(f) \dashrightarrow \overline{E}[1].$$

Therefore, there exists a (non-unique) quasi-isomorphism $\overline{C}(f) \dashrightarrow \mathrm{cone}(f)$ such that the following diagram (where the rows are distinguished triangles)

$$\begin{array}{ccccccc}
\overline{E} & \dashrightarrow & \overline{E}' & \dashrightarrow & \overline{C}(f) & \dashrightarrow & \overline{E}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F} & \dashrightarrow & \mathcal{G} & \dashrightarrow & \mathrm{cone}(f) & \dashrightarrow & \mathcal{F}[1]
\end{array}$$

commutes. Then h_f is defined as the Hermitian structure given by $\overline{C}(f) \dashrightarrow \mathrm{cone}(f)$. By Theorem 3.17 below, this Hermitian structure does not depend on the particular choice of arrow $\overline{C}(f) \dashrightarrow \mathrm{cone}(f)$. Moreover, by Theorem 3.21, the Hermitian structure will not depend on the choices of representatives of Hermitian structures nor on the choice of $\overline{E}''$.

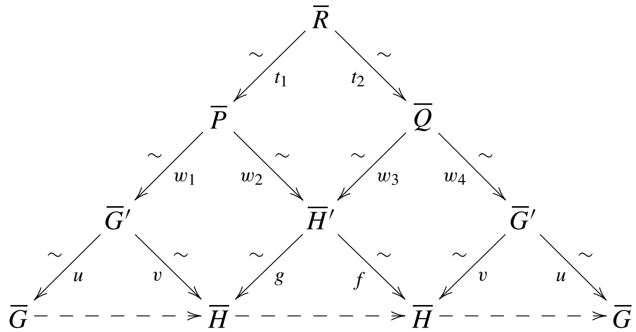
Remark 3.16. Alternatively, in Definition 3.14, we can choose the diagram (3.15) with α not being a tight morphism. In this case we just have to define $\overline{C}(f) = \overline{\mathrm{cone}}(\alpha)[1] \oplus \overline{\mathrm{cone}}(\beta)$. This is equivalent to force α to be tight by changing $\overline{E}''$ to $\overline{E}'' \oplus \overline{\mathrm{cone}}(\alpha)$ and α to $(\alpha, 0)$.

Theorem 3.17. *Let*

$$\begin{array}{ccccccccc}
\mathcal{F} & \dashrightarrow & \mathcal{G} & \dashrightarrow & \mathcal{H} & \dashrightarrow & \mathcal{F}[1] & \dashrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathcal{F} & \dashrightarrow & \mathcal{G} & \dashrightarrow & \mathcal{H} & \dashrightarrow & \mathcal{F}[1] & \dashrightarrow & \cdots
\end{array}$$

be a commutative diagram in $\mathbf{D}^b(X)$, where the rows are the same distinguished triangle. Let $\overline{H} \dashrightarrow \mathcal{H}$ be any Hermitian structure. Then $\alpha : (\overline{H} \dashrightarrow \mathcal{H}) \dashrightarrow (\overline{H} \dashrightarrow \mathcal{H})$ is a tight isomorphism.

Proof. First of all, we claim that if $\gamma : \overline{\mathcal{B}} \dashrightarrow \overline{\mathcal{H}}$ is any isomorphism, then $\gamma^{-1} \circ \alpha \circ \gamma$ is tight if, and only if, α is tight. Indeed, denote by $\overline{G} \dashrightarrow \mathcal{B}$ a representative of the Hermitian structure on $\overline{\mathcal{B}}$. Then there is a diagram



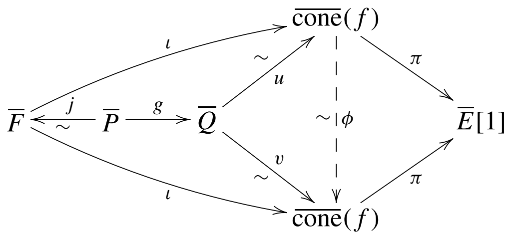
for the liftings of γ^{-1}, α, γ to representatives, as well as for their composites, all whose squares are commutative up to homotopy. By acyclic calculus, we have the following chain of equalities

$$\begin{aligned}
&\big[\overline{\mathrm{cone}}(u \circ w_1 \circ t_1)[1]\big] + \big[\overline{\mathrm{cone}}(u \circ w_4 \circ t_2)\big] \\
&\quad = \big[\overline{\mathrm{cone}}(u)[1]\big] + \big[\overline{\mathrm{cone}}(v)\big] + \big[\overline{\mathrm{cone}}(g)[1]\big] + \big[\overline{\mathrm{cone}}(f)\big] + \big[\overline{\mathrm{cone}}(v)[1]\big] + \big[\overline{\mathrm{cone}}(u)\big] \\
&\quad = \big[\overline{\mathrm{cone}}(g)[1]\big] + \big[\overline{\mathrm{cone}}(f)\big].
\end{aligned}$$

Thus, the right-hand side vanishes if and only if the left-hand side vanishes, proving the claim. This observation allows to reduce the proof of the lemma to the following situation: consider a diagram of complexes of Hermitian vector bundles

$$\begin{array}{ccccccccc}
\overline{E} & \xrightarrow{f} & \overline{F} & \xrightarrow{\iota} & \overline{\mathrm{cone}}(f) & \xrightarrow{\pi} & \overline{E}[1] & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \sim \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\overline{E} & \xrightarrow{f} & \overline{F} & \xrightarrow{\iota} & \overline{\mathrm{cone}}(f) & \xrightarrow{\pi} & \overline{E}[1] & \longrightarrow & \cdots,
\end{array}$$

which commutes in $\mathbf{D}^b(X)$. We need to show that ϕ is a tight isomorphism. The commutativity of the diagram translates into the existence of bounded complexes of Hermitian vector bundles $\overline{P}$ and $\overline{Q}$ and a diagram



fulfilling the following properties: (a) j, u, v are quasi-isomorphisms; (b) the squares formed by ι, j, g, u and ι, j, g, v are commutative up to homotopy; (c) the morphisms u, v induce ϕ in the derived category. We deduce a commutative up to homotopy square

$$\begin{array}{ccc}
\overline{\mathrm{cone}}(g) & \xrightarrow[\sim]{\tilde{u}} & \overline{\mathrm{cone}}(\iota) \\
\tilde{v}\downarrow\sim & & \sim\downarrow\tilde{\pi} \\
\overline{\mathrm{cone}}(\iota) & \xrightarrow[\sim]{\tilde{\pi}} & \overline{E}[1].
\end{array}$$

The arrows $\tilde{u}$, $\tilde{v}$ are induced by j, u and j, v respectively. Observe they are quasi-isomorphisms. Also the natural projection $\tilde{\pi}$ is a quasi-isomorphism. By acyclic calculus, we have

$$\big[\overline{\mathrm{cone}}(\tilde{\pi})\big] + \big[\overline{\mathrm{cone}}(\tilde{u})\big] = \big[\overline{\mathrm{cone}}(\tilde{\pi})\big] + \big[\overline{\mathrm{cone}}(\tilde{v})\big].$$

Therefore we find

$$\big[\overline{\mathrm{cone}}(\tilde{u})\big] = \big[\overline{\mathrm{cone}}(\tilde{v})\big]. \tag{3.18}$$

Finally, notice there is an exact sequence

$$0 \to \overline{\mathrm{cone}}(u) \to \overline{\mathrm{cone}}(\tilde{u}) \to \overline{\mathrm{cone}}\big(j[1]\big) \to 0,$$

whose rows are orthogonally split. Therefore,

$$\big[\overline{\mathrm{cone}}(\tilde{u})\big] = \big[\overline{\mathrm{cone}}(u)\big] + \big[\overline{\mathrm{cone}}\big(j[1]\big)\big]. \tag{3.19}$$

Similarly we prove

$$\big[\overline{\mathrm{cone}}(\tilde{v})\big] = \big[\overline{\mathrm{cone}}(v)\big] + \big[\overline{\mathrm{cone}}\big(j[1]\big)\big]. \tag{3.20}$$

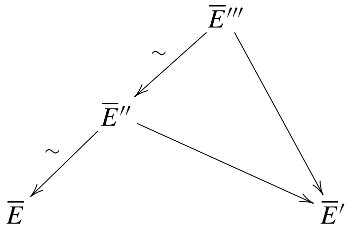
From Eqs. (3.18)–(3.20) we infer

$$\big[\overline{\mathrm{cone}}(u)[1]\big] + \big[\overline{\mathrm{cone}}(v)\big] = 0,$$

as was to be shown. □

Theorem 3.21. *The object $\overline{C}(f)$ of Definition* 3.14 *is well defined up to tight isomorphism.*

Proof. We first show the independence on the choice of $\overline{E}''$, up to tight isomorphism. To this end, it is enough to assume that there is a diagram

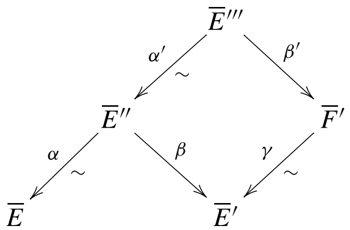


such that the quasi-isomorphisms are tight morphisms and the triangle commutes up to homotopy. Fix such a homotopy. Then

$$\big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}\big(\overline{E}''', \overline{E}'\big), \overline{\mathrm{cone}}\big(\overline{E}'', \overline{E}'\big)\big)\big] = -\big[\overline{\mathrm{cone}}\big(E''', E''\big)\big] = 0.$$

Thus, the morphism $\overline{\mathrm{cone}}(\overline{E}''', \overline{E}') \to \overline{\mathrm{cone}}(\overline{E}'', \overline{E}')$ is tight.

We now prove the independence on the choice of the representative $\overline{E}'$. Let $\gamma : \overline{F}' \to \overline{E}'$ be a tight morphism. Then we can construct a diagram



where the square commutes up to homotopy and α' is tight. Choose one homotopy for the square. Taking into account Lemma 2.4, we find

$$\big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}\big(\beta'\big), \overline{\mathrm{cone}}(\beta)\big)\big] = \big[\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}\big(\alpha'\big), \overline{\mathrm{cone}}(\gamma)\big)\big] = 0.$$

Hence the definitions of $\overline{C}(f)$ using $\overline{E}'$ or $\overline{F}'$ agree up to tight isomorphism. The remaining possible choices of representatives are treated analogously. □

Remark 3.22. The construction of $\overline{\mathrm{cone}}(f)$ involves the choice of $\mathrm{cone}(f)$, which is unique up to isomorphism. Since the construction of $\overline{C}(f)$ in Definition 3.14 does not depend on the choice of $\mathrm{cone}(f)$, by Theorem 3.17, we see that different choices of $\mathrm{cone}(f)$ give rise to tightly isomorphic Hermitian cones. Therefore $\overline{\mathrm{cone}}(f)$ is well defined up to tight isomorphism and we will usually call it *the* Hermitian cone of f. When the morphism is clear, we will also write $\overline{\mathrm{cone}}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ to refer to it.

The Hermitian cone satisfies the same relations than the usual cone.

Proposition 3.23. *Let $f : \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ be a morphism in $\overline{\mathbf{D}}^b(X)$. Then, the natural morphisms*

$$\overline{\mathrm{cone}}\big(\overline{\mathcal{G}}, \overline{\mathrm{cone}}(f)\big) \dashrightarrow \overline{\mathcal{F}}[1],$$
$$\overline{\mathcal{G}} \dashrightarrow \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(f)[-1], \overline{\mathcal{F}}\big)$$

are tight isomorphisms.

Proof. After choosing representatives, there are isometries

$$\overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}\big(\overline{\mathcal{G}}, \overline{\mathrm{cone}}(f)\big), \overline{\mathcal{F}}[1]\big) \cong \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(\mathrm{id}_{\mathcal{F}}), \overline{\mathrm{cone}}(\mathrm{id}_{\mathcal{G}})\big) \cong \overline{\mathrm{cone}}\big(\overline{\mathcal{G}}, \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(f)[-1], \overline{\mathcal{F}}\big)\big).$$

Since the middle term is meager, the same is true for the other two. □

We next extend some basic constructions in $\mathbf{D}^b(X)$ to $\overline{\mathbf{D}}^b(X)$.

Derived tensor product. Let $\overline{\mathcal{F}}_i = (\overline{E}_i \dashrightarrow \mathcal{F}_i)$, $i = 1, 2$, be objects of $\overline{\mathbf{D}}^b(X)$. The derived tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ is endowed with a natural Hermitian structure

$$\overline{E}_1 \otimes \overline{E}_2 \dashrightarrow \mathcal{F}_1 \otimes \mathcal{F}_2, \tag{3.24}$$

that is well defined by Theorem 2.18(iii). We write $\overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2$ for the resulting object in $\overline{\mathbf{D}}^b(X)$.

Derived internal Hom and dual objects. Let $\overline{\mathcal{F}}_i = (\overline{E}_i \dashrightarrow \mathcal{F}_i)$, $i = 1, 2$, be objects of $\overline{\mathbf{D}}^b(X)$. The derived internal Hom, $\underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2)$ is endowed with a natural Hermitian structure

$$\underline{\mathrm{Hom}}(\overline{E}_1, \overline{E}_2) \dashrightarrow \underline{\mathrm{Hom}}(\mathcal{F}_1, \mathcal{F}_2), \tag{3.25}$$

that is well defined by Theorem 2.18(iii). We write $\underline{\mathrm{Hom}}(\overline{\mathcal{F}}_1, \overline{\mathcal{F}}_2)$ for the resulting object in $\overline{\mathbf{D}}^b(X)$.

In particular, denote by $\overline{\mathcal{O}}_X$ the structural sheaf with the metric $\|1\| = 1$. Then, for every object $\overline{\mathcal{F}} \in \overline{\mathbf{D}}^b(X)$, the *dual object* is defined to be

$$\overline{\mathcal{F}}^{\vee} = \underline{\mathrm{Hom}}(\overline{\mathcal{F}}, \overline{\mathcal{O}}_X). \tag{3.26}$$

Left derived inverse image. Let $g : X' \to X$ be a morphism of smooth algebraic varieties over $\mathbb{C}$ and $\overline{\mathcal{F}} = (\overline{E} \dashrightarrow \mathcal{F}) \in \mathrm{Ob}\,\overline{\mathbf{D}}^b(X)$. Then the left derived inverse image $g^*(\mathcal{F})$ is equipped with the Hermitian structure $g^*(\overline{E}) \dashrightarrow g^*(\mathcal{F})$, that is well defined up to tight isomorphism by Theorem 2.18(iv). As it is customary, we will pretend that g^* is a functor. The notation for the corresponding object in $\overline{\mathbf{D}}^b(X')$ is $g^*(\overline{\mathcal{F}})$. If $f : \overline{\mathcal{F}}_1 \dashrightarrow \overline{\mathcal{F}}_2$ is a morphism in $\overline{\mathbf{D}}^b(X)$, we denote by $g^*(f) : g^*(\overline{\mathcal{F}}_1) \dashrightarrow g^*(\overline{\mathcal{F}}_2)$ its left derived inverse image by g.

The functor g^* preserves the structure of principal fibered category with flat connection and the formation of Hermitian cones. Namely we have the following result that is easily proved.

Theorem 3.27. *Let $g : X' \to X$ be a morphism of smooth algebraic varieties over $\mathbb{C}$ and let $f : \overline{\mathcal{F}}_1 \dashrightarrow \overline{\mathcal{F}}_2$ be a morphism in $\overline{\mathbf{D}}^b(X)$.*

(i) *The functor g^* preserves the forgetful functor*:
$$\mathfrak{F} \circ g^* = g^* \circ \mathfrak{F}.$$
(ii) *The restriction $g^* : \overline{\mathbf{KA}}(X) \to \overline{\mathbf{KA}}(X')$ is a group homomorphism.*
(iii) *The functor g^* is equivariant with respect to the actions of $\overline{\mathbf{KA}}(X)$ and $\overline{\mathbf{KA}}(X')$.*
(iv) *The functor g^* preserves parallel transport*: *if f is an isomorphism, then*
$$g^* \circ \mathfrak{t}_f = \mathfrak{t}_{g^*(f)} \circ g^*.$$
(v) *The functor g^* preserves Hermitian cones*:
$$g^*\big(\overline{\mathrm{cone}}(f)\big) = \overline{\mathrm{cone}}\big(g^*(f)\big).$$

Classes of isomorphisms and distinguished triangles. Let $f : \overline{\mathcal{F}} \overset{\sim}{\dashrightarrow} \overline{\mathcal{G}}$ be an isomorphism in $\overline{\mathbf{D}}^b(X)$. To it, we attach a class $[f] \in \overline{\mathbf{KA}}(X)$ that measures the default of being a tight isomorphism. This class is defined using the Hermitian cone.

$$[f] = \big[\overline{\mathrm{cone}}(f)\big]. \tag{3.28}$$

Observe the abuse of notation: we wrote $[\overline{\mathrm{cone}}(f)]$ for the class in $\overline{\mathbf{KA}}(X)$ of the Hermitian structure of a Hermitian cone of f. This is well defined, since the Hermitian cone is unique up to tight isomorphism. Alternatively, we can construct $[f]$ using parallel transport as follows. There is a unique element $\overline{A} \in \overline{\mathbf{KA}}(X)$ such that

$$\overline{\mathcal{G}} = \mathfrak{t}_f \overline{\mathcal{F}} + \overline{A}.$$

We denote this element by $\overline{\mathcal{G}} - \mathfrak{t}_f \overline{\mathcal{F}}$. Then

$$[f] = \overline{\mathcal{G}} - \mathfrak{t}_f \overline{\mathcal{F}}.$$

By the definition of parallel transport, both definitions agree.

Definition 3.29. A *distinguished triangle in* $\overline{\mathbf{D}}^b(X)$, consists in a diagram

$$\overline{\tau} = (u, v, w) : \overline{\mathcal{F}} \overset{u}{\dashrightarrow} \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{u}{\dashrightarrow} \cdots \tag{3.30}$$

in $\overline{\mathbf{D}}^b(X)$, whose underlying morphisms in $\mathbf{D}^b(X)$ form a distinguished triangle. We will say that it is *tightly distinguished* if there is a commutative diagram

$$\begin{array}{ccccccccc}
\overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\operatorname{cone}}(\overline{\mathcal{F}}, \overline{\mathcal{G}}) & \dashrightarrow & \overline{\mathcal{F}}[1] & \dashrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{F}}[1] & \dashrightarrow & \cdots,
\end{array} \tag{3.31}$$

with α a tight isomorphism.

To every distinguished triangle in $\overline{\mathbf{D}}^b(X)$ we can associate a class in $\overline{\mathbf{KA}}(X)$ that measures the default of being tightly distinguished. Let $\overline{\tau}$ be a distinguished triangle as in (3.30). Then there is a diagram as (3.31), but with α an isomorphism non-necessarily tight. Then we define

$$[\overline{\tau}] = [\alpha]. \tag{3.32}$$

By Theorem 3.17, the class $[\alpha]$ does not depend on the particular choice of morphism α in $\overline{\mathbf{D}}^b(X)$ for which (3.31) commutes. Hence (3.32) only depends on $\overline{\tau}$.

Theorem 3.33.

(i) *Let f be an isomorphism in $\overline{\mathbf{D}}^b(X)$ (respectively $\overline{\tau}$ a distinguished triangle). Then $[f] = 0$ (respectively $[\overline{\tau}] = 0$) if and only if f is a tight isomorphism (respectively $\overline{\tau}$ is tightly distinguished).*

(ii) *Let $g : X' \to X$ be a morphism of smooth complex varieties, let f be an isomorphism in $\overline{\mathbf{D}}^b(X)$ and $\overline{\tau}$ a distinguished triangle in $\overline{\mathbf{D}}^b(X)$. Then*

$$g^*[f] = [g^* f], \qquad g^*[\overline{\tau}] = [g^* \overline{\tau}].$$

In particular, tight isomorphisms and tightly distinguished triangles are preserved under left derived inverse images.

(iii) *Let $f : \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ and $h : \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{H}}$ be two isomorphisms in $\overline{\mathbf{D}}^b(X)$. Then:*

$$[h \circ f] = [h] + [f].$$

In particular, $[f^{-1}] = -[f]$.

(iv) *For any distinguished triangle $\overline{\tau}$ in $\overline{\mathbf{D}}^b(X)$ as in Definition 3.29, the rotated triangle*

$$\overline{\tau}' : \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{-u[1]}{\dashrightarrow} \overline{\mathcal{G}}[1] \overset{v[1]}{\dashrightarrow} \cdots$$

satisfies $[\overline{\tau}'] = -[\overline{\tau}]$. In particular, rotating preserves tightly distinguished triangles.

(v) *For any acyclic complex $\overline{\mathcal{F}}$, we have a distinguished triangle $\overline{\mathcal{F}} \dashrightarrow \bar{0} \dashrightarrow \bar{0} \dashrightarrow \overline{\mathcal{F}}[1] \dashrightarrow \cdots$. Then*

$$[\overline{\mathcal{F}} \dashrightarrow \bar{0} \dashrightarrow \bar{0} \dashrightarrow \overline{\mathcal{F}}[1] \dashrightarrow \cdots] = [\overline{\mathcal{F}}].$$

(vi) *If $f : \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ is an isomorphism in $\overline{\mathbf{D}}^b(X)$, then*

$$[\bar{0} \dashrightarrow \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}} \dashrightarrow \bar{0}[1] \dashrightarrow \cdots] = [f].$$

(vii) *For a commutative diagram of distinguished triangles*

$$\begin{array}{cccccccccc}
\overline{\tau} & \overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{F}}[1] & \dashrightarrow & \cdots \\
\downarrow & \downarrow{\scriptstyle \sim f} & & \downarrow{\scriptstyle \sim g} & & \downarrow{\scriptstyle \sim h} & & \downarrow{\scriptstyle \sim f[1]} & & \\
\overline{\tau}' & \overline{\mathcal{F}}' & \dashrightarrow & \overline{\mathcal{G}}' & \dashrightarrow & \overline{\mathcal{H}}' & \dashrightarrow & \overline{\mathcal{F}}'[1] & \dashrightarrow & \cdots
\end{array}$$

the following relation holds:

$$[\overline{\tau}'] - [\overline{\tau}] = [f] - [g] + [h].$$

(viii) *For a commutative diagram of distinguished triangles*

$$\begin{array}{ccccccccccc}
\overline{\tau} & & \overline{\mathcal{F}} & \dashrightarrow & \overline{\mathcal{G}} & \dashrightarrow & \overline{\mathcal{H}} & \dashrightarrow & \overline{\mathcal{F}}[1] & \dashrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\overline{\tau}' & & \overline{\mathcal{F}}' & \dashrightarrow & \overline{\mathcal{G}}' & \dashrightarrow & \overline{\mathcal{H}}' & \dashrightarrow & \overline{\mathcal{F}}'[1] & \dashrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\overline{\tau}'' & & \overline{\mathcal{F}}'' & \dashrightarrow & \overline{\mathcal{G}}'' & \dashrightarrow & \overline{\mathcal{H}}'' & \dashrightarrow & \overline{\mathcal{F}}''[1] & \dashrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \overline{\mathcal{F}}[1] & \dashrightarrow & \overline{\mathcal{G}}[1] & \dashrightarrow & \overline{\mathcal{H}}[1] & \dashrightarrow & \overline{\mathcal{F}}[2] & \dashrightarrow & \cdots, \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & & & & & & & & & \\
 & & \overline{\eta} & \dashrightarrow & \overline{\eta}' & \dashrightarrow & \overline{\eta}'' & & & &
\end{array} \tag{3.34}$$

the following relation holds:

$$[\overline{\tau}] - [\overline{\tau}'] + [\overline{\tau}''] = [\overline{\eta}] - [\overline{\eta}'] + [\overline{\eta}''].$$

Proof. The first two statements are clear. For the third, we may assume that f and g are realized by quasi-isomorphisms

$$f : \overline{F} \to \overline{G}, \qquad g : \overline{G} \to \overline{H}.$$

Then the result follows from Theorem 2.30(vii). The fourth assertion is a consequence of Proposition 3.23. Then (v), (vi) and (vii) follow from Eq. (3.32) and the fourth statement. The last property is derived from (vii) by comparing the diagram to a diagram of tightly distinguished triangles. □

As an application of the class in $\overline{\mathbf{KA}}(X)$ attached to a distinguished triangle, we exhibit a natural morphism $K_1(X) \to \overline{\mathbf{KA}}(X)$. This is included for the sake of completeness, but won't be needed in the sequel.

Proposition 3.35. *There is a natural morphism of groups* $K_1(X) \to \overline{\mathbf{KA}}(X)$.

Proof. We follow the definitions and notations of [14]. From [14] we know it is enough to construct a morphism of groups

$$H_1\big(\widetilde{\mathbb{Z}}C(X)\big) \to \overline{\mathbf{KA}}(X). \tag{3.36}$$

By definition, the piece of degree n of the homological complex $\widetilde{\mathbb{Z}}C(X)$ is

$$\widetilde{\mathbb{Z}}C_n(X) = \mathbb{Z}C_n(X)/D_n.$$

Here $\mathbb{Z}C_n(X)$ stands for the free abelian group on metrized exact n-cubes and D_n is the subgroup of degenerate elements. A metrized exact 1-cube is a short exact sequence of Hermitian vector bundles. Hence, for such a 1-cube $\overline{\varepsilon}$, there is a well defined class in $\overline{\mathbf{KA}}(X)$. Observe that this class coincides with the class of $\overline{\varepsilon}$ thought as a distinguished triangle in $\overline{\mathbf{D}}^b(X)$. Because $\overline{\mathbf{KA}}(X)$ is an abelian group, it follows the existence of a morphism of groups

$$\mathbb{Z}C_1(X) \to \overline{\mathbf{KA}}(X).$$

From the definition of degenerate cube [14, Definition 3.3] and the construction of $\overline{\mathbf{KA}}(X)$, this morphism clearly factors through $\widetilde{\mathbb{Z}}C_1(X)$. The definition of the differential d of the complex $\widetilde{\mathbb{Z}}C(X)$ [14, (3.2)] and Theorem 3.33(viii) ensure that $d\mathbb{Z}C_2(X)$ is in the kernel of the morphism. Hence we derive the existence of a morphism (3.36). □

Classes of complexes and of direct images of complexes. In [13, Section 2] the notion of homological exact sequences of metrized coherent sheaves is treated. We provide now the link between the point of view of [13] and the formalism adopted here. The reader will find no difficulty to translate it to cohomological complexes.

Consider a homological complex

$$\bar{\varepsilon}: 0 \to \overline{\mathcal{F}}_m \to \cdots \to \overline{\mathcal{F}}_l \to 0$$

of metrized coherent sheaves, namely coherent sheaves provided with Hermitian structures $\overline{\mathcal{F}}_i = (\mathcal{F}_i, \overline{F}_i \dashrightarrow \mathcal{F}_i)$. We may equivalently see $\bar{\varepsilon}$ as a cohomological complex, by the usual relabeling $\overline{\mathcal{F}}^{-i} = \overline{\mathcal{F}}_i$. This will be freely used in the sequel, especially in cone constructions.

Definition 3.37. The complex $\bar{\varepsilon}$ defines an object $[\bar{\varepsilon}] \in \operatorname{Ob} \overline{\mathbf{D}}^b(X)$ that is determined inductively by the condition

$$[\bar{\varepsilon}] = \overline{\operatorname{cone}}\big(\overline{\mathcal{F}}_m[m], [\sigma_{<m}\bar{\varepsilon}]\big).$$

Here $\sigma_{<m}$ is the homological bête filtration and $\overline{\mathcal{F}}_m$ denotes a cohomological complex concentrated in degree zero. Hence, $\overline{\mathcal{F}}_m[m]$ is a cohomological complex concentrated in degree $-m$.

If $\overline{E}$ is a Hermitian vector bundle on X, then $[\bar{\varepsilon} \otimes \overline{E}] = [\bar{\varepsilon}] \otimes \overline{E}$. According to Definition 3.10, if ε is an acyclic complex, then we also have the corresponding class $[\![\bar{\varepsilon}]\!]$ in $\overline{\mathbf{KA}}(X)$. We will employ the lighter notation $[\bar{\varepsilon}]$ for this class.

Given a morphism $\varphi: \bar{\varepsilon} \to \bar{\mu}$ of bounded complexes of metrized coherent sheaves, the pieces of the complex $\operatorname{cone}(\varepsilon, \mu)$ are naturally endowed with Hermitian metrics. We thus get a complex of metrized coherent sheaves $\overline{\operatorname{cone}(\varepsilon, \mu)}$. Hence Definition 3.37 provides an object $[\overline{\operatorname{cone}(\varepsilon, \mu)}]$ in $\overline{\mathbf{D}}^b(X)$. On the other hand, Definition 3.14 attaches to φ the Hermitian cone $\overline{\operatorname{cone}}([\bar{\varepsilon}], [\bar{\mu}])$, which is well defined up to tight isomorphism. Both constructions actually agree.

Lemma 3.38. *Let $\bar{\varepsilon} \to \bar{\mu}$ be a morphism of bounded complexes of metrized coherent sheaves on X. Then there is a tight isomorphism*

$$\overline{\operatorname{cone}}\big([\bar{\varepsilon}], [\bar{\mu}]\big) \cong \big[\overline{\operatorname{cone}(\varepsilon, \mu)}\big].$$

Proof. The case when ε and μ are both concentrated in a single degree d is clear. The general case follows by induction taking into account Definition 3.37. $\square$

Assume now that $f: X \to Y$ is a morphism of smooth complex varieties and, for each complex $f_*\mathcal{F}_i$, we have chosen a Hermitian structure $\overline{f_*\mathcal{F}_i} = (\overline{E}_i \dashrightarrow f_*\mathcal{F}_i)$. Denote by $\overline{f_*\varepsilon}$ this choice of metrics. Recall that by f_* we mean the derived direct image, therefore $f_*\mathcal{F}_i$ are objects on the derived category and not coherent sheaves. Thus we are not in the situation of Definition 3.37.

Definition 3.39. The family of Hermitian structures $\overline{f_*\varepsilon}$ defines an object $[\overline{f_*\varepsilon}] \in \operatorname{Ob} \overline{\mathbf{D}}^b(Y)$ that is determined inductively by the condition

$$[\overline{f_*\varepsilon}] = \overline{\operatorname{cone}}\big(\overline{f_*\mathcal{F}_m}[m], [\overline{f_*\sigma_{<m}\varepsilon}]\big),$$

where the morphism $f_*\mathcal{F}_m[m] \dashrightarrow f_*\sigma_{<m}\varepsilon$ is the one induced by the morphism $\mathcal{F}_m[m] \to \sigma_{<m}\varepsilon$.

We remark that the notation $\overline{f_*\varepsilon}$ means that the Hermitian structure is chosen after taking the direct image and it is not determined by the Hermitian structure on $\bar{\varepsilon}$.

Notice also that if ε is an acyclic complex on X, we have the class $[\overline{f_*\varepsilon}] \in \overline{\mathbf{KA}}(Y)$.

Let $\varepsilon \to \mu$ be a morphism of bounded complexes of coherent sheaves on X and $f: X \to Y$ a morphism of smooth complex varieties. Fix choices of metrics $\overline{f_*\varepsilon}$ and $\overline{f_*\mu}$. Then there is an obvious choice of metrics on $f_* \operatorname{cone}(\varepsilon, \mu)$, that we denote by $\overline{f_* \operatorname{cone}(\varepsilon, \mu)}$, and hence an object $[\overline{f_* \operatorname{cone}(\varepsilon, \mu)}]$ in $\overline{\mathbf{D}}^b(Y)$. On the other hand, we also have the Hermitian cone $\overline{\operatorname{cone}}([\overline{f_*\varepsilon}], [\overline{f_*\mu}])$. Again both definitions agree.

Lemma 3.40. *Let $\varepsilon \to \mu$ be a morphism of bounded complexes of coherent sheaves on X and $f : X \to Y$ a morphism of smooth complex varieties. Assume that families of metrics $\overline{f_*\varepsilon}$ and $\overline{f_*\mu}$ are chosen. Then there is a tight isomorphism*

$$\overline{\mathrm{cone}}\big([\overline{f_*\varepsilon}], [\overline{f_*\mu}]\big) \cong \big[\overline{f_* \operatorname{cone}(\varepsilon, \mu)}\big].$$

Proof. If ε and μ are concentrated in a single degree d, then the statement is obvious. The proof follows by induction and Definition 3.39. □

The objects we have defined are compatible with short exact sequences, in the sense of the following statement.

Proposition 3.41. *Consider a commutative diagram of exact sequences of coherent sheaves on X*

$$\begin{array}{ccccccccc}
 & & & 0 & & & 0 & & \\
 & & & \downarrow & & & \downarrow & & \\
\mu' & & 0 \longrightarrow & \mathcal{F}'_m & \longrightarrow \cdots \longrightarrow & & \mathcal{F}'_l & \longrightarrow 0 & \\
 & & & \downarrow & & & \downarrow & & \\
\mu & & 0 \longrightarrow & \mathcal{F}_m & \longrightarrow \cdots \longrightarrow & & \mathcal{F}_l & \longrightarrow 0 & \\
 & & & \downarrow & & & \downarrow & & \\
\mu'' & & 0 \longrightarrow & \mathcal{F}''_m & \longrightarrow \cdots \longrightarrow & & \mathcal{F}''_l & \longrightarrow 0 & \\
 & & & \downarrow & & & \downarrow & & \\
 & & & 0 & & & 0 & & \\
 & & & & & & & & \\
 & & & \xi_m & \cdots & & \xi_l. & &
\end{array}$$

Let $f : X \to Y$ be a morphism of smooth complex varieties and choose Hermitian structures on the sheaves $\mathcal{F}'_j$, $\mathcal{F}_j$, $\mathcal{F}''_j$ and on the objects $f_*\mathcal{F}'_j$, $f_*\mathcal{F}_j$ and $f_*\mathcal{F}''_j$, $j = l, \dots, m$. Then the following equalities hold in $\overline{\mathbf{KA}}(X)$ and $\overline{\mathbf{KA}}(Y)$, respectively:

$$\sum_j (-1)^j [\bar{\xi}_j] = [\bar{\mu}'] - [\bar{\mu}] + [\bar{\mu}''],$$
$$\sum_j (-1)^j [\overline{f_*\xi}_j] = [\overline{f_*\mu}'] - [\overline{f_*\mu}] + [\overline{f_*\mu}''].$$

Proof. The lemma follows inductively taking into account Definitions 3.37 and 3.39 and Theorem 3.33(viii). □

Corollary 3.42. *Let $\bar{\varepsilon} \to \bar{\mu}$ be a morphism of exact sequences of metrized coherent sheaves. Let $f : X \to Y$ be a morphism of smooth complex varieties and fix families of metrics $\overline{f_*\varepsilon}$ and $\overline{f_*\mu}$. Then the following equalities in $\overline{\mathbf{KA}}(X)$ and $\overline{\mathbf{KA}}(Y)$, respectively, hold*

$$\big[\overline{\operatorname{cone}(\varepsilon, \mu)}\big] = [\bar{\mu}] - [\bar{\varepsilon}], \tag{3.43}$$
$$\big[\overline{f_* \operatorname{cone}(\varepsilon, \mu)}\big] = [\overline{f_*\mu}] - [\overline{f_*\varepsilon}]. \tag{3.44}$$

Proof. The result readily follows from Lemmas 3.38, 3.40 and Proposition 3.41. □

Hermitian structures on cohomology. Let $\mathcal{F}$ be an object of $\mathbf{D}^b(X)$ and $(\mathcal{F}, \mathrm{d})$ an actual complex representing it. Denote by $\mathcal{H}$ its cohomology complex. Observe that $\mathcal{H}$ is a bounded complex with 0 differentials. Assume that Hermitian structures are given on each non-zero individual piece $\mathcal{H}^i$. We show that there is a natural Hermitian structure on the complex $\mathcal{F}$ attached to these data. This situation arises in [11] when considering cohomology sheaves endowed with L^2 metric structures. Since, in general, $\mathcal{F}$ and $\mathcal{H}$ are not isomorphic objects in $\mathbf{D}^b(X)$, we cannot just define a Hermitian structure on $\mathcal{H}$ and transfer it to $\mathcal{F}$.

The construction is recursive. If the cohomology complex is trivial, then $\mathcal{F}$ is acyclic and we endow $\mathcal{F}$ with the trivial Hermitian structure. Otherwise, let $\mathcal{H}^m$ be the highest non-zero cohomology sheaf. The canonical filtration $\tau^{\leqslant m}$ is given by

$$\tau^{\leqslant m}\mathcal{F}: \cdots \to \mathcal{F}^{m-2} \to \mathcal{F}^{m-1} \to \ker\bigl(\mathrm{d}^m\bigr) \to 0.$$

By the condition on the highest non-vanishing cohomology sheaf, the natural inclusion is a quasi-isomorphism:

$$\tau^{\leqslant m}\mathcal{F} \xrightarrow{\sim} \mathcal{F}. \tag{3.45}$$

We also introduce the subcomplex

$$\widetilde{\mathcal{F}}: \cdots \to \mathcal{F}^{m-2} \to \mathcal{F}^{m-1} \to \operatorname{Im}\bigl(\mathrm{d}^{m-1}\bigr) \to 0.$$

Observe that the cohomology complex of $\widetilde{\mathcal{F}}$ is the bête truncation $\mathcal{H}/\sigma^{\geqslant m}\mathcal{H}$. By induction, $\widetilde{\mathcal{F}}$ carries an induced Hermitian structure. We also have an exact sequence

$$0 \to \widetilde{\mathcal{F}} \to \tau^{\leqslant m}\mathcal{F} \to \mathcal{H}^m[-m] \to 0. \tag{3.46}$$

Taking into account the quasi-isomorphism (3.45) and the exact sequence (3.46), we construct a natural commutative diagram of distinguished triangles in $\mathbf{D}^b(X)$

$$\begin{array}{ccccccc}
\mathcal{H}^m[-m-1] & \overset{0}{\dashrightarrow} & \widetilde{\mathcal{F}} & \dashrightarrow & \mathcal{F} & \dashrightarrow & \mathcal{H}^m[m] \\
\big\downarrow \mathrm{id} & & \big\downarrow \mathrm{id} & & \big\downarrow \sim & & \big\downarrow \mathrm{id} \\
\mathcal{H}^m[-m-1] & \overset{0}{\dashrightarrow} & \widetilde{\mathcal{F}} & \dashrightarrow & \operatorname{cone}(\mathcal{H}^m[-m-1], \widetilde{\mathcal{F}}) & \dashrightarrow & \mathcal{H}^m[m].
\end{array}$$

By the Hermitian cone construction and Theorem 3.17, we see that Hermitian structures on $\widetilde{\mathcal{F}}$ and $\mathcal{H}^m$ induce a well defined Hermitian structure on $\mathcal{F}$.

Definition 3.47. Let $\mathcal{F}$ be an object of $\mathbf{D}^b(X)$ with cohomology complex $\mathcal{H}$. Assume the pieces $\mathcal{H}^i$ are endowed with Hermitian structures. The Hermitian structure on $\mathcal{F}$ constructed above will be called the *Hermitian structure induced by the Hermitian structure on the cohomology complex* and will be denoted by $(\mathcal{F}, \overline{\mathcal{H}})$.

The following proposition is a direct consequence of the definitions.

Proposition 3.48. *Let $\varphi: \mathcal{F}_1 \dashrightarrow \mathcal{F}_2$ be an isomorphism in $\mathbf{D}^b(X)$. Assume the pieces of the cohomology complexes $\mathcal{H}_1$, $\mathcal{H}_2$ of $\mathcal{F}_1$, $\mathcal{F}_2$ are endowed with Hermitian structures. If the induced isomorphism in cohomology $\varphi_*: \mathcal{H}_1 \to \mathcal{H}_2$ is tight, then φ is tight for the induced Hermitian structures on $\mathcal{F}_1$ and $\mathcal{F}_2$.*

4. Bott–Chern classes for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^b(X)$

In this section we will define Bott–Chern classes for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^b(X)$. The natural context where one can define the Bott–Chern classes is that of Deligne complexes. For details about Deligne complexes the reader is referred to [10] and [12]. In this section we will use the same notations as in [13, §1]. In particular, the *Deligne algebra of differential forms* on X is denoted by $\mathcal{D}^*(X, *)$, and we use the notation

$$\widetilde{\mathcal{D}}^n(X, p) = \mathcal{D}^n(X, p)/\mathrm{d}_{\mathcal{D}}\mathcal{D}^{n-1}(X, p).$$

When characterizing axiomatically Bott–Chern classes, the basic tool to exploit the functoriality axiom is to use a deformation parametrized by $\mathbb{P}^1$. This argument leads to the following lemma that will be used to prove the uniqueness of the Bott–Chern classes introduced in this section.

Lemma 4.1. *Let X be a smooth complex variety. Let $\widetilde{\varphi}$ be an assignment that, to each smooth morphism of complex varieties $g : X' \to X$ and each acyclic complex $\overline{A}$ of Hermitian vector bundles on X' assigns a class*

$$\widetilde{\varphi}(\overline{A}) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^n\big(X', p\big)$$

fulfilling the following properties:

(i) (*Differential equation*) *The equality* $\mathrm{d}_{\mathcal{D}}\widetilde{\varphi}(\overline{A}) = 0$ *holds*;
(ii) (*Functoriality*) *For each morphism of smooth complex varieties $h : X'' \to X'$ with $g \circ h$ smooth, we have* $h^*\widetilde{\varphi}(\overline{A}) = \widetilde{\varphi}(h^*\overline{A})$;
(iii) (*Normalization*) *If $\overline{A}$ is orthogonally split, then* $\widetilde{\varphi}(\overline{A}) = 0$.

Then $\widetilde{\varphi} = 0$.

Proof. The argument of the proof of [13, Theorem 2.3] applies *mutatis mutandis* to the present situation. □

Definition 4.2. An *additive genus in Deligne cohomology* is a characteristic class φ for vector bundles of any rank in the sense of [13, Definition 1.5] that satisfies the equation

$$\varphi(E_1 \oplus E_2) = \varphi(E_1) + \varphi(E_2). \tag{4.3}$$

Let $\mathbb{D}$ denote the base ring for Deligne cohomology. That is, $\mathbb{D} = \bigoplus H^n_{\mathcal{D}}(\mathrm{Spec}(\mathbb{C}), p)$, where

$$H^n_{\mathcal{D}}\big(\mathrm{Spec}(\mathbb{C}), p\big) = \begin{cases} \mathbb{R}(p) := (2\pi i)^p\mathbb{R}, & \text{if } n = 0,\ p \leqslant 0, \\ \mathbb{R}(p-1) := (2\pi i)^{p-1}\mathbb{R}, & \text{if } n = 1,\ p > 0, \\ \{0\}, & \text{otherwise.} \end{cases}$$

The product structure is the bigraded product given by complex number multiplication when the degrees allow the product to be non-zero.

A consequence of [13, Theorem 1.8] is that there is a bijection between the set of additive genera in Deligne cohomology and the set of power series in one variable $\mathbb{D}[\![x]\!]$. To each power series $\varphi \in \mathbb{D}[\![x]\!]$ it corresponds the unique additive genus such that

$$\varphi(L) = \varphi\big(c_1(L)\big)$$

for every line bundle L.

Definition 4.4. A *real additive genus* is an additive genus such that the corresponding power series belong to $\mathbb{R}[\![x]\!]$ with $\mathbb{R} = H^0_{\mathcal{D}}(\mathrm{Spec}(\mathbb{C}), 0) \subset \mathbb{D}$.

Remark 4.5. It is clear that, if φ is a real additive genus, then for each vector bundle E we have

$$\varphi(E) \in \bigoplus_p H^{2p}_{\mathcal{D}}\big(X, \mathbb{R}(p)\big).$$

We now focus on additive genera, for instance, the Chern character is a real additive genus. Let φ be such a genus. Using Chern–Weil theory, to each Hermitian vector bundle $\overline{E}$ on X we can attach a closed characteristic form

$$\varphi(\overline{E}) \in \bigoplus_{n,p} \mathcal{D}^n(X, p).$$

If $\overline{E}$ is an object of $\overline{\mathbf{V}}^b(X)$, then we define

$$\varphi(\overline{E}) = \sum_i (-1)^i \varphi\big(\overline{E}^i\big).$$

If $\overline{E}$ is acyclic, following [13, Section 2], we associate to it a Bott–Chern characteristic class

$$\widetilde{\varphi}(\overline{E}) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

that satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}}\widetilde{\varphi}(\overline{E}) = \varphi(\overline{E}).$$

In fact, [13, Theorem 2.3] for additive genera can be restated as follows.

Proposition 4.6. *Let φ be an additive genus. Then there is a unique group homomorphism*

$$\widetilde{\varphi}: \overline{\mathbf{KA}}(X) \to \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

satisfying the properties:

(i) (*Differential equation*) $\mathrm{d}_{\mathcal{D}}\widetilde{\varphi}(\overline{E}) = \varphi(\overline{E})$.
(ii) (*Functoriality*) *If $f: X \to Y$ is a morphism of smooth complex varieties, then $\widetilde{\varphi}(f^*(\overline{E})) = f^*(\widetilde{\varphi}(\overline{E}))$.*

Proof. For the uniqueness, we observe that, if $\widetilde{\varphi}$ is a group homomorphism then $\widetilde{\varphi}(\bar{0}) = 0$. Hence, if $\overline{E}$ is an orthogonally split complex, then it is meager and therefore $\widetilde{\varphi}(\overline{E}) = 0$. Thus, the assignment that, to each acyclic bounded complex $\overline{E}$, associates the class $\widetilde{\varphi}([\overline{E}])$ satisfies the conditions of [13, Theorem 2.3], hence is unique. For the existence, we note that Bott–Chern classes for additive genera satisfy the hypothesis of Theorem 2.35. Hence the result follows. □

Remark 4.7. If

$$\bar{\varepsilon}: 0 \to \overline{\mathcal{F}}_m \to \cdots \to \overline{\mathcal{F}}_l \to 0$$

is an acyclic complex of coherent sheaves on X provided with Hermitian structures $\overline{\mathcal{F}}_i = (\mathcal{F}_i, \overline{F}_i \dashrightarrow \mathcal{F}_i)$, by Definition 3.37 we have an object $[\bar{\varepsilon}] \in \overline{\mathbf{KA}}(X)$, hence a class $\widetilde{\varphi}([\bar{\varepsilon}])$. In the case of the Chern character, in [13, Theorem 2.24], a class $\widetilde{\mathrm{ch}}(\bar{\varepsilon})$ is defined. It follows from [13, Theorem 2.24] that both classes agree. That is, $\widetilde{\mathrm{ch}}([\bar{\varepsilon}]) = \widetilde{\mathrm{ch}}(\bar{\varepsilon})$. For this reason we will denote $\widetilde{\varphi}([\bar{\varepsilon}])$ by $\widetilde{\varphi}(\bar{\varepsilon})$.

Definition 4.8. Let $\overline{\mathcal{F}} = (\overline{E} \overset{\sim}{\dashrightarrow} \mathcal{F})$ be an object of $\overline{\mathbf{D}}^b(X)$. Let φ denote an additive genus. We denote the form

$$\varphi(\overline{\mathcal{F}}) = \varphi(\overline{E}) \in \bigoplus_{n,p} \mathcal{D}^n(X,p)$$

and the class

$$\varphi(\mathcal{F}) = \big[\varphi(\overline{E})\big] \in \bigoplus_{n,p} H^n_{\mathcal{D}}\big(X, \mathbb{R}(p)\big).$$

Note that the form $\varphi(\overline{\mathcal{F}})$ only depends on the Hermitian structure and not on a particular representative thanks to Propositions 3.3 and 4.6. The class $\varphi(\mathcal{F})$ only depends on the object $\mathcal{F}$ and not on the Hermitian structure.

Remark 4.9. The reason to restrict to additive genera when working with the derived category is now clear: there is no canonical way to attach a rank to $\bigoplus_{i \text{ even}} \mathcal{F}^i$ (respectively $\bigoplus_{i \text{ odd}} \mathcal{F}^i$). The naive choice $\mathrm{rk}(\bigoplus_{i \text{ even}} E^i)$ (respectively $\mathrm{rk}(\bigoplus_{i \text{ odd}} E^i)$) does depend on $E \dashrightarrow \mathcal{F}$. Thus we can not define Bott–Chern classes by the general rule from [13, Theorem 2.3] for arbitrary invariant power series. The case of a multiplicative genus such as the Todd genus will be considered later.

Next we will construct Bott–Chern classes for isomorphisms in $\overline{\mathbf{D}}^b(X)$.

Definition 4.10. Let $f: \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ be a morphism in $\overline{\mathbf{D}}^b(X)$ and φ an additive genus. We define the differential form

$$\varphi(f) = \varphi(\overline{\mathcal{G}}) - \varphi(\overline{\mathcal{F}}).$$

Theorem 4.11. *Let φ be an additive genus. There is a unique way to attach to every isomorphism in $\overline{\mathbf{D}}^b(X)$ $f:(\overline{F} \dashrightarrow \mathcal{F}) \overset{\sim}{\dashrightarrow} (\overline{G} \dashrightarrow \mathcal{G})$ a Bott–Chern class*

$$\widetilde{\varphi}(f) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X,p)$$

such that the following axioms are satisfied:

(i) (*Differential equation*) $\mathrm{d}_{\mathcal{D}}\widetilde{\varphi}(f) = \varphi(f)$.
(ii) (*Functoriality*) *If $g : X' \to X$ is a morphism of smooth Noetherian schemes over $\mathbb{C}$, then*

$$\widetilde{\varphi}\big(g^*(f)\big) = g^*\big(\widetilde{\varphi}(f)\big).$$

(iii) (*Normalization*) *If f is a tight isomorphism, then $\widetilde{\varphi}(f) = 0$.*

Proof. For the existence we define

$$\widetilde{\varphi}(f) = \widetilde{\varphi}\big([f]\big), \tag{4.12}$$

where $[f] \in \overline{\mathbf{KA}}(X)$ is the class of f given by Eq. (3.28). That $\widetilde{\varphi}$ satisfies the axioms follows from Proposition 4.6 and Theorem 3.27.

We now focus on the uniqueness. Assume such a theory $f \mapsto \widetilde{\varphi}_0(f)$ exists. Fix f as in the statement. Since $\widetilde{\varphi}_0$ is well defined, by replacing $\overline{F}$ by one that is tightly related, we may assume that f is realized by a morphism of complexes

$$f : \overline{F} \to \overline{G}.$$

We factorize f as

$$\overline{F} \xrightarrow{\alpha} \overline{G} \oplus \overline{\mathrm{cone}}(\overline{F}, \overline{G})[-1] \xrightarrow{\beta} \overline{G},$$

where both arrows are zero on the second factor of the middle complex. Since α is a tight morphism and $\overline{\mathrm{cone}}(\overline{F}, \overline{G})[-1]$ is acyclic, we are reduced to the case when $\overline{F} = \overline{G} \oplus \overline{A}$, with $\overline{A}$ an acyclic complex and f is the projection onto the first factor.

For each smooth morphism $g : X' \to X$ and each acyclic complex of vector bundles $\overline{E}$ on X', we denote

$$\widetilde{\varphi}_1(\overline{E}) = \widetilde{\varphi}_0\big(g^*\overline{G} \oplus \overline{E} \to g^*\overline{G}\big) + \widetilde{\varphi}(\overline{E}),$$

where $\widetilde{\varphi}$ is the usual Bott–Chern form for acyclic complexes of Hermitian vector bundles associated to φ. Then $\widetilde{\varphi}_1$ satisfies the hypothesis of Lemma 4.1, so $\widetilde{\varphi}_1 = 0$. Therefore $\widetilde{\varphi}_0(f) = -\widetilde{\varphi}(\overline{A})$. □

Proposition 4.13. *Let $f : \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ and $g : \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{H}}$ be two isomorphisms in $\overline{\mathbf{D}}^b(X)$. Then*:

$$\widetilde{\varphi}(g \circ f) = \widetilde{\varphi}(g) + \widetilde{\varphi}(f).$$

In particular, $\widetilde{\varphi}(f^{-1}) = -\widetilde{\varphi}(f)$.

Proof. The statement follows from Theorem 3.33(iii). □

The Bott–Chern classes behave well under shift.

Proposition 4.14. *Let $f : \overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{G}}$ be an isomorphism in $\overline{\mathbf{D}}^b(X)$. Let $f[i] : \overline{\mathcal{F}}[i] \dashrightarrow \overline{\mathcal{G}}[i]$ be the shifted isomorphism. Then*

$$(-1)^i \widetilde{\varphi}\big(f[i]\big) = \widetilde{\varphi}(f).$$

Proof. The assignment $f \mapsto (-1)^i \widetilde{\varphi}(f[i])$ satisfies the characterizing properties of Theorem 4.11. Hence it agrees with $\widetilde{\varphi}$. □

The following notation will be sometimes used.

Notation 4.15. Let $\mathcal{F}$ be an object of $\mathbf{D}^b(X)$ and consider two choices of Hermitian structures $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}'$. Then we write

$$\widetilde{\varphi}(\overline{\mathcal{F}}, \overline{\mathcal{F}}') = \widetilde{\varphi}(\overline{\mathcal{F}} \overset{\mathrm{id}}{\dashrightarrow} \overline{\mathcal{F}}').$$

Thus $\mathrm{d}_{\mathcal{D}}\widetilde{\varphi}(\overline{\mathcal{F}}, \overline{\mathcal{F}}') = \varphi(\overline{\mathcal{F}}') - \varphi(\overline{\mathcal{F}})$.

Example 4.16. Let $\overline{\mathcal{F}} = (\mathcal{F}, \mathcal{F} \dashrightarrow \overline{E})$ be an object of $\overline{\mathbf{D}}^b(X)$. Let $\mathcal{H}^i$ denote the cohomology sheaves of $\mathcal{F}$ and assume that we have chosen Hermitian structures $\overline{\mathcal{H}}^i$ of each $\mathcal{H}^i$. In the case when the sheaves $\mathcal{H}^i$ are vector bundles and the Hermitian structures are Hermitian metrics, X. Ma, in the paper [24], has associated to these data a Bott–Chern class, that we denote $M(\overline{\mathcal{F}}, \overline{\mathcal{H}})$. By the characterization given by Ma of $M(\overline{\mathcal{F}}, \overline{\mathcal{H}})$, it is immediate that

$$M(\overline{\mathcal{F}}, \overline{\mathcal{H}}) = \widetilde{\mathrm{ch}}(\overline{\mathcal{F}}, (\mathcal{F}, \overline{\mathcal{H}})),$$

where $(\mathcal{F}, \overline{\mathcal{H}})$ is as in Definition 3.47.

Our next aim is to construct Bott–Chern classes for distinguished triangles.

Definition 4.17. Let $\overline{\tau}$ be a distinguished triangle in $\overline{\mathbf{D}}^b(X)$,

$$\overline{\tau} : \overline{\mathcal{F}} \overset{u}{\dashrightarrow} \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{u}{\dashrightarrow} \cdots.$$

For an additive genus φ, we attach the differential form

$$\varphi(\overline{\tau}) = \varphi(\overline{\mathcal{F}}) - \varphi(\overline{\mathcal{G}}) + \varphi(\overline{\mathcal{H}}).$$

Notice that if $\overline{\tau}$ is tightly distinguished, then $\varphi(\overline{\tau}) = 0$. Moreover, for any distinguished triangle $\overline{\tau}$ as above, the rotated triangle

$$\overline{\tau}' : \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{-u[1]}{\dashrightarrow} \overline{\mathcal{G}}[1] \overset{v[1]}{\dashrightarrow} \cdots$$

satisfies $\varphi(\overline{\tau}') = -\varphi(\overline{\tau})$.

Theorem 4.18. *Let φ be an additive genus. There is a unique way to attach to every distinguished triangle in $\overline{\mathbf{D}}^b(X)$*

$$\overline{\tau} : \overline{\mathcal{F}} \overset{u}{\dashrightarrow} \overline{\mathcal{G}} \overset{v}{\dashrightarrow} \overline{\mathcal{H}} \overset{w}{\dashrightarrow} \overline{\mathcal{F}}[1] \overset{u[1]}{\dashrightarrow} \cdots$$

a Bott–Chern class

$$\widetilde{\varphi}(\overline{\tau}) \in \bigoplus_{n,p} \widetilde{\mathcal{D}}^{n-1}(X, p)$$

such that the following axioms are satisfied:

(i) (*Differential equation*) $\mathrm{d}_{\mathcal{D}}\widetilde{\varphi}(\overline{\tau}) = \varphi(\overline{\tau})$.
(ii) (*Functoriality*) *If $g : X' \to X$ is a morphism of smooth Noetherian schemes over $\mathbb{C}$, then*

$$\widetilde{\varphi}(g^*(\overline{\tau})) = g^*\widetilde{\varphi}(\overline{\tau}).$$

(iii) (*Normalization*) *If $\overline{\tau}$ is tightly distinguished, then $\widetilde{\varphi}(\overline{\tau}) = 0$.*

Proof. To show the existence we write

$$\widetilde{\varphi}(\overline{\tau}) = \widetilde{\varphi}([\overline{\tau}]). \tag{4.19}$$

Theorem 3.33 implies that it satisfies the axioms.

To prove the uniqueness, observe that, by replacing representatives of the Hermitian structures by tightly related ones, we may assume that the distinguished triangle is represented by

$$\overline{F} \to \overline{G} \to \overline{\mathrm{cone}}(\overline{F}, \overline{G}) \oplus \overline{K} \to \overline{F}[1],$$

with $\overline{K}$ acyclic. Then Lemma 4.1 shows that the axioms imply $\widetilde{\varphi}(\overline{\tau}) = \widetilde{\varphi}(\overline{K})$. □

Remark 4.20. The normalization axiom can be replaced by the apparently weaker condition that $\widetilde{\varphi}(\overline{\tau}) = 0$ for all distinguished triangles of the form

$$\overline{\mathcal{F}} \dashrightarrow \overline{\mathcal{F}} \overset{\perp}{\oplus} \overline{\mathcal{G}} \dashrightarrow \overline{\mathcal{G}} \dashrightarrow$$

where the maps are the natural inclusion and projection.

Theorem 3.33(iv)–(viii) can be easily translated to Bott–Chern classes.

5. Multiplicative genera, the Todd genus and the category $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$

Let ψ be a multiplicative genus, such that the piece of degree zero is $\psi^0 = 1$, and

$$\varphi = \log(\psi).$$

It is a well defined additive genus, because, by the condition above, the power series $\log(\psi)$ contains only finitely many terms in each degree.

If $\bar{\theta}$ is either a Hermitian vector bundle, a complex of Hermitian vector bundles, a morphism in $\overline{\mathbf{D}}^b(X)$ or a distinguished triangle in $\overline{\mathbf{D}}^b(X)$ we can write

$$\psi(\bar{\theta}) = \exp\bigl(\varphi(\bar{\theta})\bigr).$$

All the results of the previous sections can be translated to the multiplicative genus ψ. In particular, if $\bar{\theta}$ is an acyclic complex of Hermitian vector bundles, an isomorphism in $\overline{\mathbf{D}}^b(X)$ or a distinguished triangle in $\overline{\mathbf{D}}^b(X)$, we define a Bott–Chern class

$$\widetilde{\psi}_m(\bar{\theta}) = \frac{\exp(\varphi(\bar{\theta})) - 1}{\varphi(\bar{\theta})} \widetilde{\varphi}(\bar{\theta}). \tag{5.1}$$

Theorem 5.2. *The characteristic class $\widetilde{\psi}_m(\bar{\theta})$ satisfies*:

(i) (*Differential equation*) $\mathrm{d}_{\mathcal{D}} \widetilde{\psi}_m(\bar{\theta}) = \psi(\bar{\theta}) - 1$.
(ii) (*Functoriality*) *If $g : X' \to X$ is a morphism of smooth Noetherian schemes over $\mathbb{C}$, then*

$$\widetilde{\psi}_m\bigl(g^*(\bar{\theta})\bigr) = g^* \widetilde{\psi}_m(\bar{\theta}).$$

(iii) (*Normalization*) *If $\bar{\theta}$ is either a meager complex, a tight isomorphism or a tightly distinguished triangle, then* $\widetilde{\psi}_m(\bar{\theta}) = 0$.

Moreover $\widetilde{\psi}_m$ is uniquely characterized by these properties.

Remark 5.3. For an acyclic complex of vector bundles $\overline{E}$, using the general procedure for arbitrary symmetric power series, we can associate a Bott–Chern class $\widetilde{\psi}(\overline{E})$ (see for instance [13, Theorem 2.3]) that satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}} \widetilde{\psi}(\overline{E}) = \prod_{k \text{ even}} \psi\bigl(\overline{E}^k\bigr) - \prod_{k \text{ odd}} \psi\bigl(\overline{E}^k\bigr),$$

whereas $\widetilde{\psi}_m$ satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}} \widetilde{\psi}_m(\overline{E}) = \prod_{k} \psi\bigl(\overline{E}^k\bigr)^{(-1)^k} - 1. \tag{5.4}$$

In fact both Bott–Chern classes are related by

$$\widetilde{\psi}_m(\overline{E}) = \widetilde{\psi}(\overline{E}) \prod_{k \text{ odd}} \psi\bigl(\overline{E}^k\bigr)^{-1}. \tag{5.5}$$

The relationship between additive, multiplicative and general secondary characteristic classes has been studied by Berthomieu. For instance (5.5) is [3, Definition 12] and (5.1) is [3, Theorem 12].

The main example of a multiplicative genus with the above properties is the Todd genus Td. From now on we will treat only this case. Following the above procedure, to the Todd genus we can associate two Bott–Chern classes for acyclic complexes of vector bundles: the one given by the general theory, denoted by $\widetilde{\mathrm{Td}}$, and the one given by the theory of multiplicative genera, denoted by $\widetilde{\mathrm{Td}}_m$. Both are related by Eq. (5.5). Note however that, for isomorphisms and distinguished triangles in $\overline{\mathbf{D}}^b(X)$, we only have the multiplicative version.

We now consider morphisms between smooth complex varieties and relative Hermitian structures.

Definition 5.6. Let $f: X \to Y$ be a morphism of smooth complex varieties. The *tangent complex* of f is the complex

$$T_f: 0 \to T_X \xrightarrow{df} f^*T_Y \to 0$$

where T_X is placed in degree 0 and f^*T_Y is placed in degree 1. It defines an object $T_f \in \operatorname{Ob} \mathbf{D}^b(X)$. A *relative Hermitian structure on* f is the choice of an object $\overline{T}_f \in \overline{\mathbf{D}}^b(X)$ over T_f.

The following particular situations are of special interest:

- Suppose $f: X \hookrightarrow Y$ is a closed immersion. Let $N_{X/Y}[-1]$ be the normal bundle to X in Y, considered as a complex concentrated in degree 1. By definition, there is a natural quasi-isomorphism $p: T_f \xrightarrow{\sim} N_{X/Y}[-1]$ in $\mathbf{C}^b(X)$, and hence an isomorphism $p^{-1}: N_{X/Y}[-1] \overset{\sim}{\dashrightarrow} T_f$ in $\mathbf{D}^b(X)$. Therefore, a Hermitian metric h on the vector bundle $N_{X/Y}$ naturally induces a Hermitian structure $p^{-1}: (N_{X/Y}[-1], h) \dashrightarrow T_f$ on T_f. Let $\overline{T}_f$ be the corresponding object in $\overline{\mathbf{D}}^b(X)$. Then we have
$$\mathrm{Td}(\overline{T}_f) = \mathrm{Td}\big(N_{X/Y}[-1], h\big) = \mathrm{Td}(N_{X/Y}, h)^{-1};$$
- Suppose $f: X \to Y$ is a smooth morphism. Let $T_{X/Y}$ be the relative tangent bundle on X, considered as a complex concentrated in degree 0. By definition, there is a natural quasi-isomorphism $\iota: T_{X/Y} \xrightarrow{\sim} T_f$ in $\mathbf{C}^b(X)$. Any choice of Hermitian metric h on $T_{X/Y}$ naturally induces a Hermitian structure $\iota: (T_{X/Y}, h) \dashrightarrow T_f$. If $\overline{T}_f$ denotes the corresponding object in $\overline{\mathbf{D}}^b(X)$, then we find
$$\mathrm{Td}(\overline{T}_f) = \mathrm{Td}(T_{X/Y}, h).$$

Let now $g: Y \to Z$ be another morphism of smooth varieties over $\mathbb{C}$. The tangent complexes T_f, T_g and $T_{g\circ f}$ fit into a distinguished triangle in $\mathbf{D}^b(X)$

$$\mathcal{T}: T_f \dashrightarrow T_{g\circ f} \dashrightarrow f^*T_g \dashrightarrow T_f[1].$$

Definition 5.7. We denote $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ the following data:

(i) The class $\operatorname{Ob} \overline{\mathbf{Sm}}_{*/\mathbb{C}}$ of smooth complex varieties.

(ii) For each $X, Y \in \operatorname{Ob} \overline{\mathbf{Sm}}_{*/\mathbb{C}}$, a set of morphisms $\overline{\mathbf{Sm}}_{*/\mathbb{C}}(X, Y)$ whose elements are pairs $\overline{f} = (f, \overline{T}_f)$, where $f: X \to Y$ is a projective morphism and $\overline{T}_f$ is a Hermitian structure on T_f. When $\overline{f}$ is given we will denote the Hermitian structure by $T_{\overline{f}}$. A Hermitian structure on T_f will also be called a Hermitian structure on f.

(iii) For each pair of morphisms $\overline{f}: X \to Y$ and $\overline{g}: Y \to Z$, the composition defined as
$$\overline{g} \circ \overline{f} = \big(g \circ f, \overline{\mathrm{cone}}\big(f^*T_{\overline{g}}[-1], T_{\overline{f}}\big)\big).$$

We shall prove (Theorem 5.11) that $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ is a category. Before this, we proceed with some examples emphasizing some properties of the composition rule.

Example 5.8. Let $f: X \to Y$ and $g: Y \to Z$, be projective morphisms of smooth complex varieties. Assume that we have chosen Hermitian metrics on the tangent vector bundles T_X, T_Y and T_Z. Denote by $\overline{f}$, $\overline{g}$ and $\overline{g \circ f}$ the morphism of $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ determined by these metrics. Then

$$\overline{g} \circ \overline{f} = \overline{g \circ f}.$$

This is seen as follows. By the choice of metrics, there is a tight isomorphism

$$\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}}) \to f^*T_{\overline{g}}.$$

Then the natural maps

$$T_{\overline{g}\circ\overline{f}} \to \overline{\mathrm{cone}}\big(f^*T_{\overline{g}}[-1], T_{\overline{f}}\big) \to \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1], T_{\overline{f}}\big) \to T_{\overline{g\circ f}}$$

are tight isomorphisms.

Example 5.9. Let $f: X \to Y$ and $g: Y \to Z$ be smooth projective morphisms of smooth complex varieties. Choose Hermitian metrics on the relative tangent vector bundles T_f, T_g and $T_{g\circ f}$. Denote by $\overline{f}$, $\overline{g}$ and $\overline{g\circ f}$ the morphism of $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ determined by these metrics. There is a short exact sequence of Hermitian vector bundles

$$\bar{\varepsilon}: 0 \to \overline{T}_f \to \overline{T}_{g\circ f} \to f^*\overline{T}_g \to 0,$$

that we consider as an acyclic complex declaring $f^*\overline{T}_g$ of degree 0. The morphism $f^*T_{\overline{g}}[-1] \dashrightarrow T_{\overline{f}}$ is represented by the diagram

$$\begin{array}{ccc} & \overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1] & \\ \overset{\sim}{\swarrow} & & \searrow \\ f^*T_{\overline{g}}[-1] & & T_{\overline{f}}. \end{array}$$

Thus, by the definition of a composition we have

$$T_{\overline{g}\circ\overline{f}} = \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1], f^*T_{\overline{g}}[-1]\big)[1] \oplus \overline{\mathrm{cone}}\big(\overline{\mathrm{cone}}(T_{\overline{f}}, T_{\overline{g\circ f}})[-1], T_{\overline{f}}\big).$$

In general this Hermitian structure is different to $T_{\overline{g\circ f}}$.

Claim. *The equality of Hermitian structures*

$$T_{\overline{g}\circ\overline{f}} = T_{\overline{g\circ f}} + [\bar{\varepsilon}] \tag{5.10}$$

holds.

Proof. We have a commutative diagram of distinguished triangles

$$\begin{array}{ccccccccc} \bar{\varepsilon} & & T_{\overline{f}} & \longrightarrow & T_{\overline{g\circ f}} & \longrightarrow & f^*T_{\overline{g}} & \dashrightarrow & T_{\overline{f}}[1] \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \bar{\tau} & & T_{\overline{f}} & \longrightarrow & T_{\overline{g}\circ\overline{f}} & \longrightarrow & f^*T_{\overline{g}} & \dashrightarrow & T_{\overline{f}}[1]. \end{array}$$

By construction the triangle $\bar{\tau}$ is tightly distinguished, hence $[\bar{\tau}] = 0$. Therefore, according to Theorem 3.33(vii), we have

$$[T_{\overline{g\circ f}} \to T_{\overline{g}\circ\overline{f}}] = [\bar{\varepsilon}].$$

The claim follows. □

Note that the class $\widetilde{\mathrm{Td}}(\bar{\varepsilon})$ is the Bott–Chern secondary class introduced by Bismut, Gillet and Soulé in [6] and used by Ma in [24] when studying the analytic torsion of a composition of submersions.

Theorem 5.11. $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ *is a category.*

Proof. The only non-trivial fact to prove is the associativity of the composition, given by the following lemma:

Lemma 5.12. *Let $\overline{f}: X \to Y$, $\bar{g}: Y \to Z$ and $\bar{h}: Z \to W$ be projective morphisms together with Hermitian structures. Then $\bar{h} \circ (\bar{g} \circ \overline{f}) = (\bar{h} \circ \bar{g}) \circ \overline{f}$.*

Proof. First of all we observe that if the Hermitian structures on $\overline{f}$, $\overline{g}$ and $\bar{h}$ come from fixed Hermitian metrics on T_X, T_Y, T_Z and T_W, Example 5.8 ensures that the proposition holds. For the general case, it is enough to see that if the proposition holds for a fixed choice of Hermitian structures $\overline{f}$, $\overline{g}$, $\bar{h}$, and we change the metric on f, g or h, then the proposition holds for the new choice of metrics. We treat, for instance, the case when we change the Hermitian structure on g, the proof of the other cases being analogous. Denote by $\overline{g}'$ the new Hermitian structure on g. Then there exists a unique class $\varepsilon \in \overline{\mathbf{KA}}(Y)$ such that $T_{\overline{g}'} = T_{\overline{g}} + \varepsilon$. According to the definitions, we have

$$T_{\bar{h}\circ(\overline{g}'\circ\overline{f})} = \overline{\operatorname{cone}}\big((g\circ f)^* T_{\bar{h}}[-1], \overline{\operatorname{cone}}\big(f^*(T_{\overline{g}}+\varepsilon)[-1], T_{\overline{f}}\big)\big) = T_{\bar{h}\circ(\overline{g}\circ\overline{f})} + f^*\varepsilon.$$

Similarly, we find

$$T_{(\bar{h}\circ\overline{g}')\circ\overline{f}} = \overline{\operatorname{cone}}\big(f^*\overline{\operatorname{cone}}\big(g^*T_{\bar{h}}[-1], T_{\overline{g}}\big)[-1] + f^*(-\varepsilon), T_{\overline{f}}\big) = T_{(\bar{h}\circ\overline{g})\circ\overline{f}} + f^*\varepsilon.$$

By assumption, $T_{\bar{h}\circ(\overline{g}\circ\overline{f})} = T_{(\bar{h}\circ\overline{g})\circ\overline{f}}$. Hence the relations above show

$$T_{\bar{h}\circ(\overline{g}'\circ\overline{f})} = T_{(\bar{h}\circ\overline{g}')\circ\overline{f}}.$$

This concludes the proofs of Lemma 5.12 and of Theorem 5.11. □

Let $f: X \to Y$ and $g: Y \to Z$ be projective morphisms of smooth complex varieties. By the definition of composition, Hermitian structures on f and g determine a Hermitian structure on $g\circ f$. Conversely we have the following result.

Lemma 5.13. *Let $\overline{g}$ and $\overline{g\circ f}$ be Hermitian structures on g and $g\circ f$. Then there is a unique Hermitian structure $\overline{f}$ on f such that*

$$\overline{g\circ f} = \overline{g}\circ\overline{f}. \tag{5.14}$$

Proof. We have the distinguished triangle

$$T_f \dashrightarrow T_{g\circ f} \dashrightarrow f^*T_g \dashrightarrow T_f[1].$$

The unique Hermitian structure that satisfies Eq. (5.14) is $\overline{\operatorname{cone}}(T_{\overline{g\circ f}}, f^*T_{\overline{g}})[-1]$. □

Remark 5.15. By contrast with the preceding result, it is not true in general that Hermitian structures $\overline{f}$ and $\overline{g\circ f}$ determine a unique Hermitian structure $\overline{g}$ that satisfies Eq. (5.14). For instance, if $X = \emptyset$, then any Hermitian structure on g will satisfy this equation.

If $\mathbf{Sm}_{*/\mathbb{C}}$ denotes the category of smooth complex varieties and projective morphisms and $\mathfrak{F}: \overline{\mathbf{Sm}}_{*/\mathbb{C}} \to \mathbf{Sm}_{*/\mathbb{C}}$ is the forgetful functor, for any object X we have that

$$\operatorname{Ob}\mathfrak{F}^{-1}(X) = \{X\},$$
$$\operatorname{Hom}_{\mathfrak{F}^{-1}(X)}(X, X) = \overline{\mathbf{KA}}(X).$$

To any arrow $\overline{f}: X \to Y$ in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ we associate a Todd form

$$\operatorname{Td}(\overline{f}) := \operatorname{Td}(T_{\overline{f}}) \in \bigoplus_p \mathcal{D}^{2p}(X, p). \tag{5.16}$$

The following simple properties of $\operatorname{Td}(\overline{f})$ follow directly from the definitions.

Proposition 5.17.

(i) *Let $\overline{f}: X \to Y$ and $\overline{g}: Y \to Z$ be morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$. Then*

$$\operatorname{Td}(\overline{g}\circ\overline{f}) = f^*\operatorname{Td}(\overline{g}) \bullet \operatorname{Td}(\overline{f}).$$

(ii) *Let $f, f': X \to Y$ be two morphisms in $\overline{\mathbf{Sm}}_{*/\mathbb{C}}$ with the same underlying algebraic morphism. There is an isomorphism $\bar{\theta}: T_{\overline{f}} \to T_{\overline{f}'}$ whose Bott–Chern class $\widetilde{\operatorname{Td}}_m(\bar{\theta})$ satisfies*

$$\mathrm{d}_{\mathcal{D}}\widetilde{\operatorname{Td}}_m(\bar{\theta}) = \operatorname{Td}(T_{\overline{f}'})\operatorname{Td}(T_{\overline{f}})^{-1} - 1.$$

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