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Mixed lower bounds for quantum transport

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Abstract

Let H be a self-adjoint operator on a separable Hilbert space $\mathscr{H}$, $\psi \in \mathscr{H}, ||\psi|| = 1$. Given an orthonormal basis $\mathscr{B} = \{e_n\}$ of $\mathscr{H}$, we consider the time-averaged moments $\langle |X|_\psi^p \rangle (T)$ of the position operator associated to $\mathscr{B}$. We derive lower bounds for the moments in terms of both spectral measure μ_ψ and generalized eigenfunctions $u_\psi(n, x)$ of the state ψ. As a particular corollary, we generalize the recently obtained lower bound in terms of multifractal dimensions of μ_ψ and give some equivalent forms of it which can be useful in applications. We establish, in particular, the relations between the L^q-norms ($q > 1/2$) of the imaginary part of Borel transform of probability measures and the corresponding multifractal dimensions.

Keywords: Quantum transport; Lower bounds; Spectral measure; Generalized eigenfunctions; Multifractal dimensions

1. Introduction

Let H be a self-adjoint operator on a separable Hilbert space $\mathscr{H}$. Let $\psi \in \mathscr{H}, ||\psi|| = 1$. The time evolution of the state ψ is given by $\psi_t = \exp(-itH)\psi$. Consider an orthonormal basis $\mathscr{B} = \{e_n\}$ of $\mathscr{H}$ (in fact, one can also take any orthonormal basis of the cyclic subspace of ψ). The vectors e_n are labelled by $n \in \mathbf{N}$ or by $n \in \mathbf{Z}^d$ (in the specific case $\mathscr{H} = l^2(\mathbf{Z}^d)$ and $e_n = \delta_n$, where δ_n is the canonical basis of $l^2(\mathbf{Z}^d)$).

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We define the time-averaged moments of order p of abstract position operator (associated to the basis $\mathscr{B}$) as

$$\langle |X|_{\psi}^{p} \rangle (T) = \frac{1}{T} \int_{0}^{T} dt \sum_{n} (|n| + 1)^{p} |\langle \exp(-itH)\psi, e_n \rangle|^{2}.$$

These quantities describe the spreading of the wave packet over the basis $\mathscr{B}$. Many recent papers were devoted to the following problem: what is the relation between the transport properties of the operator H (represented by the behaviour of $\langle |X|_{\psi}^{p} \rangle (T), T \to +\infty)$ and its spectral properties (represented by the spectral measure μ_{ψ} and the generalized eigenfunctions $u_{\psi}(n, x)$ associated to the state ψ). The problem is to establish the links between the growth exponents

$$\alpha_{\psi}^{+}(p) = \limsup_{T \to +\infty} \frac{\log \langle |X|_{\psi}^{p} \rangle (T)}{\log T}, \quad \alpha_{\psi}^{-}(p) = \liminf_{T \to +\infty} \frac{\log \langle |X|_{\psi}^{p} \rangle (T)}{\log T}$$

and the spectral properties of H. Most general results obtained so far deal with the lower bounds for $\alpha_{\psi}^{\pm}(p)$. The results obtained in 1989–1999 can be summarized as follows: the lower bounds for $\alpha_{\psi}^{\pm}(p)$ are determined by "the most continuous" part of the spectral measure μ_{ψ}. Namely, it was shown [1,8,14,15,21] that

$$\alpha_{\psi}^{-}(p) \geqslant \frac{p}{d} \dim_{H}(\mu_{\psi}) \tag{1.1}$$

and [16] that

$$\alpha_{\psi}^{+}(p) \geqslant \frac{p}{d} \dim_{P}(\mu_{\psi}). \tag{1.2}$$

Here $d = 1$ for abstract basis labelled by $n \in \mathbf{N}$ and $d \geqslant 1$ in the special case $l^{2}(\mathbf{Z}^{d})$, and $\dim_{H}(\mu)$, $\dim_{P}(\mu)$ are the Hausdorff and packing dimension of the measure μ, respectively (for definitions, see Appendix A).

In [20] it was shown that bound (1.1) can be improved if one has some additional information about the decay of generalized eigenfunctions $u_{\psi}(n, x)$ as $|n| \to \infty$. One can then take smaller value of d (in the case of α-continuous μ_{ψ}).

As to general upper bounds for $\alpha_{\psi}^{\pm}(p)$, there are no results available (except from trivial ballistic upper bound in most cases where $\langle \psi, e_n \rangle$ is fast decaying in n). In available examples with nontrivial upper bounds [5,9] one uses rather the methods specific to the considered quantum system.

Bounds (1.1) and (1.2) clearly are not optimal in many cases. Let the measure μ_{ψ} be pure point, so that $\dim_{H}(\mu_{\psi}) = \dim_{P}(\mu_{\psi}) = 0$. While for many models with pure point spectrum one has dynamical localization (so that $\alpha_{\psi}^{\pm}(p) = 0$ for any $p > 0$), it is possible that $\alpha_{\psi}^{\pm}(p) > 0$. In the well-known "pathological" example of [10] one even has $\alpha_{\psi}^{+}(p) = p$ for $\psi = \delta_0$ for any $p > 0$ (quasiballistic behaviour on the sequence of times).

Further, bounds (1.1) and (1.2) are always linear in p. At the same time, numerical calculations show that for some quantum systems [9,22,23] $\alpha_\psi^\pm(p)$ grow faster than linearly (i.e. $\alpha_\psi^\pm(p)/p$ is growing with p). This phenomenon is called by physicists "quantum intermittency".

It was thus clear that one should find some new general lower bounds, improving (1.1) and (1.2). In particular, the nature of the spectrum (pure point or singular continuous) should not be so important as in (1.1) and (1.2). Some intermediate results were obtained in [6]. Probably, most important in [6] was the understanding that one should optimize the lower bound for a given time T rather than for all T simultaneously. The developement of the ideas of [6] allowed to achieve a significant breakthrough [2,3], where the new lower bounds were obtained (similar result was simultaneously obtained in [11] in the special case $l^2(\mathbf{N})$ by different methods and under more restrictive assumptions). Namely, under some assumptions on μ_ψ (in particular, for any μ_ψ with compact support),

$$\alpha_\psi^\pm(p) \geqslant \frac{p}{d} D_{\mu_\psi}^\pm(1/(1+p/d)). \tag{1.3}$$

Here $D_\mu^\pm(q)$ are the nonincreasing functions called multifractal dimensions (or generalized fractal dimensions) defined for any Borel probability measure μ for $q \neq 1$ as follows:

$$D_\mu^+(q) = \limsup_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon}, \quad D_\mu^-(q) = \liminf_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log\varepsilon}, \tag{1.4}$$

where

$$I_\mu(q,\varepsilon) = \int_{\mathbf{R}} (\mu([x-\varepsilon, x+\varepsilon]))^{q-1}\, d\mu(x).$$

It was pointed out in [3] that one cannot expect better general bound of this kind (for example, (1.3) with $\frac{p}{d} D^\pm(1-p/d)$ on the r.h.s.).

The bound like (1.3) was expected to hold, because the multifractal dimensions earlier appeared in upper bounds for dynamics for Julia matrices [5] and in heuristic form in [22,23]. The importance of this theoretical result follows from three observations:

1. For any $q \in (0,1)$ one has [3] $D_\mu^-(q) \geqslant \dim_H(\mu)$, $D_\mu^+(q) \geqslant \dim_P(\mu)$, so that (1.3) implies (1.1) and (1.2). At the same time it is possible that $\lim_{q\to 1, q<1} D_\mu^-(q) > \dim_H(\mu)$, and similarly for D^+ and $\dim_P(\mu)$.
2. There exist pure point measures whose dimensions are positive for some or even all $q \in (0,1)$.
3. If $D_\mu^-(q)$ or $D_\mu^+(q)$ is nonconstant on $(0,1)$ (i.e. strictly decaying), then the corresponding lower bound (1.3) is nonlinear in p.

At the moment when [3] was written, bounds (1.3) were rather of theoretical interest. Now, 3 years later, the things are different. There are actually some examples [12,27] where one obtains nontrivial lower bounds using (1.3) in one of its equivalent forms (our Theorem 4.3).

The first is the Schrödinger operator with barrier sparse potentials (this model was suggested in [19]), where one gets [27] nonlinear lower bound for $\alpha_\psi^-(p)$. Together with the upper bound of Combes–Mantica [9], it gives the first example of quantum system where the phenomenon of quantum intermittency is rigorously proved. Next, in [12] one considers Schrödinger operators with random slowly decaying potential. In the case of pure point spectrum, one obtains a quasiballistic lower bound for the moments of high order. One shows also in [12] for the famous "pathological" example of [10] that $D_\mu^+(q) = 1$ for any $q \in (0,1)$, and thus bound (1.3) yields the quasiballistic behaviour for the moments of any order $p > 0$ on some sequence of times (the result proved by different methods in [10]). All these results in preparation cannot be obtained using "classical" lower bounds (1.1) and (1.2).

What has also changed in last years, is the better understanding of multifractal dimensions $D_\mu^\pm(q)$. In particular, the basic properties and many equivalent definitions of $D_\mu^\pm(q)$ were established in [4] (see also Theorem 4.3), which appeared very useful in applications.

In the present paper we go further and obtain lower bounds for the time-averaged moments which take into account not only the properties of the spectral measure μ_ψ but also of the generalized eigenfunctions $u_\psi(n,x)$ associated to ψ. For this reason we call them mixed lower bounds.

The paper is organized as follows. In Section 2 we show our basic lower bound (Theorem 2.2) which takes into account both the spectral measure and generalized eigenfunctions:

$$\langle |X|_\psi^p \rangle (T) \geqslant C(r,p) \int_{\mathbf{R}} d\mu_\psi(x)(N(x,T)+1)^r,$$

where $0 < r < p$,

$$N(x,T) = \sup\{N > 0 \mid b(x,T) S_N(x) \leqslant 1/16\},$$

$$S_N(x) = \sum_{|n| \leqslant N} |u_\psi(n,x)|^2, \quad b(x,T) = \int_{\mathbf{R}} d\mu_\psi(y) R(T(x-y)),$$

and $R(u)$ is some fast decaying function such that $R(u) = 1$ for any $u : |u| \leqslant 1$ (one should think to $b(x,T)$ as $\mu_\psi([x - 1/T, x + 1/T])$).

In Section 3 we derive some simplified versions of this general lower bound, where the functions $S_N(x)$ do not appear explicitly. First, for any $s > 0$ let $U_s(x) = \sup_{N>0}((N+1)^{-s} S_N(x))$, so that $S_N(x) \leqslant U_s(x)(N+1)^s$ (it is possible that $U_s(x) = +\infty$). Then Theorem 3.1 states that

$$\langle |X|_\psi^p \rangle (T) \geqslant C(r,p,\delta) \int_{\mathbf{R}} d\mu(x)(b(x,T) U_{s(x)}(x))^{-r/s(x)}, \tag{1.5}$$

where $0<r<p$ and $s(x)$ is any Borel positive function such that $s(x)\geqslant\delta>0$. Bound (1.5) can be considered (to some extent) as the generalization of the result of Kiselev and Last [20] established for α-continuous spectral measures.

Next, we improve the bounds of [2,3]. We show that whatever the spectral measure μ_ψ is, for any $T>0$ the following bound holds:

$$\langle |X|_\psi^p\rangle(T)\geqslant C(p,q)(I_{\mu_\psi}(q,1/T))^{1/q}, \tag{1.6}$$

where $q>1/(1+p/s_0)$ and s_0 is some positive number depending on $\psi,\mathscr{B}$. (One always has $s_0\leqslant d$, so that (1.6) holds for any $q>1/(1+p/d)$.) Bounds (1.6) are obtained as a direct corollary of (1.5). The proof of (1.6) we give is simpler rather than the technical proof of [3].

In Section 4 we recall definition (1.4) of multifractal dimensions of Borel measures and establish the lower bounds (1.3) without any assumptions on the measure μ_ψ. We give also in Theorem 4.3 many quantities equivalent to $I_\mu(q,\varepsilon)$ which give rise to the same multifractal dimensions $D_\mu^\pm(q)$. In particular, for $q>1/2$ one can take instead of I_μ the integrals

$$\varepsilon^{q-1}\int_{\mathbf{R}} dx(\operatorname{Im} F_\mu(x+i\varepsilon))^q,$$

where dx is Lebesgue measure and $F_\mu(z)$ is the Borel transform of μ.

If m is some integer, then for any $q>1/(2m)$ one can take

$$\varepsilon^{2qm-1}\int_{\mathbf{R}} dx||R^m(x+i\varepsilon)\psi||^{2q},$$

where $R^m(z)=(H-z)^{-m}$ and the measure μ is the spectral measure associated to the state ψ and self-adjoint operator H. As an interesting corollary of Theorem 4.3 we show in Theorem 4.4 that the lower bound

$$\mu([x-\varepsilon,x+\varepsilon])\geqslant C(x)\varepsilon^\gamma,\quad \gamma\geqslant 1,\ C(x)>0,\ x\in A$$

uniform in $\varepsilon\in(0,1)$ on the set A of positive Lebesgue measure yields nontrivial dynamical information about the moments of order p high enough. Such a possibility was never investigated before.

We establish also some related (but more general) result concerning the behaviour of $D_\mu^\pm(q)$ as $q\to 0$. Namely, assume that there exist two finite positive constants C,A such that

$$\mu([x-\varepsilon,x+\varepsilon])\geqslant C\varepsilon^A \text{ for all } x\in\operatorname{supp}\mu,\ \varepsilon\in(0,1). \tag{1.7}$$

Then Theorem 4.5 yields

$$\lim_{q\to 0+0} D_\mu^\pm(q)=\dim_B^\pm(\operatorname{supp}\mu),$$

where $\dim_B^{\pm}$ are the box-counting dimensions of the support of the measure. Thus, under condition (1.7), these quantities give information about the behaviour of the moments of high order p.

In Appendix A we discuss relations between the multifractal dimensions and the Hausdorff and packing dimensions of finite Borel measures. In Theorem A.1 we generalize some result of [10] and give the lower bounds for the lower Hausdorff and packing dimensions in terms of the L^q-norms ($q>1$) of the imaginary part of the Borel transform of the measure.

In Appendix B we consider the lower bounds of Sections 2 and 3 in the case of pure point spectrum and discuss their relation with the problem of dynamical localization.

Finally, in Appendix C we derive from Theorem 2.2 some lower bounds in the case of uniformly α-Hölder continuous measures. These bounds show that the nonlinear behaviour of $\alpha_\psi^{\pm}(p)$ may come from the generalized eigenfunctions even if the multifractal dimensions are constant.

2. General mixed lower bounds

Let H be a self-adjoint operator on separable Hilbert space $\mathscr{H}$ and ψ some vector from $\mathscr{H}$ with $||\psi|| = 1$. We denote by $\mathscr{H}_\psi$ the cyclic subspace spanned by H and ψ and by P_ψ the orthogonal projection on $\mathscr{H}_\psi$. Let $\mathscr{B} = \{e_n\}$ be any orthonormal system in $\mathscr{H}$ such that $\mathscr{H}_\psi \subset \mathscr{L}(\mathscr{B})$, where $\mathscr{L}(\mathscr{B})$ is the subspace spanned by $\mathscr{B}$. For example, one can take as $\mathscr{B}$ any orthonormal basis of $\mathscr{H}$. Remark that $||\phi||^2 = \sum_n |\langle \phi, e_n \rangle|^2$ for any $\phi \in \mathscr{H}_\psi$. Most of the time, we shall assume that the system $\mathscr{B}$ is labelled by $n \in \mathbf{N}$. However, in the particular case $\mathscr{H} = l^2(\mathbf{Z}^d)$ where $\mathscr{B} = \{\delta_n\}$, $n \in \mathbf{Z}^d$, is the canonical basis of $l^2(\mathbf{Z}^d)$, we shall use the labelling $n \in \mathbf{Z}^d$. In the first case we shall define the dimension of the orthonormal system $\mathscr{B}$ equal to 1, and in the second case equal to d. The moments of the abstract position operator associated to the vector ψ and the system $\mathscr{B}$ are defined by

$$|X|_\psi^p(t) = \sum_n (|n|+1)^p |\langle \exp(-itH)\psi, e_n \rangle|^2, \quad p>0,$$

where $|n|$ is the $\mathbf{Z}^d$-norm of vector n in the case of $\mathbf{Z}^d$-labelling and the summation is carried over $\mathbf{N}$ or $\mathbf{Z}^d$. We shall be interested by the lower bounds for the Cesaro averages of the position operator. At the beginning we follow the same strategy as in [3]. Let h be some positive function from $C_0^\infty([0,1])$ such that $\int_0^1 h(z)\,dz = 1$. The constants in the estimates we shall obtain will depend on h. As the function h is fixed, we omit it in notations. For any $z \in [0,1]$ we have $h(z) \leqslant ||h||_\infty$, so for $T>0$

$$\langle |X|_\psi^p \rangle(T) \geqslant \frac{1}{||h||_\infty} \frac{1}{T} \int_0^{+\infty} |X|_\psi^p(t) h(t/T)\,dt. \tag{2.1}$$

For any couple of vectors ψ, ϕ and any $N>0$ define

$$D_{\psi,\phi}(T,N) = \frac{1}{T}\int_0^{+\infty} \sum_{|n|\leqslant N} \langle \exp(-itH)\psi, e_n\rangle \overline{\langle \exp(-itH)\phi, e_n\rangle} h(t/T)\,dt.$$

In particular, if $\phi = \psi$, consider

$$B_\phi(T,N) \equiv D_{\phi,\phi}(T,N) = \frac{1}{T}\int_0^{+\infty} \sum_{|n|\leqslant N} |\langle \exp(-itH)\phi, e_n\rangle|^2 h(t/T)\,dt$$

Clearly,

$$\sum_n (|n|+1)^p |\langle \exp(-itH)\psi, e_n\rangle|^2 \geqslant (N+1)^p \sum_{|n|>N} |\langle \exp(-itH)\psi, e_n\rangle|^2 \tag{2.2}$$

for any $t>0, N>0$. Since $\int_0^\infty h(z)\,dz = 1$ and $||\exp(-itH)\psi|| = ||\psi|| = 1$, bounds (2.1) and (2.2) yield

$$\langle |X|_\psi^p\rangle(T) \geqslant C(N+1)^p(1 - B_\psi(T,N)). \tag{2.3}$$

Let us take $\phi = P_\Omega(H)\psi$ where $\Omega \subset \mathbf{R}$ is some Borel set and $P_\Omega(H)$ is the corresponding spectral projector of operator H. Obviously, $\psi = \phi + \chi$, where $\phi, \chi \in \mathscr{H}_\psi$ and $\langle \phi, \chi\rangle = 0$. It is easy to see that

$$\begin{aligned} B_\psi(T,N) &= B_\phi(T,N) + B_\chi(T,N) + 2\,\mathrm{Re}\, D_{\chi,\phi}(T,N) \\ &= -\, B_\phi(T,N) + B_\chi(T,N) + 2\,\mathrm{Re}\, D_{\psi,\phi}(T,N) \\ &\leqslant B_\chi(T,N) + 2\mathrm{Re} D_{\psi,\phi}(T,N). \end{aligned} \tag{2.4}$$

As $1/T \int_0^{+\infty} h(t/T)dt = 1$ and $h(z) \geqslant 0$,

$$B_\chi(T,N) \leqslant ||\chi||^2 = 1 - ||\phi||^2. \tag{2.5}$$

Finally, inequalities (2.3)–(2.5) give the following lower bound for the time-averaged moments:

$$\begin{aligned} \langle |X|_\psi^p\rangle(T) &\geqslant C(N+1)^p(||\phi||^2 - 2\,\mathrm{Re}\, D_{\psi,\phi}(T,N)) \\ &\geqslant C(N+1)^p(||\phi||^2 - 2|D_{\psi,\phi}(T,N)|). \end{aligned} \tag{2.6}$$

Suppose that T, N are such that

$$|D_{\psi,\phi}(T,N)| \leqslant ||\phi||^2/4. \tag{2.7}$$

Then (2.6) yields

$$\langle |X|_{\psi}^{p} \rangle(T) \geq C(N+1)^{p}||\phi||^{2}/2 \tag{2.8}$$

with some positive constant C. To obtain a good lower bound for the Cesaro averaged moments, one should first estimate from above $D_{\psi,\phi}(T,N)$ and then optimize bound (2.8) choosing ϕ, N depending on ψ and T.

Some estimate of $D_{\psi,\phi}(T,N)$ was obtained in [3, Theorem 3.2]. This result, however, will not be sufficient for our purposes. To derive a better bound for $D_{\psi,\phi}(T,N)$, we shall use again the standard spectral Theorem for a self-adjoint operator H. Namely, there exist a Borel measure μ_{ψ} (spectral measure of vector ψ) and a unitary map W_{ψ} from $\mathscr{H}_{\psi}$ to $L^{2}(\mathbf{R}, d\mu_{\psi})$ such that $(W_{\psi}(g(H)\psi))(x) = g(x)$ for any Borel bounded function g. In particular, $(W_{\psi}(\exp(-itH)\psi))(x) = \exp(-itx)$ and for any Borel set $\Omega \subset \mathbf{R}$,

$$(W_{\psi}(\phi))(x) = \chi_{\Omega}(x), \quad \phi = P_{\Omega}\psi, \tag{2.9}$$

where $\chi_{\Omega}(x)$ is the characteristic function of the set Ω. Recall that we denote by P_{ψ} the orthogonal projection on $\mathscr{H}_{\psi}$. For any n define

$$u_{\psi}(n,x) = (W_{\psi}(P_{\psi}e_{n}))(x). \tag{2.10}$$

Later on in this section, the state ψ is fixed and we shall omit it in notations (so that $\mu \equiv \mu_{\psi}$ and $u(n,x) \equiv u_{\psi}(n,x)$). Since for a fixed n each function $u(n,x)$ is defined μ-everywhere, the sequence $\{u(n,x)\}$ is well defined for μ-a.e. $x \in \mathbf{R}$. One calls $u(n,x)$ generalized eigenfunctions of H corresponding to the vector ψ and the energy x in the representation $\mathscr{B} = \{e_{n}\}$. Two choices of $\mathscr{B}$ and thus of $u(n,x)$ are of particular interest.

1. In the case $\mathscr{H} = l^{2}(\mathbf{Z}^{d})$, $H = -\Delta + Q(n)$, where $-\Delta f(n) = \sum_{m:|m-n|=1} f(m)$, one takes as $\mathscr{B}$ the canonical basis of $l^{2}(\mathbf{Z}^{d})$. Then for a fixed x the function $u(n,x)$ is a solution to the generalized eigenvalue equation

$$-\Delta u(n,x) + Q(n)u(n,x) = xu(n,x). \tag{2.11}$$

One should make here an important observation. Typically, the eigenvalue equation (2.11) has many linearly independent solutions, some growing as $|n| \to \infty$, and moreover any solution can be multiplied by some constant $K(x)$. On the other hand, given ψ, the functions $u(n,x)$ are uniquely defined due to (2.10). Therefore, for μ-a.e.x, $u(n,x)$ is some solution to (2.11) depending on ψ but not any solution. In particular, since

$$||u(n,\cdot)||_{L^{2}(\mathbf{R},d\mu)} = ||P_{\psi}e_{n}|| \leq 1 \tag{2.12}$$

for any n, one can easily show, integrating with μ, that

$$\sum_n (|n|+1)^{-d}(\ln^2|n|+1)^{-1}|u(n,x)|^2 < +\infty$$

for μ-a.e. x. Therefore, $u(n,x)$ cannot have power growth as $|n| \to \infty$.

2. Suppose that the linear combinations of $\{H^k\psi\}$ are dense in $\mathscr{H}_\psi$ (this is always true if H is bounded). Then one can construct the basis $\mathscr{B}$ of $\mathscr{H}_\psi$ by orthonormalization of $\{H^k\psi\}$. This basis consists of vectors $e_n = W_\psi^{-1}(R_n(x))$, $n \in \mathbf{N}$, where $R_n(x) = u(n,x)$ are the orthogonal polynomials of the spectral measure μ. The restriction of H on $\mathscr{H}_\psi$ has a tridiagonal matrix representation in this basis.

Let us continue the estimation of $D(T,N)$. As the map W_ψ is unitary, we have

$$\langle \exp(-itH)\psi, e_n \rangle = \langle \exp(-itH)\psi, P_\psi e_n \rangle$$

$$= \langle W_\psi(\exp(-itH)\psi, W_\psi(P_\psi e_n) \rangle_{L^2(\mathbf{R},d\mu)} = \int_{\mathbf{R}} d\mu(x) e^{-itx}\overline{u(n,x)} \qquad (2.13)$$

and in the similar manner using (2.9),

$$\langle \exp(-itH)\phi, e_n \rangle = \int_\Omega d\mu(x) e^{-itx}\overline{u(n,x)}, \qquad (2.14)$$

It follows directly from definition of $D_{\psi,\phi}(T,N)$ and (2.13)–(2.14) that

$$D_{\psi,\phi}(T,N) = \int_\Omega \int_{\mathbf{R}} d\mu(x)\, d\mu(y) \hat{h}(T(x-y)) S_N(x,y), \qquad (2.15)$$

where $\hat{h}$ is the Fourier transform of h and

$$S_N(x,y) = \sum_{|n| \leqslant N} \overline{u(n,x)} u(n,y).$$

Consider the following positive function on $\mathbf{R}$:

$$R(u) = 1, \ |u| \leqslant 1 \quad \text{and } R(u) = \sup_{z:|z| \geqslant |u|} |\hat{h}(z)|^2, \quad |u| > 1.$$

Since $h \in C_0^\infty([0,1])$, the function R is bounded, even, monotonous in $|u|$ and fast decaying at infinity. Since $|\hat{h}(u)| \leqslant 1$,

$$|\hat{h}(u)|^2 \leqslant R(u) \leqslant 1 \qquad (2.16)$$

for any $u \in \mathbf{R}$. Define the function

$$b(x,T) = \int_{\mathbf{R}} d\mu(y) R(T(x-y)), \quad x \in \mathbf{R}, \ T > 0. \qquad (2.17)$$

It is clear from definition of R that $b(x,T)$ is decaying in T and $\mu([x-1/T, x+1/T]) \leqslant b(x,T) \leqslant 1$ for all x, T.

Lemma 2.1. *Let*

$$S_N(x) = \sum_{|n| \leqslant N} |u(n,x)|^2.$$

For any $\phi = P_\Omega \psi$ the following estimate holds:

$$|D_{\psi,\phi}(T,N)| \leqslant \int_\Omega d\mu(x) \sqrt{b(x,T) S_N(x)}. \tag{2.18}$$

Proof. Applying the Cauchy–Schwartz inequality to the integral over y for a fixed x and using (2.16), we obtain:

$$|D_{\psi,\phi}(T,N)| \leqslant \int_\Omega d\mu(x) \sqrt{b(x,T) G_N(x)}, \tag{2.19}$$

where

$$G_N(x) = \int_{\mathbf{R}} d\mu(y) |S_N(x,y)|^2.$$

For any $x \in \mathbf{R}, N > 0$ consider the following vector in $\mathscr{H}$:

$$f(x,N) = \sum_{|n| \leqslant N} \overline{u(n,x)} e_n.$$

Let $g(y) = W_\psi(P_\psi f(x,N))(y)$. One can easily check that

$$g(y) = \sum_{|n| \leqslant N} \overline{u(n,x)} W_\psi(P_\psi e_n)(y) = \sum_{|n| \leqslant N} \overline{u(n,x)} u(n,y) = S_N(x,y).$$

Therefore,

$$G_N(x) = ||g||^2_{L^2(\mathbf{R},d\mu)} = ||P_\psi f(x,N)||^2 \leqslant ||f(x,N)||^2. \tag{2.20}$$

The system $\{e_n\}$ being orthonormal in $\mathscr{H}$,

$$||f(x,N)||^2 = \sum_{|n| \leqslant N} |u(n,x)|^2 \equiv S_N(x). \tag{2.21}$$

The result of the lemma follows from (2.19)–(2.21). $\square$

We can obtain now the basic lower bound for the moments.

Theorem 2.2. *For any $T, N > 0$ consider the set*

$$\Omega(T,N) = \{x \mid b(x,T)S_N(x) \leqslant 1/16\}.$$

For any $x \in \mathbf{R}$, $T > 0$ define the numbers

$$N(x,T) = \sup\{N > 0 \mid x \in \Omega(T,N)\}.$$

We set $N(x,T) = 0$ if $x \notin \Omega(T,N)$ for all $N > 0$.

For all $p > 0$, $0 < r < p$ the following estimates hold with positive constants $C(r,p)$:

$$\langle |X|_\psi^p \rangle(T) \geqslant C(r,p) \int_0^{+\infty} dN (N+1)^{r-1} \mu(\Omega(T,N)), \tag{2.22}$$

$$\langle |X|_\psi^p \rangle(T) \geqslant C(r,p) \int_{\mathbf{R}} d\mu(x) (N(x,T)+1)^r. \tag{2.23}$$

Proof. One first proves that

$$\langle |X|_\psi^p(T) \rangle \geqslant C(N+1)^p \mu(\Omega(T,N)) \tag{2.24}$$

for any $N > 0, T > 0$. If the set $\Omega(T,N)$ is of measure 0 (in particular, this is the case if $\Omega(T,N)$ is empty), the inequality is trivially true. If $\mu(\Omega(T,N)) > 0$, then consider $\phi = \chi_{\Omega(T,N)}\psi$, $||\phi||^2 = \mu(\Omega(T,N))$. The definition of $\Omega(T,N)$ and Lemma 2.1 yield

$$|D_{\psi,\phi}(T,N)| \leqslant 1/4\mu(\Omega(T,N)) = 1/4||\phi||^2.$$

Bound (2.24) then follows directly from (2.7)–(2.8). Define now the function

$$L_p(T) = \sup_{N>0} ((N+1)^p \mu(\Omega(T,N))).$$

It follows from (2.24) that

$$\langle |X|_\psi^p \rangle(T) \geqslant C L_p(T) \tag{2.25}$$

with $C > 0$ uniform in p, T. From definition of $L_p(T)$ we have

$$\mu(\Omega(T,N)) \leqslant (N+1)^{-p} L_p(T)$$

for any $N > 0$. Therefore, if $r < p$,

$$\int_0^{+\infty} dN (N+1)^{r-1} \mu(\Omega(T,N)) \leqslant L_p(T) \int_0^{+\infty} dN (N+1)^{r-p-1} = C(p-r) L_p(T), \tag{2.26}$$

where $C(p-r) < +\infty$. The first bound of the theorem follows from (2.25)–(2.26).

To prove the second bound, we write the integral in (2.22) as

$$I_r(T) = \int_{\mathbf{R}} d\mu(x) \int_0^{+\infty} dN(N+1)^{r-1}\chi_{\Omega(T,N)}(x).$$

Since $S_N(x)$ are growing with N, it is clear from the definition of $N(x,T)$ that $x \in \Omega(T,N)$ for all $N \in [0, N(x,T))$. Therefore,

$$\begin{aligned} I_r(T) &\geqslant \int_{\mathbf{R}} d\mu(x) \int_0^{N(x,T)} dN(N+1)^{r-1} \\ &= C(r) \int_{\mathbf{R}} d\mu(x)((N(x,T)+1)^r - 1) \\ &= C(r)\left(\int_{\mathbf{R}} d\mu(x)(N(x,T)+1)^r - 1\right). \end{aligned} \tag{2.27}$$

Since $\langle |X|^p_\psi \rangle(T) \geqslant 1$, bound (2.23) follows from (2.22) and (2.27). □

Remark 2.1. One can obtain slightly better bounds taking $(N+1)^{p-1}(\log(N+2))^{-1-\delta}, \delta > 0$, instead of $(N+1)^{r-1}$ in (2.26).

Remark 2.2. It follows from the results of [7,20] that the spectral measure and the generalized eigenfunctions are not completely independent. Therefore, there is also some relation between $b(x,T)$ and $S_N(x)$.

Remark 2.3. One can give the following (not rigorous) interpretation of the numbers $N(x,T)$. Consider the part of the wave packet ψ with the energy x. Then for $t \in [0,T]$ this part of the wave packet spends at least half a time outside the ball of radius $N(x,T)$.

Remark 2.4. The proof can be adapted to obtain the lower bounds for more general quantities like

$$\frac{1}{T}\int_0^T dt \sum_n f(|n|)|\langle \exp(-itH)\psi, e_n \rangle|^2,$$

where $f(z)$ is some growing function such that $\lim_{z\to\infty} f(z) = +\infty$. The particular choices of interest different from $f(z) = (z+1)^p$ are $f(z) = \log^p z$ or $f(z) = \exp(pz)$, where $p > 0$.

3. Simplified lower bounds

Let $u(n,x)$ be the functions defined for a given vector ψ and an orthonormal family $\mathscr{B}$ in the previous section: $u(n,x) = (W_\psi(P_\psi e_n))(x)$, and $S_N(x) = \sum_{|n|\leqslant N} |u(n,x)|^2$. For any $s>0$ define two functions

$$U_s(x) = \sup_{N>0} ((N+1)^{-s} S_N(x)), \quad Y_s(x) = \sum_n (|n|+1)^{-s} |u(n,x)|^2,$$

where it is possible that $U_s(x) = +\infty$ and $Y_s(x) = +\infty$ for some (or even all) $x \in \mathbf{R}$. The functions $U_s(x)$, $Y_s(x)$ are finite for some $s>0$ if the generalized eigenfunctions $u(n,x)$ have sufficiently fast decay at infinity (it was pointed out in the previous section that they cannot grow faster than logarithmically). It is clear that

$$S_N(x) \leqslant (N+1)^s U_s(x) \tag{3.1}$$

for any N, x. One can easily see that

$$Y_s(x) \geqslant \sum_{|n|\leqslant N} (|n|+1)^{-s} |u(n,x)|^2 \geqslant (N+1)^{-s} S_N(x)$$

for any $N>0$, so that

$$U_s(x) \leqslant Y_s(x). \tag{3.2}$$

It is also straightforward to show that $Y_s(x) \leqslant C(\delta) U_{s-\delta}(x)$ for any $\delta>0$. Therefore, considering $Y_s(x)$ or $U_s(x)$ is virtually equivalent.

Since

$$\int_{\mathbf{R}} d\mu(x) |u(n,x)|^2 = ||P_\psi e_n||^2 \leqslant 1$$

for all n, we get for any $s>d$

$$\int_{\mathbf{R}} d\mu(x) Y_s(x) = \sum_n (|n|+1)^{-s} ||P_\psi e_n||^2 \leqslant \sum_n (|n|+1)^{-s} < +\infty. \tag{3.3}$$

We see that if $s>d$, then $Y_s(x) < +\infty$ for a.e. x. As the growth of $S_N(x)$ as $N \to \infty$ (determined by the rate of decay of $u(n,x)$) may depend on x, it will be convenient to take s depending on x. Assume that $s(x)$ is some positive Borel function such that $s(x) \geqslant \delta > 0$ for μ-a.e. x. In principle, it is reasonable to define for any x

$$s^*(x) = \inf\{s>0 \mid Y_s(x) < +\infty\}$$

and to take $s(x)$ slightly bigger than $s^*(x)$. But one can also take as $s(x)$ any value, even such that $Y_{s(x)}(x) = +\infty$. One particular choice is to take $s(x)$ constant: $s(x) = a$, $a>0$. If $a>d$, then one is sure that $Y_a(x) < +\infty$ for μ-a.e. x. It is possible, however, that for some $a<d$ the function $Y_a(x)$ is also finite, at least on some set of positive measure. In this case the bound of the theorem below remains nontrivial.

Theorem 3.1. *Let* $p>0$ *and the function* $b(x,T)$ *defined by* (2.17). *Let* $s(x)$ *be the positive function defined above. For any* $0<r<p$ *the uniform in* T *estimate holds*:

$$\langle |X|_\psi^p \rangle(T) \geqslant C(r,p,\delta) \int_{\mathbf{R}} d\mu(x)(b(x,T)G(x))^{-r/s(x)}, \tag{3.4}$$

where $C(r,p,\delta)>0$ *and* $G(x)$ *is one of the two functions* $U_{s(x)}(x)$, $Y_{s(x)}(x)$ (*we adopt the convention that* $(+\infty)^{-\gamma}=0,\ \gamma>0$).

Proof. Let us prove the statement of the theorem for $G(x)=U_{s(x)}(x)$. For $Y_{s(x)}(x)$ the result will then follow from (3.2). For any $T>0, x\in\mathbf{R}$ define

$$M(x,T)=(16b(x,T)U_{s(x)}(x))^{-1/s(x)}-1, \quad M(x,T)\geqslant -1.$$

Consider the set

$$A(T)=\{x \mid M(x,T)>0\}.$$

If $x\in A(T)$, then for any $N\in[0,M(x,T)]$ by (3.1),

$$\begin{aligned} b(x,T)S_N(x) &\leqslant b(x,T)(N+1)^{s(x)}U_{s(x)}(x) \\ &\leqslant b(x,T)(M(x,T)+1)^{s(x)}U_{s(x)}(x)=\tfrac{1}{16}. \end{aligned}$$

Therefore, $x\in\Omega(T,N)$ for all $N\in[0,M(x,T)]$ and thus $N(x,T)\geqslant M(x,T)$ for all $x\in A(T)$ ($\Omega(T,N)$ and $N(x,T)$ were defined in Theorem 2.2). The second bound of Theorem 2.2 yields:

$$\langle |X|_\psi^p \rangle(T) \geqslant C(r,p) \int_{A(T)} d\mu(x)(M(x,T)+1)^r, \quad 0<r<p. \tag{3.5}$$

On the other hand, it follows directly from definition of the set $A(T)$ that

$$\int_{\mathbf{R}\setminus A(T)} d\mu(x)(16b(x,T)U_{s(x)}(x))^{-r/s(x)} \leqslant 1 \leqslant \langle |X|_\psi^p \rangle(T). \tag{3.6}$$

Since $s(x)\geqslant\delta>0$, the result of the theorem follows from definition of $M(x,T)$ and (3.5)–(3.6). □

Remark 3.1. If $G(x)=+\infty$ for μ-a.e. x for some choice of $s(x)$, the theorem is empty.

Remark 3.2. The result can be considered as a generalization of the result of [20], established for α-continuous measures μ. In fact, if the measure is α-continuous, then

$\mu([x-\varepsilon, x+\varepsilon]) \leqslant K(x)\varepsilon^{\alpha}$ with some finite $K(x)$ for μ-a.e. x and any $\varepsilon \in (0,1)$. One can easily see that $b(x,T) \leqslant LK(x)T^{-\alpha}$ with some uniform $L>0$. Assume that $U_a(x) < +\infty$ for some $a>0$ for all $x \in \Omega$, $\mu(\Omega)>0$. Then (3.4) immediately yields for any $0<r<p$

$$\begin{aligned}\langle |X|_{\psi}^{p} \rangle (T) &\geqslant C(p,r,a)L^{-r/a}T^{r/a} \int_{\Omega} d\mu(x)(K(x)U_a(x))^{-r/a} \\ &= D(p,r,a)T^{r/a}, \quad D>0. \end{aligned} \tag{3.7}$$

This is virtually the bound of [20]. In fact, in [20] it is proved for $r=p$, which is slightly better than (3.7). However, the result of Theorem 3.1 is more general.

Remark 3.3. Eliminating N in bounds (3.4), we may lose the intermittency due to the generalized eigenfunctions (see the discussion in Appendix C).

We shall derive now the lower bounds where the kernels $u(n,x)$ and thus the functions $Y_s(x), U_s(x)$ do not appear explicitly. In particular, are of interest the bounds independent of the choice of the orthonormal family $\mathscr{B}$ (we shall call them "basis-independent bounds"). The results we obtain below as a direct corollary of Theorem 3.1, improve the recent results of [2,3,17]. One can note that the proof we present is simpler than that of [3].

Let

$$\begin{aligned} s_0 &= \inf\left\{ s>0 \,\middle|\, \int_{\mathbf{R}} d\mu(x)\, Y_s(x) < +\infty \right\} \\ &= \inf\left\{ s>0 \,\middle|\, \sum_n (|n|+1)^{-s} ||P_{\psi} e_n||^2 < +\infty \right\}. \end{aligned}$$

The constant s_0 depends on ψ and $\mathscr{B}$. As it was mentioned above, one always has $s_0 \leqslant d$. If $\mathscr{L}(\mathscr{B}) = \mathscr{H}_{\psi}$, then $||P_{\psi}e_n|| = 1$ for all n and it is clear that $s_0 = d$. However, if the vector ψ is not cyclic and $\mathscr{L}(\mathscr{B})$ is considerably "bigger" than $\mathscr{H}_{\psi}$, it is possible that $s_0 < d$, because $||P_{\psi}e_n|| \to 0$ as $|n| \to \infty$. One can expect that it may happen if the subspace $\mathscr{H}_{\psi}$ is "thin" and $\mathscr{B}$ is, for example, a basis of $\mathscr{H}$. Thus, one could have $s_0 = 1$ for some operators with absolutely continuous spectrum of infinite multiplicity in $l^2(\mathbf{Z}^d)$, $d>1$, which could give the ballistic lower bounds for the moments.

One should stress that the condition

$$\int_{\mathbf{R}} d\mu(x)\, Y_s(x) < +\infty \tag{3.8}$$

is stronger than $Y_s(x) < +\infty$ for μ-a.e. x.

For any $q \in \mathbf{R}$, $\varepsilon > 0$ and any Borel measure μ define the integrals

$$I_\mu(q,\varepsilon) = \int_{\mathbf{R}} d\mu(x)(\mu([x-\varepsilon, x+\varepsilon]))^{q-1}.$$

In fact, it is sufficient to integrate over $\operatorname{supp}\mu$ and it is possible that $I(q,\varepsilon) = +\infty$.

Theorem 3.2. *Let* $\psi \in \mathscr{H}$, $\|\psi\| = 1$ *and* μ *the corresponding spectral measure. Let* s_0 *be the positive number defined above,* $p > 0$, $q_0 = \frac{1}{1+p/s_0}$. *For any* $q \in (q_0, 1)$ *the uniform in* T *estimate holds*:

$$\langle |X|_\psi^p \rangle(T) \geqslant C I_\mu^{1/q}(q, T^{-1}). \tag{3.9}$$

In particular, this is always true for $q > 1/(1+p/d)$ (*basis-independent lower bounds*), *and in this case the constant* C *depends only on* p, q *but not on* ψ *and* μ.

Proof. Let $q \in (q_0, 1)$, $\beta = 1/q - 1$. Since $\beta < p/s_0$, one can represent it as $\beta = r/s$ with some $r < p$, $s > s_0$. Consider the integrals

$$J_\mu(q,\varepsilon) = \int_{\mathbf{R}} d\mu(x)(b(x,\varepsilon^{-1}))^{q-1}. \tag{3.10}$$

Let $\varepsilon = 1/T$. Applying the Hölder inequality, one can estimate

$$\begin{aligned}(J_\mu(q,\varepsilon))^{1+\beta} &\leqslant \int_{\mathbf{R}} d\mu(x)(b(x,T)Y_s(x))^{-\beta}\left(\int_{\mathbf{R}} d\mu(x)Y_s(x)\right)^\beta \\ &\leqslant C(s,\beta)\int_{\mathbf{R}} d\mu(x)(b(x,T)Y_s(x))^{-\beta},\end{aligned} \tag{3.11}$$

since $s > s_0$ and thus $\int_{\mathbf{R}} d\mu(x) Y_s(x) < +\infty$. The result of Theorem 3.1 with $s(x) \equiv s$ and (3.11) yield

$$\langle |X|_\psi^p \rangle(T) \geqslant C J_\mu^{1/q}(q, T^{-1}). \tag{3.12}$$

The integral $J_\mu(q,\varepsilon)$ can be written as

$$J_\mu(q,\varepsilon) = \int_{\mathbf{R}} d\mu(x)(\rho(x,\varepsilon))^{q-1},$$

where

$$\rho(x,\varepsilon) = \int_{\mathbf{R}} d\mu(x) R\left(\frac{x-y}{\varepsilon}\right).$$

Recall that the function $R(u)$ is positive, fast decaying at infinity and $R(u) = 1$ for all $u : |u| \leqslant 1$. Recent result of [4] established the equivalence of integrals $I_\mu(q,\varepsilon)$ and $J_\mu(q,\varepsilon)$ for $q \in (0,1)$ in full generality (i.e. for any Borel probability measure μ).

Namely, the two integrals are either both finite or both equal to $+\infty$. Moreover, there exist finite positive constants (depending only on q and on the choice of the function R, i.e. of h) such that

$$C_1 I_\mu(q,\varepsilon) \leqslant J_\mu(q,\varepsilon) \leqslant C_2 I_\mu(q,\varepsilon) \tag{3.13}$$

for any $\varepsilon \in (0,1)$. The statement of the theorem follows from (3.12) and (3.13). □

Remark. One can see that the r.h.s. of (3.9) decays in q for a fixed T. Therefore, one has interest to take q close to q_0.

4. Multifractal dimensions of spectral measures

The growth exponents of integrals $I_\mu(q,\varepsilon)$ are closely related with the multifractal dimensions of probability measures. One defines them for any $q \in \mathbf{R}$, $q \neq 1$ as follows:

$$D_\mu^+(q) = \limsup_{\varepsilon \to 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log \varepsilon},$$

$$D_\mu^-(q) = \liminf_{\varepsilon \to 0} \frac{\log I_\mu(q,\varepsilon)}{(q-1)\log \varepsilon}.$$

In fact, one can adopt this definition for any finite measure μ, and the dimensions of μ are identical with the dimensions of probability measure $v = \mu/\mu(\mathbf{R})$.

Defined in such a way, the quantities $D_\mu^+(q)$, $D_\mu^-(q)$ are decreasing with q and $0 \leqslant D_\mu^-(q) \leqslant D_\mu^+(q) \leqslant +\infty$ for any q. Some basic properties of the functions $D_\mu^\pm(q)$, such as continuity on the set of q's where they are finite (except maybe $q = 1$), are established in [4].

Let $f(q)$ be some monotonous function. We define

$$f(r+0) = \lim_{q \to r, q > r} f(q)$$

and in the similar way for $f(r-0)$. As an immediate consequence of Theorem 3.2, we obtain the following.

Corollary 4.1. *Under the conditions of Theorem* 3.2, *for any* $p > 0$

$$\alpha_\psi^\pm(p) \geqslant \frac{p}{s_0} D_\mu^\pm(1/(1+p/s_0)+0). \tag{4.1}$$

In particular, the following basis-independent bounds hold:

$$\alpha_\psi^\pm(p) \geqslant \frac{p}{d} D_\mu^\pm(1/(1+p/d)+0). \tag{4.2}$$

Proof. The bound of Theorem 3.2 and the definitions of $\alpha_\psi^\pm(p), D_\mu^\pm(q)$ yield

$$\alpha_\psi^\pm(q) \geqslant (1/q - 1) D_\mu^\pm(q)$$

for any $q > q_0 = 1/(1 + p/s_0)$. Taking the limit $q \to q_0$, $q > q_0$, we obtain the result. □

The result of this corollary generalizes the bounds

$$\alpha_\psi^\pm(p) \geqslant \frac{p}{d} D_\mu^\pm(1/(1 + p/d)) \tag{4.3}$$

obtained in [3] under the assumption that the moments of μ of order high enough are finite (which is always true if supp μ is compact). The proofs of [3] can be generalized to obtain (4.3) with s_0 instead of d. Bounds (4.3) were also obtained by different methods in [17] in the special case $\mathscr{H} = l^2(\mathbf{N})$, $e_n = \delta_n$ under rather restrictive assumptions that $D_\mu^+(q) = D_\mu^-(q) = D(q)$ for all $q \in \mathbf{R}$ and $D(q) < +\infty$ for some $q < 1$. One should stress that the result of our corollary holds in any Hilbert space for any orthonormal family $\mathscr{B}$ and any spectral measure μ. In particular, it is possible that $D_\mu^\pm(q) = +\infty$ for some or even for all $q < 1$.

As it was pointed out in [3,17], one can obtain better bound under assumption that

$$S_N(x) \leqslant C N^s \tag{4.4}$$

for some $s > 0$ with constant C uniform in x, N. This condition implies $U_s(x) \leqslant C$. Taking $s(x) = s$ in Theorem 3.1, we obtain immediately that

$$\langle |X|_\psi^p \rangle (T) \geqslant C(r,p) J_\mu(1 - r/s, T^{-1}),$$

where $J_\mu(q, \varepsilon)$ was defined in the previous section. As we said above, the integrals $J_\mu(q, \varepsilon)$ are equivalent to $I_\mu(q, \varepsilon)$ for $q > 0$ in full generality. In the case $q \leqslant 0$ this equivalence can be easily established (the proof is the same as in [3]) for $q > \tilde{q}$, where

$$\tilde{q} = \inf\{q \in \mathbf{R} \mid D_\mu^+(q) < +\infty\}.$$

If $\tilde{q} = -\infty$, then we obtain for any $p > 0$, using the continuity [4] of $D_\mu^\pm(q)$ on $(\tilde{q}, 1)$:

$$\alpha_\psi^\pm(p) \geqslant \frac{p}{s} D_\mu^\pm(1 - p/s).$$

These bounds may be better than (4.3) (even if $s > d$) provided $D_\mu^\pm(q)$ are essentially nonconstant for $q < 1$.

Let us return to bounds (4.1) and (4.2). In fact, under the assumptions of [3] the dimensions are always finite and continuous at $q = q_0$, so in this case our bound (4.2) is identical with (4.3). Let μ be any Borel probability measure on $\mathbf{R}$. Define the following number $q^* \in [0,1]$:

$$q^* = \inf\left\{ q>0 \,\middle|\, \sum_{k\in\mathbf{Z}} (\mu([k,k+1))^q < +\infty \right\}.$$

If supp μ is compact, then always $q^* = 0$. For measures with noncompact support q^* may take any value from $[0,1]$. It was shown in [4] that the dimensions $D_\mu^\pm(q)$ are continuous on $(q^*, +\infty)\backslash\{1\}$ and there $0 \leqslant D_\mu^\pm(q) \leqslant 1$. On the other hand, if $q<q^*$, then $I_\mu(q,\varepsilon) = +\infty$ for ε small enough, in particular, $D_\mu^\pm(q) = +\infty$. The next statement follows directly from what is said above, Theorem 3.2 and Corollary 4.1.

Theorem 4.2. (1) *Assume that* $q^*<1$. *Then for any* $p \in (0, s_0(1/q^*-1))$ (*any* $p>0$ *if* $q^*=0$),

$$\alpha_\psi^\pm(p) \geqslant \frac{p}{d} D_\mu^\pm(1/(1+p/d)) \tag{4.3}$$

(2) *Assume that* $q^*>0$. *Then for any* $p>s_0(1/q^*-1)$ *for* T *large enough* $\langle |X|_\psi^p \rangle(T) = +\infty$.

Remark 4.1. We do not control the upper bounds for the moments. Therefore, it is possible that in the second case the moments are infinite from the beginning, i.e. for the state ψ itself.

Remark 4.2. If $p = s_0(1/q^*-1)$, one always has bound (4.1), but one cannot say whether the moments are finite or not. This case is more delicate.

Remark 4.3. In some cases one has a priori ballistic upper bound for the moments $|X|_\psi^p(t)$ for any $p>0$. It follows from Theorem 4.2 that $q^*=0$ for the corresponding measure μ_ψ.

To apply the results of Theorem 3.2, Corollary 4.1 and Theorem 4.2 to concrete models, one should be able to calculate or rather to estimate from below the integrals $I_\mu(q,\varepsilon)$. In fact, these quantities can be represented in many equivalent forms. We hope that the following theorem will be useful in applications (its first statement is used in [12,27]). Let $f(\varepsilon), g(\varepsilon)$ be two functions from $(0,1)$ to $[0,+\infty]$. We shall say that $f \sim g$ if they are either both finite or both equal to $+\infty$ and there exist two finite positive constants C_1, C_2 such that

$$C_1 f(\varepsilon) \leqslant g(\varepsilon) \leqslant C_2 f(\varepsilon).$$

Theorem 4.3. *Let* μ *be any Borel probability measure. The following statements hold, where the constants* C_1, C_2 *depend on the parameters such as* q, m, δ *but do not* *depend on the choice of* μ:

1. *For any* $q>0$

$$L_\mu(q,\varepsilon)\equiv\frac{1}{\varepsilon}\int_{\mathbf{R}}dx(\mu([x-\varepsilon,x+\varepsilon]))^q\sim I_\mu(q,\varepsilon),$$

$$S_\mu(q,\varepsilon)\equiv\sum_{j\in\mathbf{Z}}(\mu([j\varepsilon,(j+1)\varepsilon)))^q\sim I_\mu(q,\varepsilon).$$

2. *Let* $R(u)$ *be some Borel positive function on* $\mathbf{R}$ *such that* $\inf_{[-1,1]}R(u)=\delta>0$ *and*

$$R(u)\leqslant\frac{C}{|u|^m+1}$$

for some real $m>1$. *Define*

$$\rho(x,\varepsilon)=\int_{\mathbf{R}}d\mu(y)R\Big(\frac{x-y}{\varepsilon}\Big).$$

Then for any $q>\frac{1}{m}$

$$L_\mu^{(R)}(q,\varepsilon)\equiv\frac{1}{\varepsilon}\int_{\mathbf{R}}dx(\rho(x,\varepsilon))^q\sim I_\mu(q,\varepsilon).$$

3. *For any* $q>\frac{1}{2}$

$$K_\mu(q,\varepsilon)\equiv\varepsilon^{q-1}\int_{\mathbf{R}}dx(\operatorname{Im}F_\mu(x+i\varepsilon))^q\sim I_\mu(q,\varepsilon),$$

where $F_\mu(z)$ is the Borel transform of μ:

$$F_\mu(z)=\int_{\mathbf{R}}\frac{d\mu(y)}{y-z},\quad \operatorname{Im}z>0.$$

4. *Let* H *be a self-adjoint operator on* $\mathscr{H},\psi\in\mathscr{H},||\psi||=1$ *and* μ *the corresponding spectral measure. Let* $m\in\mathbf{N}$. *For any* $q>\frac{1}{2m}$,

$$M_\psi(q,\varepsilon)\equiv\varepsilon^{2qm-1}\int_{\mathbf{R}}dx||R^m(x+i\varepsilon)\psi||^{2q}\sim I_\mu(q,\varepsilon),$$

where $R^m(z)=(H-z)^{-m}$.

Proof. The first statement is proved in [4]. The proof for $q>1$ is trivial and well known. It was conjectured many years ago that the result should also hold for $q\in(0,1)$. However, unlike it was stated in many physicist's papers, the proof for $q\in(0,1)$ is not the same (except the case of the measures verifying the doubling condition) and rather nontrivial. Only in [4] it was rigorously proved in all generality (for any Borel probability measure).

The third statement follows from the second if one takes $R(u) = (|u|^2 + 1)^{-1}$, because

$$\operatorname{Im} F_\mu(x + i\varepsilon) = \frac{1}{\varepsilon} \int_{\mathbf{R}} \frac{d\mu(y)}{((x-y)/\varepsilon)^2 + 1}.$$

The same is true for the fourth statement, where one takes $R(u) = (|u|^{2m} + 1)^{-1}$.

Let us show the second statement of the theorem. First, as R is positive and $R(u) \geqslant \delta > 0$ for $u \in [-1, 1]$,

$$\rho(x, \varepsilon) \geqslant \delta\mu([x - \varepsilon, x + \varepsilon]).$$

Therefore, using the first statement,

$$L_\mu^{(R)}(q, \varepsilon) \geqslant \delta^q L_\mu(q, \varepsilon) \sim I_\mu(q, \varepsilon). \tag{4.5}$$

It is thus sufficient to show that $L_\mu^{(R)}(q, \varepsilon) \leqslant C I_\mu(q, \varepsilon) \sim S_\mu(q, \varepsilon)$, Let $I_j = [j\varepsilon, (j+1)\varepsilon)$ and $a_j = \mu(I_j)$, so that $S_\mu(q, \varepsilon) = \sum_{j \in Z} a_j^q$. One can write $L_\mu^{(R)}(q, \varepsilon)$ as follows:

$$L_\mu^{(R)}(q, \varepsilon) = \frac{1}{\varepsilon} \sum_j \int_{I_j} dx (\rho(x, \varepsilon))^q, \tag{4.6}$$

where

$$\rho(x, \varepsilon) = \sum_k \int_{I_k} d\mu(y) R\left(\frac{x - y}{\varepsilon}\right).$$

As $R(u) \leqslant \frac{C}{|u|^m + 1}$, it is easy to see that for any $x \in I_j,\ y \in I_k$ the bound holds:

$$R\left(\frac{x - y}{\varepsilon}\right) \leqslant \frac{K}{|j - k|^m + 1}$$

with some uniform constant K. Therefore, if $x \in I_j$, one can estimate

$$\rho(x, \varepsilon) \leqslant K \sum_k \frac{a_k}{|j - k|^m + 1}.$$

Since $|I_j| = \varepsilon$, we obtain from (4.6)

$$L_\mu^{(R)}(q, \varepsilon) \leqslant K^q \sum_j \left(\sum_k \frac{a_k}{|j - k|^m + 1} \right)^q. \tag{4.7}$$

Consider first the case when $\frac{1}{m} < q \leqslant 1$. Using the elementary bound $(\sum_k b_k)^q \leqslant \sum_k b_k^q$, we get

$$L_\mu^{(R)}(q,\varepsilon) \leqslant K^q \sum_{j,k} \frac{a_k^q}{(|j-k|^m+1)^q} = DK^q S_\mu(q,\varepsilon), \tag{4.8}$$

where $D = \sum_n 1/(|n|^m+1)^q < +\infty$. Bounds (4.5) and (4.8) yield the second statement of the theorem.

Let now $q>1$. One can write the r.h.s. of (4.7) as

$$\sum_j \left(\sum_n \frac{a_{j+n}}{|n|^m+1} \right)^q.$$

Using the bound $\|\sum_n h_n\| \leqslant \sum_n \|h_n\|$, where $\|\cdot\|$ is the $l^q(\mathbf{Z})$ norm, one sees that

$$(L_\mu^{(R)}(q,\varepsilon))^{1/q} \leqslant K \sum_n \frac{1}{|n|^m+1} \left(\sum_j a_{j+n}^q \right)^{1/q} = KDS_\mu^{1/q}(q,\varepsilon), \tag{4.9}$$

where $D = \sum_n \dfrac{1}{|n|^m+1} < +\infty$. Bounds (4.5) and (4.9) give the statement of the theorem for $q>1$. □

Remark. In [25] the behaviour of L^q-norms of Borel transform was related to the absolute continuity of the measure ($q>1$) or the absence of the a.c. part for μ (if $q\in(0,1)$). The third statement of Theorem 4.3 shows the relation of such L^q-norms (for $q>1/2$) with multifractal dimensions and thus their importance (especially for $q\in(1/2,1)$) in quantum dynamics.

As an example of application of this theorem we shall prove that the *lower* bound for the measure of intervals $[x-\varepsilon, x+\varepsilon]$ may give some nontrivial dynamical information. It is well known that the upper bound

$$\mu([x-\varepsilon,x+\varepsilon]) \leqslant C(x)\varepsilon^\alpha, \quad C(x) < +\infty, \ \alpha\in[0,1] \tag{4.10}$$

is important for dynamics. If bound (4.10) uniform in $\varepsilon\in(0,1)$ holds for any x from the set A of positive measure $\mu(A)$, then it is easy to show that $\dim_H(\mu) \geqslant \alpha$, and thus we have bound (1.1) for the moments. If (4.10) holds for any $x\in A$ for some sequence $\varepsilon_k \to 0$ (may be depending on x), then the similar bound is true for $\dim_P(\mu)$. What is surprising, is the fact that the lower bound uniform in $\varepsilon\in(0,1)$

$$\mu([x-\varepsilon,x+\varepsilon]) \geqslant C(x)\varepsilon^\gamma, \quad C(x)>0, \quad \gamma\geqslant 1, \tag{4.11}$$

on the set of positive *Lebesgue* measure also yields rather nontrivial dynamical information for the moments of high order of position operator (and the corresponding lower bounds for $\alpha_\psi^\pm(p)$ are always nonlinear in p). Such a possibility was never supposed before. One can observe that bound (4.11) may hold for pure point measures.

Theorem 4.4. *Let μ be a Borel probability measure. Assume that for some $\gamma \geqslant 1$ bound* (4.11) *holds for any $x \in A$, where A is a set of positive Lebesgue measure and $C(x)$ is strictly positive Borel function. Then for any $q \in (0, \frac{1}{\gamma})$,*

$$D_\mu^-(q) \geqslant \frac{1 - q\gamma}{1 - q}. \tag{4.12}$$

If the spectral measure of some state ψ verifies the conditions of this theorem, then

$$\alpha_\psi^-(p) \geqslant \frac{p}{s_0} - (\gamma - 1)$$

for all $p > s_0(\gamma - 1)$. In particular, in one dimension one always has $\alpha_\psi^-(p) \geqslant p - (\gamma - 1)$ and thus the behaviour of the moments of high order is quasiballistic.

Bound (4.12) *also holds for $q > 1/m$ if the function $\rho(x, \varepsilon)$ from the second statement of Theorem* 4.3 *verifies the same lower bound* (4.11).

Proof. The definition of $L_\mu(q, \varepsilon)$ and the first statement of Theorem 4.3 yield

$$I_\mu(q, \varepsilon) \sim L_\mu(q, \varepsilon) \geqslant \frac{1}{\varepsilon} \int_A dx(\mu([x - \varepsilon, x + \varepsilon]))^q \geqslant \varepsilon^{q\gamma - 1} \int_A dx C^q(x) = K\varepsilon^{q\gamma - 1},$$

where $K > 0$ because the set A has positive Lebesgue measure and $C(x) > 0$ for all $x \in A$.The definition of $D_\mu^\pm(q)$ and Corollary 4.1 give the result. For $\rho(x, \varepsilon)$ the proof is the same. □

Remark. One cannot have (4.11) with $\gamma < 1$ on the set of positive Lebesgue measure because μ is finite.

To understand better this result, consider the sums $S_\mu(q, \varepsilon)$, which are equivalent to $I_\mu(q, \varepsilon)$ and thus also yield lower bounds for the moments of position operator:

$$S_\mu(q, \varepsilon) = \sum_{j \in \mathbf{Z}} a_j^q, \quad a_j = \mu([j\varepsilon, (j + 1)\varepsilon)), \quad \sum_j a_j = 1.$$

The behaviour of $S_\mu(q, \varepsilon)$ as $\varepsilon \to 0$ and thus the multifractal dimensions $D_\mu^\pm(q)$ are determined by the distribution of numbers a_j depending on ε for small ε. In particular, the upper and the lower bounds for a_j imply some lower bounds for $S_\mu(q, \varepsilon)$. Assume, for example, that $a_j \leqslant C\varepsilon^\alpha$ with uniform constant C (uniformly α-continuous measure). Then, as $q - 1 < 0$, one can estimate:

$$S_\mu(q, \varepsilon) = \sum_j a_j^{q-1} a_j \geqslant (C\varepsilon^\alpha)^{q-1} \sum_j a_j = C(q)\varepsilon^{\alpha(q-1)},$$

thus, $D_\mu^-(q) \geqslant \alpha$.

Assume now that for all $I_j \subset [0,1]$ the uniform in ε, j lower bound holds $a_j \geqslant C\varepsilon^{\gamma}$. Then immediately

$$S_\mu(q,\varepsilon) \geqslant C(q)\varepsilon^{q\gamma-1}$$

and $D_\mu^-(q) \geqslant \frac{1-q\gamma}{1-q}$. That is how one can interpret the result of Theorem 4.4. To obtain a good lower bound for $D_\mu^\pm(q)$ for any $q \in (0,1)$, in general, however, the whole statistics of the numbers a_j is necessary.

An interesting question related to the result of Theorem 4.4 is the following: what properties of the measure μ determine the behaviour of $D_\mu^\pm(q)$ as $q \to 0$ (which gives lower bound for the moments for large values of p)? We provide below a partial answer to it. Let us recall the definition (one of two equivalent) of the support of the measure:

$$\operatorname{supp} \mu = \{x \in \mathbf{R} \mid \forall \varepsilon > 0, \quad \mu([x-\varepsilon, x+\varepsilon]) > 0\}.$$

One notes that $\mu(\mathbf{R} \backslash \operatorname{supp} \mu) = 0$.

We define also the box-counting dimensions of the set $\Omega \subset \mathbf{R}$ (see [11]):

$$\dim_B^+(\Omega) = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

and similarly for $\dim_B^-(\Omega)$, where

$$N(\varepsilon) = \operatorname{card}\{j \in \mathbf{Z} \mid [j\varepsilon, (j+1)\varepsilon) \cap \Omega \neq \emptyset\}.$$

The following statement follows from more general results of [13]. However, for the sake of completeness, we shall give below a simple direct proof of it.

Theorem 4.5. *Assume that there exist two positive constants C, A such that*

$$\mu([x-\varepsilon, x+\varepsilon]) \geqslant C\varepsilon^A \tag{4.13}$$

for all $x \in \operatorname{supp} \mu$. (This is possible [13] only if $\operatorname{supp} \mu$ is compact). Then for all $q \in (0,1)$

$$\frac{\dim_B^\pm(\operatorname{supp} \mu) - qA}{1-q} \leqslant D_\mu^\pm(q) \leqslant \frac{\dim_B^\pm(\operatorname{supp} \mu)}{1-q}.$$

In particular,

$$\lim_{q \to 0} D_\mu^\pm(q) = \dim_B^\pm(\operatorname{supp} \mu).$$

Proof. Consider the sums

$$S(q,\varepsilon) = \sum_{j \in \mathbf{Z}} a_j^q, \quad a_j = \mu(I_j), \; I_j = [j\varepsilon, (j+1)\varepsilon),$$

where $q \geqslant 0$ and the summation is carried only over j such that $a_j > 0$. For $q > 0$ these sums are equivalent to $I_\mu(q, \varepsilon)$ due to Theorem 4.3. One observes that

$$S(0, \varepsilon) \geqslant S(q, \varepsilon), \quad q > 0, \ \varepsilon \in (0, 1) \tag{4.14}$$

and

$$S(0, \varepsilon) = \mathrm{card}(L), \quad L = \{j \in \mathbf{Z} \mid \mu(I_j) > 0\}.$$

Consider the set

$$M = \{j \in \mathbf{Z} \mid I_j \cap \mathrm{supp}\, \mu \neq \emptyset\},$$

where $N(\varepsilon) = \mathrm{card}(M)$ is the number in the definition of $\dim_B^{\pm}(\mathrm{supp}\, \mu)$. Let $j \in L$. It is clear that $I_j \cap \mathrm{supp}\, \mu \neq \emptyset$, thus, $j \in M$ and $L \subset M$. Define

$$L^- = \{j \in \mathbf{Z} \mid j - 1 \in L\} = L + 1.$$

Let $j \in M$. Due to the definition of $\mathrm{supp}\, \mu$ and the fact that $I_j = [j\varepsilon, (j+1)\varepsilon)$, one of the number a_{j-1}, a_j is positive. Therefore, $j \in L^- \cup L$ and thus $M \subset (L^- \cup L)$. We see that $\mathrm{card}(L) \leqslant \mathrm{card}(M) \leqslant 2\, \mathrm{card}(L)$. Therefore, one can take $\mathrm{card}(L)$ instead of $\mathrm{card}(M) = N(\varepsilon)$ in the definition of $\dim_B^{\pm}(\mathrm{supp}\, \mu)$. Finally, we obtain

$$\dim_B^{+}(\mathrm{supp}\, \mu) = \limsup_{\varepsilon \to 0} \frac{\log S(0, \varepsilon)}{\log 1/\varepsilon} \tag{4.15}$$

and similarly for the lower limits.

Let us minorate now $S(q, \varepsilon)$ for some $q \in (0, 1)$. Let j be such that $a_j > 0$. Then $[j\varepsilon, (j+1)\varepsilon) \cap \mathrm{supp}\, \mu \neq \emptyset$ and thus one can find $x_0 \in [j\varepsilon, (j+1)\varepsilon) \cap \mathrm{supp}\, \mu$. Since $[x_0 - \varepsilon, x_0 + \varepsilon] \subset (I_{j-1} \cup I_j \cup I_{j+1})$, condition (4.13) implies

$$a_{j-1} + a_j + a_{j+1} \geqslant C\varepsilon^A.$$

Therefore, taking the q's power and summating over j such that $a_j > 0$,

$$\begin{aligned} C^q \varepsilon^{qA} S(0, \varepsilon) &\leqslant \sum_{j: a_j > 0} (a_{j-1} + a_j + a_{j+1})^q \\ &\leqslant \sum_{j \in \mathbf{Z}} (a_{j-1} + a_j + a_{j+1})^q \\ &\leqslant \sum_{j \in \mathbf{Z}} (a_{j-1}^q + a_j^q + a_{j+1}^q) \\ &= 3S(q, \varepsilon). \end{aligned} \tag{4.16}$$

Finally, (4.14) and (4.16) yield

$$C(q)\varepsilon^{qA} S(0, \varepsilon) \leqslant S(q, \varepsilon) \leqslant S(0, \varepsilon) \tag{4.17}$$

with some constant $C(q)$ uniform in ε. The definition of $D_\mu^\pm(q)$, the first statement of Theorem 4.3, (4.17) and (4.15) prove the statement of the theorem. $\square$

Remark 4.4. Without condition (4.13) the result of the theorem may be not true. One can give an abstract example, but we prefer to consider an example from quantum mechanics. Let H be a self-adjoint operator with dense pure point spectrum on some interval $[a,b]$ (for example, this is the case for some random Schrödinger operators). Assume that for some cyclic state ψ from the subspace of the pure point spectrum the dynamical localization holds. Then $\alpha_\psi^\pm(p) = 0$ for all $p > 0$ and thus $D_{\mu_\psi}^\pm(q) = 0$ for all $q \in (0,1)$. On the other hand, $[a,b] \subset \operatorname{supp} \mu_\psi$, therefore $\dim_B^\pm(\operatorname{supp} \mu_\psi) = 1$. It is clear that the result of Theorem 4.5 is not true in this case and, condition (4.13) fails for μ_ψ.

Remark 4.5. Condition (4.13) plays an important role [13] in the behaviour of multifractal dimensions $D_\mu^\pm(q)$ for $q < 0$. In particular, under this condition the dimensions are finite for all $q < 0$ and $\lim_{q\to-\infty} D_\mu^\pm(q) \leqslant A < +\infty$.

Remark 4.6. One says that the measure μ is doubling if there exist $\varepsilon_0 > 0, K > 0$ such that

$$\mu([x - 2\varepsilon, x + 2\varepsilon]) \leqslant K\mu([x - \varepsilon, x + \varepsilon])$$

for all $x \in \operatorname{supp} \mu$, $\varepsilon < \varepsilon_0$. One can show [13] that any doubling measure with compact support verifies (4.13) (the converse in general being not true).

Appendix A. Hausdorff and packing dimensions of measures

In this appendix we shall discuss the relations between multifractal dimensions and Hausdorff and packing dimensions of Borel measures. For any Borel set S we denote by $\dim(S)$ and $\operatorname{Dim}(S)$ the Hausdorff and packing dimension of S, respectively (for the definition of Hausdorff and packing measures and dimensions see, for example, [11]). Let μ be some finite Borel measure. We define the lower and upper Hausdorff dimensions of μ by

$$\dim_*(\mu) = \inf\{\dim(S) \mid \mu(S) > 0\},$$

$$\dim^*(\mu) \equiv \dim_H(\mu) = \inf\{\dim(S) \mid \mu(S) = 1\}$$

and similarly for the packing dimensions

$$\operatorname{Dim}_*(\mu) = \inf\{\operatorname{Dim}(S) \mid \mu(S) > 0\},$$

$$\operatorname{Dim}^*(\mu) \equiv \dim_P(\mu) = \inf\{\operatorname{Dim}(S) \mid \mu(S) = 1\}.$$

All these dimensions lie in $[0,1]$. One can interpret them in the following way: the measure μ gives zero weight to any set S with $\dim(S) < \dim_*(\mu)$ and is supported by a set with Hausdorff dimension less than $\dim^*(\mu) + \varepsilon$ for any $\varepsilon > 0$. Similarly for the packing dimensions.

The following inequalities hold: $\dim_*(\mu) \leqslant \dim^*(\mu)$, $\mathrm{Dim}_*(\mu) \leqslant \mathrm{Dim}^*(\mu)$, $\dim_*(\mu) \leqslant \mathrm{Dim}_*(\mu)$ and $\dim^*(\mu) \leqslant \mathrm{Dim}^*(\mu)$ (the two last follow from the fact that $\dim(S) \leqslant \mathrm{Dim}(S)$ for any set S). The dimensions $\dim^*(\mu) \equiv \dim_H(\mu)$ and $\mathrm{Dim}^*(\mu) \equiv \dim_P(\mu)$ are often called by physicists Hausdorff and packing dimension of μ, respectively (we adopt this definition in this paper). Mathematicians have a different definition. They say that the measure is of exact Hausdorff dimension α if $\dim_*(\mu) = \dim^*(\mu) = \alpha$ and in the same way for packing dimensions. It is clear that not all measures have exact Hausdorff or packing dimensions.

Define now the local exponents of the measure μ:

$$\gamma^-(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu(x-\varepsilon, x+\varepsilon)}{\log \varepsilon}, \quad \gamma^+(x) = \limsup_{\varepsilon \to 0} \frac{\log \mu(x-\varepsilon, x+\varepsilon)}{\log \varepsilon},$$

where $x \in \operatorname{supp} \mu$. Obviously, $0 \leqslant \gamma^-(x) \leqslant \gamma^+(x)$ for any x. It is known [1] that $\gamma^\pm(x) \in [0,1]$ for μ-a.e. x. It was proved in [11,18] that

$$\dim_*(\mu) = \mu - \operatorname{essinf} \gamma^-(x) = \sup\{\alpha \mid \gamma^-(x) \geqslant \alpha,\ \mu\text{-a.s.}\},$$

$$\dim_H(\mu) = \dim^*(\mu) = \mu\text{-esssup}\, \gamma^-(x) = \inf\{\alpha \mid \gamma^-(x) \leqslant \alpha,\ \mu\text{-a.s.}\},$$

$$\mathrm{Dim}_*(\mu) = \mu - \operatorname{essinf} \gamma^+(x) = \sup\{\alpha \mid \gamma^+(x) \geqslant \alpha,\ \mu\text{-a.s.}\},$$

$$\dim_P(\mu) = \mathrm{Dim}^*(\mu) = \mu\text{-esssup}\, \gamma^+(x) = \inf\{\alpha \mid \gamma^+(x) \leqslant \alpha, \mu\text{-a.s.}\}.$$

The Hausdorff and packing dimensions of the measure are related with its multifractal dimensions in the following way: for any $q < 1, r > 1$,

$$D_\mu^-(q) \geqslant \dim_H(\mu) = \dim^*(\mu) \geqslant \dim_*(\mu) \geqslant D_\mu^-(r), \tag{A.1}$$

$$D_\mu^+(q) \geqslant \dim_P(\mu) = \mathrm{Dim}^*(\mu) \geqslant \mathrm{Dim}_*(\mu) \geqslant D_\mu^+(r). \tag{A.2}$$

It follows from the results of [4] and the expressions of Hausdorff and packing dimensions of the measure in terms of local exponents. In particular, (A.1) implies that if the measure μ is not of exact Hausdorff dimension, then $D_\mu^-(q)$ is always discontinuous at $q = 1$, and similarly for the packing dimension.

It is interesting to note that the multifractal dimensions of measures appeared in the hidden form many years ago in the proof of absolute continuity of the spectrum of self-adjoint operators. Let us consider the well-known sufficient condition for the absolute continuity of the spectral measure μ [24]:

$$\int_{\mathbf{R}} dx |\mathrm{Im}\, F_\mu(x+i\varepsilon)|^q \leqslant C(q) < +\infty \tag{A.3}$$

for some $q>1$ uniformly in $\varepsilon \in (0,1)$. Due to Theorem 4.3, (A.3) is equivalent to

$$I(q,\varepsilon) \leqslant C\varepsilon^{q-1}. \tag{A.4}$$

Bound (A.4) implies that $D^\pm(q) \geqslant 1,\ q>1$, so that $D^\pm_\mu(q)=1$ for any $q>q^*$ (see previous section for definition of q^*). In fact, condition (A.3) or (A.4) contains more information than $D^\pm(q)=1$. It implies directly the absolute continuity of the measure.

One can generalize condition (A.3) in local form if the integrals over some interval (a,b) are bounded by some negative power of ε. The result we prove below generalizes one of the results of [10].

Theorem A.1. *Let (a,b) be some interval of $\mathbf{R}$ such that $\mu((a,b))>0$ and v is the restriction of μ* on *(a,b). We denote by $F_\mu(z)$ the Borel transform of μ.*

1. *Suppose that*

$$\int_a^b dx |\mathrm{Im}\, F_\mu(x+i\varepsilon)|^q \leqslant C\varepsilon^{-s(q-1)} \tag{A.5}$$

for some $q>1,\ s\in(0,1)$ uniformly in $\varepsilon\in(0,1)$. Then

$$\dim_*(v) \geqslant 1-s,$$

so that v gives zero weight to sets of Hausdorff dimension less than $1-s$.

2. *Suppose that for some $q>1,\ s\in(0,1)$ there exists a sequence $\varepsilon_n \to 0$ such that* (A.5) *holds for all $\varepsilon=\varepsilon_n$ with uniform constant C. Then*

$$\mathrm{Dim}_*(v) \geqslant 1-s,$$

so that v gives zero weight to sets of packing dimension less than $1-s$.

Proof. Let us estimate the integrals $I_v(q,\varepsilon)$. Due to the third statement of Theorem 4.3,

$$I_v(q,\varepsilon) \sim \varepsilon^{q-1} \int_{\mathbf{R}} dx\, g^q(x), \tag{A.6}$$

where

$$g(x) \equiv \mathrm{Im}\, F_v(x+i\varepsilon) = \varepsilon \int_{(a,b)} \frac{d\mu(y)}{(x-y)^2+\varepsilon^2} \leqslant \mathrm{Im}\, F_\mu(x+i\varepsilon). \tag{A.7}$$

The condition of the theorem and (A.7) imply that

$$\int_a^b dx\, g^q(x) \leqslant C\varepsilon^{-s(q-1)}. \tag{A.8}$$

Let us estimate the integral over $\mathbf{R}\backslash(a,b)$. First, one observes that for any $x, y \in (a,b)$,

$$\frac{1}{(2b-x-y)^2+\varepsilon^2} \leqslant \frac{1}{(x-y)^2+\varepsilon^2},$$

so that

$$g(2b-x) \leqslant g(x), \quad x \in (a,b).$$

Therefore,

$$\int_b^{2b-a} dx\, g^q(x) = \int_a^b dt\, g^q(2b-t) \leqslant \int_a^b dt\, g^q(t) \leqslant C\varepsilon^{-s(q-1)}. \tag{A.9}$$

Next, it is obvious that $g(x) \leqslant \frac{\varepsilon}{(x-b)^2}$ for all $x > b$, so that

$$\int_{2b-a}^{+\infty} dx\, g^q(x) \leqslant C(a,b)\varepsilon^q \tag{A.10}$$

As $\varepsilon \in (0,1)$, $q > 1$, bounds (A.9) and (A.10) yield

$$\int_b^{+\infty} dx\, g^q(x) \leqslant C\varepsilon^{-s(q-1)}. \tag{A.11}$$

The similar considerations give

$$\int_{-\infty}^a dx\, g^q(x) \leqslant C\varepsilon^{-s(q-1)}. \tag{A.12}$$

Finally, (A.6), (A.8), (A.11) and (A.12) imply

$$I_v(q,\varepsilon) \leqslant C\varepsilon^{(1-s)(q-1)}. \tag{A.13}$$

If bound (A.5) holds for all $\varepsilon \in (0,1)$, then (A.13) and the definition of $D_v^-(q)$ yield

$$D_v^-(q) \geqslant 1-s \tag{A.14}$$

and if (A.5) holds for some sequence $\varepsilon_n \to 0$, then

$$D_v^+(q) \geqslant 1 - s. \tag{A.15}$$

Bounds (A.1)–(A.2) and (A.14)–(A.15) yield the result. $\square$

Remark. The first statement of the theorem was proved for $q = 2$ in [10].

Appendix B. Operators with pure point spectrum

In this appendix we shall see how the general lower bounds of the paper work in the particular case of pure point spectrum. Assume that ψ belongs to the subspace of point spectrum of operator H. For simplicity we consider the particular case of $\mathscr{H} = l^2(\mathbf{Z}^d),\ \mathscr{B} = \{\delta_n\},\ n \in \mathbf{Z}^d$. One can write ψ as

$$\psi = \sum_{k=1}^{M} \gamma_k g_k, \quad \gamma_k = \langle \psi, g_k \rangle \neq 0,\ M \in \mathbf{N} \text{ or } M = +\infty,$$

where $Hg_k = x_k g_k,\ ||g_k|| = 1$ and $x_k \neq x_m$ for $k \neq m$. It is clear that the orthonormal system $\{g_k\}_{k=1}^M$ is the basis of the cyclic subspace $\mathscr{H}_\psi$ of ψ. The spectral measure of ψ is pure point:

$$\mu = \sum_{k=1}^{M} a_k \delta_{x_k},\ a_k = |\gamma_k|^2 > 0.$$

It is easy to verify that the unitary map W_ψ from $\mathscr{H}_\psi$ to $L^2(\mathbf{R}, d\mu)$ is given by

$$(W_\psi f)(x_k) = \frac{1}{\gamma_k} \langle f, g_k \rangle, \quad k \in [1, M].$$

Further, the kernels $u(n, x)$ are given by

$$u(n, x_k) = \frac{1}{\gamma_k} \overline{g_k(n)},$$

and the function $b(x, T)$ by

$$b(x_k, T) \equiv b_k(T) = \sum_{m=1}^{M} a_m R(T(x_k - x_m)).$$

One can observe that $b_k(T)$ is decreasing in k and $\lim_{T \to +\infty} b_k(T) = a_k$. Next, we have

$$Y_s(x_k) = \frac{1}{a_k} \sum_n (|n| + 1)^{-s} |g_k(n)|^2.$$

The function $N(x,T)$ of Theorem 2.2 is given by

$$N(x_k,T)\equiv N_k(T)=\sup\left\{N>0\,\middle|\,\sum_{|n|\leqslant N}|g_k(n)|^2\leqslant\frac{a_k}{16b_k(T)}\right\},\quad k\in[1,M]. \tag{B.1}$$

The lower bound (2.23) of Theorem 2.2 reads as

$$\langle |X|_\psi^p\rangle(T)\geqslant C(r,p)\sum_{k=1}^{M}a_k(N_k(T)+1)^r, \tag{B.2}$$

where $0<r<p$. This bound, of course, is of interest only if the r.h.s. of (B.2) tends to $+\infty$ as $T\to\infty$, which may happen only if $M=+\infty$. As $\lim_{T\to\infty}b_k(T)=a_k$, one has

$$N_k=\lim_{T\to+\infty}N_k(T)=\sup\left\{N>0\,\middle|\,\sum_{|n|\leqslant N}|g_k(n)|^2\leqslant\frac{1}{16}\right\}.$$

Therefore,

$$\liminf_{T\to+\infty}\langle |X|_\psi^p\rangle(T)\geqslant C(r,p)\sum_{k=1}^{\infty}a_k(N_k+1)^r. \tag{B.3}$$

If the sum in (B.3) is infinite, then $\lim_{T\to+\infty}\langle |X|_\psi^p\rangle(T)=+\infty$ and so the dynamical localization for the moment of order p (i.e. $\sup_t |X|_\psi^p(t)\leqslant C<+\infty$) fails. It is interesting to compare (B.3) with the lower bound established in [26]:

$$\liminf_{T\to+\infty}\langle |X|_\psi^p\rangle(T)\geqslant\sum_{k=1}^{\infty}a_kd_k(p), \tag{B.4}$$

where $d_k(p)=\sum_n(|n|+1)^p|g_k(n)|^2$. One can easily see from definition of N_k and $d_k(p)$ that

$$d_k(p)\geqslant 15/16(N_k+1)^p\geqslant 15/16(N_k+1)^r,$$

therefore bound (B.4) implies (B.3). In fact, if all the functions $g_k(n)$ are well localized (each around some point $n_k\in\mathbf{Z}^d$), then $d_k(p)\approx(|n_k|+1)^p$ and $N_k\approx|n_k|$, so that the two bounds are essentially the same. If the sum on the r.h.s. of (B.3) is infinite, then bound (B.2) gives us information about the rate of growth for the time-averaged moments. Such bounds are not available in [26], where one was rather interested by dynamical localization.

Consider now the bound of Theorem 3.2. In our case it takes the following form:

$$\langle |X|^p_\psi(T)\rangle \geqslant C\left(\sum_{k=1}^{\infty} a_k(c_k(T))^{q-1}\right)^{1/q}, \tag{B.5}$$

where $c_k(T)=\sum_{j:|x_j-x_k|\leqslant 1/T} a_j,\ q>1/(1+p/s_0)$. Taking the limit $T\to+\infty$ in (B.5), we obtain

$$\liminf_{T\to+\infty} \langle |X|^p_\psi\rangle(T) \geqslant C\left(\sum_{k=1}^{\infty} a_k^q\right)^{1/q}. \tag{B.6}$$

If the dynamical localization for the moment of order p holds, then (B.6) yields

$$\sum_{k=1}^{\infty} a_k^q < +\infty$$

for any $q>1/(1+p/s_0)$. This result (with $s_0=d$) was established by different methods in [26].

Bound (B.2) or (B.5) can be used to prove the growth of the moments expected for some quantum systems with pure point spectrum, provided one has necessary information about a_k, x_k and $g_k(n)$. The bounds in terms of the integrals $L_\mu(q,\varepsilon), L_\mu^{(R)}(q,\varepsilon)$ or $M_\psi(q,\varepsilon)$ (Theorem 4.3) may be useful, if one has a good control from below for $\mu([x-\varepsilon,x+\varepsilon])$ or for the powers of the resolvent $||R^m(x+i\varepsilon)\psi||$.

Appendix C. Uniformly Hölder continuous measures

Let μ be a Borel probability measure on $\mathbf{R}$. One says that μ is uniformly α-Hölder continuous ($U_\alpha H$), if there exists a finite constant C such that for any $\varepsilon\in(0,1), x\in\mathbf{R}$,

$$\mu([x-\varepsilon,x+\varepsilon])\leqslant C\varepsilon^\alpha.$$

One can easily see that for such measures $D_\mu^\pm(q)\geqslant\alpha$ for all $q\in(0,1)$. For many such measures $D_\mu^\pm(q)=\alpha$ for all $q\in(0,1)$.

In fact, the first abstract lower bound for the moments were obtained by Guarneri [14] and Combes [8] for the states ψ whose measure is $U_\alpha H$:

$$\langle |X|^p_\psi\rangle(T)\geqslant CT^{p/d},\ p>0, \tag{C.1}$$

so that $\alpha_\psi^\pm(p)\geqslant p/d$. This "classical" lower bound is linear in p. One observes, however, in numerical calculations [22,23] that $\alpha_\psi^\pm(p)$ may grow nonlinearly for some $U_\alpha H$ measures even if the multifractal dimensions are constant on $(0,1)$. It is clear that (C.1) or Theorem 4.2 cannot explain it. It was conjectured by Mantica that this intermittent behaviour may arise from the generalized eigenfunctions. The result we prove below provides some possible mechanism for such a phenomenon.

Let $\psi \in \mathscr{H}$, $||\psi|| = 1$, μ its spectral measure and $S_N(x)$ the sum of generalized eigenfunctions defined in Section 2. We shall assume that μ is $U_\alpha H$. For any $r, \gamma > 0$ we define

$$V(\gamma, N) = \int_{\mathbf{R}} d\mu(x)(S_N(x) + 1)^{-\gamma/\alpha},$$

$$W(r, \gamma) = \int_0^{+\infty} dN(N+1)^{r-1} V(\gamma, N),$$

where $V(\gamma, N) \in [0, 1]$ and it is possible that $W(r, \gamma) = +\infty$. We introduce also the numbers

$$\gamma_\psi(p) = \sup\left\{\gamma \geqslant 0 \middle| \int_1^{+\infty} dt\, t^{-1-\gamma} |X|_\psi^p(t) = +\infty\right\}$$

One can consider that $|X|_\psi^p(t) \sim t^{\gamma(p)}$, $t \to +\infty$ in average sense.

Theorem C.1. (1) *Assume that* $W(r, \gamma) = +\infty$ *for some* $0 < r < p$, $\gamma > 0$. *Then*

$$\int_1^{+\infty} dt\, t^{-1-\gamma} |X|_\psi^p(t) = +\infty. \tag{C.2}$$

(2) *For all* $p > 0$ *the inequality holds*:

$$\gamma_\psi(p) \geqslant \sup\{\gamma \geqslant 0 \mid \exists r \in (0, p),\ W(r, \gamma) = +\infty\}. \tag{C.3}$$

Proof. We start with the lower bound (2.22) of Theorem 2.2:

$$\langle |X|_\psi^p \rangle (T) \geqslant C \int_0^{+\infty} dN(N+1)^{r-1} \mu(\Omega(T, N)), \quad 0 < r < p,$$

where

$$\Omega(T, N) = \{x \mid b(x, T) S_N(x) \leqslant 1/16\},\ b(x, T) = \int_{\mathbf{R}} d\mu(x) R(T(x-y)),$$

and R is some bounded fast decaying function. Since μ is $U_\alpha H$, it is easy to show (the proof is identical with the proof of Theorem 2.5 in [21]) that

$$b(x, T) \leqslant CT^{-\alpha},\ T \geqslant 1.$$

Clearly,

$$\Omega(T,N) \supset A(T,N) \equiv \{x \mid 16CS_N(x) \leqslant T^\alpha\},$$

and thus

$$\langle |X|_\psi^p \rangle(T) \geqslant C \int_0^{+\infty} dN(N+1)^{r-1} \int_{\mathbf{R}} d\mu(x)\chi_{A(T,N)}(x). \tag{C.4}$$

Let us multiply by $(T+1)^{-1-\gamma}$ the both sides in (C.4) and integrate over $[1,+\infty)$:

$$\int_1^{+\infty} dT(T+1)^{-1-\gamma}\langle |X|_\psi^p \rangle(T) \geqslant C \int_0^{+\infty} dN(N+1)^{r-1} \int_{\mathbf{R}} d\mu(x)Z(N,x), \tag{C.5}$$

where

$$Z(N,x) = \int_1^{+\infty} dT(T+1)^{-1-\gamma}\chi_{A(T,N)}(x).$$

For given x, N one observes that $x \in A(T,N)$ iff $T \geqslant (16CS_N(x))^{1/\alpha} \equiv T_0(N,x)$. Without loss of generality, we may assume that $T_0(N,x) \geqslant 1$. Thus,

$$Z(N,x) = \frac{1}{\gamma}(T_0(N,x)+1)^{-\gamma} \geqslant K(1+S_N(x))^{-\gamma/\alpha} \tag{C.6}$$

with some positive constant K uniform in N, x. On the other hand, since $\langle |X|_\psi^p \rangle(T)$ is Cesaro average of the moments $|X|_\psi^p(t)$, one can easily show that

$$\int_1^{+\infty} dT(T+1)^{-1-\gamma}\langle |X|_\psi^p \rangle(T) \leqslant C \int_1^{+\infty} dt\, t^{-1-\gamma}|X|_\psi^p(t). \tag{C.7}$$

The first statement of the theorem follows from (C.5)–(C.7). The second statement follows directly from the first and the definition of $\gamma_\psi(p)$. The proof is completed. □

To understand better the statement of the theorem, assume that $U_s(x) = \sup_N S_N(x)N^{-s} < +\infty$ for all $x \in B$, where $\mu(B) > 0$. Then we get immediately for $N \geqslant 1$,

$$V(\gamma,N) \geqslant DN^{-s\gamma/\alpha}$$

with positive uniform constant D. Then Theorem C.1 yields

$$\gamma_\psi(p) \geqslant p\alpha/s,$$

which is virtually equivalent to the bound of Kiselev–Last [20]

$$\langle |X|_{\psi}^{p} \rangle (T) \geqslant C T^{p\alpha/s}$$

(this bound was established in [20] for a slightly larger class of α-continuous measures).

However, the function $V(\gamma, N)$ for a given $\gamma > 0$ may tend to 0 as $N \to +\infty$ slower than $N^{-s\gamma/\alpha}$ which would give larger lower bound for $\gamma_{\psi}(p)$. This lower bound may be nonlinear in p. To see why it may happen, consider the similar phenomenon in the case of integrals $I_{\mu}(q, \varepsilon)$. It is now well known that one may have

$$\lim_{\varepsilon \to 0} \frac{\log \mu([x - \varepsilon, x + \varepsilon])}{\log \varepsilon} = \alpha$$

for μ-a.e. x, but at the same time $D_{\mu}^{\pm}(q) > \alpha$ for some or even all $q \in (0, 1)$ (equivalently, $I_{\mu}(q, \varepsilon)$ grows faster than $\varepsilon^{\alpha(q-1)}$ as $\varepsilon \to 0$). This is possible due to the big contributions to the integral from the sets of energies of vanishing measure (as $\varepsilon \to 0$), where $\mu([x - \varepsilon, x + \varepsilon])/\varepsilon^{\alpha}$ is very small. Similar phenomenon may happen for $V(\gamma, N)$. If $S_N(x)/N^s$ is small on some set of vanishing measure, it may happen that $V(\gamma, N)$ tends to 0 slower than $N^{-s\gamma/\alpha}$ (although $S_N(x) \sim C(x) N^s$, $N \to +\infty$, for any fixed x). It would be interesting to check it, for example, in the case when the family $\mathscr{B}$ is obtained by orthonormalization of $H^k \psi$, because one has a good control of the generalized eigenfunctions $u_{\psi}(n, x)$ (orthogonal polynomials of the measure μ) and thus of $S_N(x)$.

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