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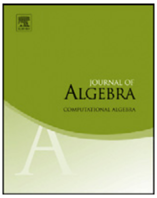

Counting Zariski chambers on Del Pezzo surfaces

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ABSTRACT

Zariski chambers provide a natural decomposition of the big cone of an algebraic surface into rational locally polyhedral subcones that are interesting from the point of view of linear series. In the present paper we present an algorithm that allows to effectively determine Zariski chambers when the negative curves on the surface are known. We show how the algorithm can be used to compute the number of chambers on Del Pezzo surfaces.

Introduction

In [2] it was shown that the big cone of an algebraic surface admits a natural locally finite decomposition into rational locally polyhedral subcones, the *Zariski chambers* on X. These chambers are of basic interest from the point of view of linear series on X: In the interior of each Zariski chamber the stable base loci are constant, and the volume function is given by a quadratic polynomial in each chamber. (See Section 1 for details on the chamber decomposition.) Understanding the behaviour of stable base loci and the volume function is also of great interest in the higher-dimensional case, where the picture is not as clear as for surfaces (see [3] and [4]).

It is an intriguing question to wonder into how many Zariski chambers the big cone decomposes on a given surface. In other words, we ask on a smooth projective surface X for the quantity

$$z(X) = \#\{\text{Zariski chambers on } X\} \in \mathbb{N} \cup \{\infty\}.$$

The number $z(X)$ is an interesting geometric invariant of the surface X, as it is the answer to the following questions (see Section 1):

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- How many different stable base loci can occur in big linear series on X?
- How many essentially different Zariski decompositions can big divisors on X have? (By "essentially different" we mean here that their negative parts have different support.)
- How many "pieces" does the volume function $\mathrm{vol}\colon \mathrm{Big}(X) \to \mathbb{R}$ have (which is a piecewise polynomial function)?

So, somewhat roughly speaking, one may think of the number $z(X)$ as measuring how complicated the surface is from the point of view of linear series.

In the present paper we provide an algorithm that allows to compute the invariant $z(X)$ whenever the irreducible curves of negative self-intersection on X are known. In particular, we will show how to apply the algorithm to Del Pezzo surfaces. Recall that a Del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, or a blow-up of $\mathbb{P}^2$ at $r \leqslant 8$ general points. As one clearly has $z(\mathbb{P}^1 \times \mathbb{P}^1) = 1$ and $z(\mathbb{P}^2) = 1$, it is enough to study the blow-ups. We show:

Theorem. *Let X_r be the blow-up of $\mathbb{P}^2$ in r general points with $1 \leqslant r \leqslant 8$.*

(i) *The number $z(X_r)$ of Zariski chambers on X_r is given by the following table*:

r	1	2	3	4	5	6	7	8
$z(X_r)$	2	5	18	76	393	2764	33 645	1 501 681

(ii) *The maximal number of curves that occur in the support of a Zariski chamber on X_r is r.*

As one might expect intuitively, the number of chambers increases as the Picard number $\rho(X_r) = r + 1$ increases. Note however that this is not automatic: On abelian surfaces, for instance, $\rho(X)$ varies between 1 and 4, but one has always $z(X) = 1$, since the intersection of the nef cone and the big cone is the only Zariski chamber. The same thing happens on suitable K3 surfaces: There are K3 surfaces X of any Picard number up to 11 with $z(X) = 1$ (see [6, Theorem 2]). On the other hand, if one considers the blow-up X_r of $\mathbb{P}^2$ in $r \geqslant 9$ general points, then the surface X_r (which is no longer a Del Pezzo surface) contains infinitely many (-1)-curves and therefore one has $z(X_r) = \infty$.

Our algorithm – to be discussed in Section 2 – is in no way specific to Del Pezzo surfaces. It applies to any surface where the irreducible curves with negative self-intersection are explicitly known. We plan to study further applications of this method in a subsequent paper.

1. Negative curves and chambers

Consider a smooth projective surface X. A divisor D on X is *big*, if its *volume*

$$\mathrm{vol}_X(D) \overset{\mathrm{def}}{=} \limsup_k \frac{h^0(X, kD)}{k^2/2}$$

is positive. The *big cone* $\mathrm{Big}(X)$ is the cone in the Néron–Severi vector space $\mathrm{NS}_{\mathbb{R}}(X)$ that is generated by the big divisors. To any big and nef $\mathbb{R}$-divisor P, one associates the *Zariski chamber* Σ_P, which by definition consists of all divisors in $\mathrm{Big}(X)$ such that the irreducible curves in the negative part of the Zariski decomposition of D are precisely the curves C with $P \cdot C = 0$. It is shown in [2, Lemma 1.6] that for any two big and nef divisors P and P', the Zariski chambers Σ_P and $\Sigma_{P'}$ are either equal or disjoint. So the Zariski chambers yield a decomposition of the big cone. If A is an ample divisor, then the chamber Σ_A is the intersection of the big cone and the nef cone, and its interior is the ample cone; in the sequel we call it the *nef chamber* for short. The main result of [2] states that the decomposition into Zariski chambers is a locally finite decomposition of $\mathrm{Big}(X)$ into rational locally polyhedral subcones, such that

- on each chamber the volume function is given by a single polynomial of degree two, and

- in the interior of each chamber the stable base loci are constant. (See Proposition 1.3 below for the general statement.)

The following characterization will be essential for our purposes.

Proposition 1.1. *The set of Zariski chambers on a smooth projective surface X that are different from the nef chamber is in bijective correspondence with the set of reduced divisors on X whose intersection matrix is negative definite.*

Proof. Given a chamber Σ_P, we consider the irreducible curves $C_1, \dots, C_r$ with $P \cdot C_i = 0$. Then the divisor $C_1 + \cdots + C_r$ has negative definite intersection matrix thanks to the index theorem.

Conversely, given a reduced divisor $C_1 + \cdots + C_r$ with negative definite intersection matrix, we consider the divisor

$$D \overset{\text{def}}{=} H + k(C_1 + \cdots + C_r),$$

where H is a fixed ample divisor and k a positive integer. This divisor is big, and we claim that for $k \gg 0$ the negative part of its Zariski decomposition will have $C_1 \cup \cdots \cup C_r$ as its support. The latter fact can for instance be seen from the computation of the Zariski decomposition according to [1]. Alternatively, consider the linear system of equations

$$\left(H + \sum_{i=1}^{r} a_i C_i\right) C_j = 0, \quad j = 1, \dots, r, \tag{1.1.1}$$

with unknowns $a_1, \dots, a_r$. If S denotes the intersection matrix $(C_i \cdot C_j)_{i,j}$, then the unique solution of (1.1.1) is given by

$$\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = -S^{-1} \begin{pmatrix} H \cdot C_1 \\ \vdots \\ H \cdot C_r \end{pmatrix}.$$

As S is by assumption negative definite, it follows that all entries of S^{-1} are $\leqslant 0$ (see [2, Lemma 4.1]), and consequently we have $a_i \geqslant 0$ for all i. The divisor $H + \sum_{i=1}^{r} a_i C_i$ is then for $k \gg 0$ clearly an effective and nef $\mathbb{Q}$-subdivisor of $H + k \sum_{i=1}^{r} C_i$ having zero intersection with all C_i. By the uniqueness of Zariski decompositions, it follows that it is the positive part in the Zariski decomposition of $H + k \sum_{i=1}^{r} C_i$, and therefore the negative part has support $C_1 \cup \cdots \cup C_r$, as claimed. □

Remark 1.2. Note that the divisor $D = H + k(C_1 + \cdots + C_r)$ that is considered in the proof of Proposition 1.1 lies in the *interior* of the chamber that corresponds to $C_1 + \cdots + C_r$. In fact, write $D = P + N$ for its Zariski decomposition, and suppose that D lies on the boundary of a chamber. Then by [2, Proposition 1.7] there must exist an irreducible curve $C \subset X$ with $P \cdot C = 0$ that does not occur as a component of N. But as P is of the form $H + a_1 C_1 + \cdots + a_r C_r$ with H ample, it is clear that $P \cdot C = 0$ can happen only if C is among the curves C_i. However, all of them are components of N.

The next statement justifies the claim made in the introduction to the effect that counting Zariski chambers is equivalent to counting stable base loci of big linear series. By way of notation, we write $\mathrm{Bs}(|D|)$ for the base locus of the linear series $|D|$, and

$$\mathbf{B}(D) \overset{\text{def}}{=} \bigcap_{m=1}^{\infty} \mathrm{Bs}\bigl(|mD|\bigr)$$

for the stable base locus of D.

Proposition 1.3. *The set of Zariski chambers on a smooth projective surface X is in bijective correspondence with the set of stable base loci that occur in big linear series on X.*

Proof. As we already said above, it follows from [2] that for a divisor D that lies in the interior of a Zariski chamber, the stable base locus $\mathbf{B}(D)$ coincides with the support of the negative part of the Zariski decomposition of D. The point to show is therefore that the big divisors whose numerical classes lie on boundaries of Zariski chambers cannot lead to stable base loci that have not been accounted for by the divisors in the interior of chambers. To see that latter, suppose that D is a big divisor on X. If A is any ample $\mathbb{Q}$-divisor A, then we have

$$\mathbf{B}(D) \subset \mathbf{B}(D-A). \tag{1.3.1}$$

For a suitable choice of A, the numerical class of the divisor $D-A$ does not lie on the boundary of any chamber. Moreover, as D is big, $D-A$ is still big when A is sufficiently small. As $D-A$ then lies in the interior of a Zariski chamber, $\mathbf{B}(D-A)$ is the support of the negative part of a Zariski decomposition, and hence it is the support of a divisor $C_1+\cdots+C_r$ with negative definite intersection matrix. But then $\mathbf{B}(D)$ is by (1.3.1) a subdivisor of this divisor, and hence has negative definite intersection matrix as well. By Proposition 1.1 this divisor corresponds to a Zariski chamber, and hence has been accounted for already. □

Remark 1.4. Note that in general the stable base locus $\mathbf{B}(D)$ does not depend only on the numerical equivalence class of D (see [7, Example 10.3.3]). In order to get a function on the big cone, one considers *augmented* base loci instead (see [7, Section 10.3]). In light of this fact it is even more surprising that by Proposition 1.3 all stable base loci on surfaces are accounted for by the Zariski chambers. For instance, in the cited example [7, 10.3.3] one has two numerically equivalent big and nef divisors D_1 and D_2 such that $\mathbf{B}(D_1)=\emptyset$ and $\mathbf{B}(D_2)$ is a curve. According to Proposition 1.3 these stable base loci correspond to two distinct Zariski chambers.

Our aim now is to study the number $z(X)$ of Zariski chambers on X. By way of terminology, the term *negative curve* will always mean an irreducible curve with negative self-intersection. Two things about $z(X)$ are clear from the outset:

(1) If X carries only a finite number N of negative curves, then one has the trivial upper bound

$$z(X) \leqslant 2^N.$$

Intuitively, it seems unlikely that $z(X)$ is equal (or close) to this upper bound, as this would mean that every (or almost every) set of negative curves occurs in a stable base locus.

(2) We have $z(X)=\infty$ if and only if there are infinitely many negative curves on X. The blow-up of $\mathbb{P}^2$ in $\geqslant 9$ general points gives such an example.

When the negative curves on X are known explicitly, then there is a way to effectively determine the number $z(X)$. To formulate the enumerative statement, we will use for a given $(n\times n)$-matrix the notion *principal submatrix* to mean as usual a submatrix that arises by deleting k corresponding rows and columns of the matrix, where $0\leqslant k<n$. The following is then an immediate consequence of Proposition 1.1:

Proposition 1.5. *Let X be a smooth projective surface that contains only finitely many negative curves.*

(i) *We have*

$$z(X) = 1 + \#\left\{\begin{array}{l}\textit{negative definite principal submatrices}\\ \textit{of the intersection matrix of the negative curves on } X\end{array}\right\}.$$

(ii) *More generally, let $C_1, \dots, C_r$ be distinct negative curves on X, and let S be their intersection matrix. Then the number of Zariski chambers that are supported by a non-empty subset of $\{C_1, \dots, C_r\}$ equals the number of negative definite principal submatrices of the matrix S.*

Strictly speaking, it is of course not actually the submatrices themselves that are to be counted, but the subsets of the index set $\{1, \dots, r\}$ that give rise to the submatrices. Nonetheless, we will generally use this shorter formulation in the sequel. Also, note that the "$1+$" in (i) accounts for the nef chamber.

Remark 1.6. Looking at Proposition 1.5, one would wish for a general matrix-theoretic result that gives information about the number of negative definite principal submatrices in terms of other (easier accessible) quantities associated with the matrix. It seems however that no results in this direction are available so far. Not even is it clear which quantities might be of relevance: The probably most naive guess might be to consider the signature (p, n) of the matrix, where p is the number of positive and n the number of negative eigenvalues. However, as the following two examples show, one cannot expect useful bounds in terms of the signature.

(i) Consider the matrix A that is diagonally composed of a $k \times k$ unit matrix and the negative of an $\ell \times \ell$ unit matrix. Its signature is $(p, n) = (k, \ell)$, and it has exactly $2^k - 1$ positive definite principal submatrices.

(ii) On the other hand, take A to be diagonally composed of a $k \times k$ unit matrix and ℓ copies of the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

It has the same number $2^k - 1$ of positive definite principal submatrices, but its signature is $(p, n) = (k + \ell, \ell)$.

So while in (i) the number of positive definite principal submatrices depends only on p, it depends in (ii) on the difference $p - n$.

2. Computing chambers

Proposition 1.5 suggests a way to effectively determine Zariski chambers when the numerical classes of the negative curves are explicitly known: Each negative definite principal submatrix of the intersection matrix of the negative curves corresponds to a chamber, supported by the curves that are represented by the chosen rows and columns. Determining the negative definite submatrices is however in practice not at all immediate: If there are many negative curves, then such work cannot be done by hand. And even when carried out by computer, it is not a viable course of action to apply brute force and check *all* submatrices for negative definiteness: For instance, on the Del Pezzo surface X_8 there are 2^{240} potential submatrices. Our algorithm exploits the following two observations, which drastically reduce the complexity of the computation:

(1) Let A be the intersection matrix of n negative curves. If the principal submatrix A_S corresponding to a subset $S \subset \{1, \dots, n\}$ is not negative definite, then none of the subsets S' with $S' \supset S$ need to be examined, since they cannot be negative definite. One can therefore use a backtracking strategy.

(2) Let S be a subset and let T be the set obtained from S by removing its largest element. If the subsets are treated in such an order that S is only examined after A_T has turned out to be negative definite, then the negative definiteness of A_S can be read off the sign of its determinant.

The algorithm below generates all *positive definite* principal submatrices of a given symmetric matrix. It will subsequently be applied to the negative of the intersection matrix.

Algorithm 2.1. The algorithm takes as input an integer $n \geqslant 1$ and a symmetric $(n \times n)$-matrix A over $\mathbb{R}$. It outputs all subsets $S \subset \{1, \dots, n\}$ having the property that the corresponding principal submatrix A_S is positive definite.

```
input $n$, $A$
$k \leftarrow 1$
$S \leftarrow \{1\}$
while $S \neq \emptyset$ do
  assert($k = \max S$ and $A_{S \setminus \{k\}}$ is positive definite)
  if $\det A_S > 0$ then
    output $S$
  else
    $S \leftarrow S \setminus \{k\}$
  end if
  assert($k \geqslant \max S$ and $A_S$ is positive definite)
  if $k < n$ then
    $k \leftarrow k + 1$
    $S \leftarrow S \cup \{k\}$
  else
    $S \leftarrow S \setminus \{k\}$
    if $S \neq \emptyset$ then
      $k \leftarrow \max S$
      $S \leftarrow S \setminus \{k\}$
      $k \leftarrow k + 1$
      $S \leftarrow S \cup \{k\}$
    end if
  end if
end while
```

Remark 2.2. The gain in efficiency compared to checking *all* principal submatrices is considerable – and in fact crucial for the algorithm to be practical at all. For instance, on the Del Pezzo surface X_6 the algorithm checks only 15 600 submatrices instead of all $2^{27} = 134\,217\,728$ submatrices, which means reducing cases to about 0.01 percent.

Proof of correctness and termination. Note first that the two assertions made within the loop are true whenever the algorithm reaches them (the empty matrix being considered positive definite). Therefore the condition that A_S be positive definite is equivalent to $\det A_S > 0$. We now have to show that the algorithm terminates and that it outputs precisely the claimed subsets. Readers familiar with backtracking algorithms might rather quickly understand the strategy of Algorithm 2.1 and can argue from there. For readers not versed in these matters we will provide an explicit alternative view as follows.

For index sets $S, S' \subset \{1, \dots, n\}$ we write $S < S'$ if for some integer $\ell \geqslant 0$ we have

$$S \cap \{1, \dots, \ell\} = S' \cap \{1, \dots, \ell\}$$

and

$$\min\bigl(S \setminus \{1, \dots, \ell\}\bigr) < \min\bigl(S' \setminus \{1, \dots, \ell\}\bigr),$$

where we set $\min(\emptyset) = -\infty$. It is immediate that "$<$" is a strict total order on the set of subsets of $\{1, \dots, n\}$. Correctness and termination follow then from the two following claims.

(i) Loop invariant: At the beginning and at the end of each loop cycle all index sets $T < S$ have been output for which A_T is positive definite.
(ii) At the end of each loop cycle either the value of S is strictly bigger than at the beginning, or $S = \emptyset$ (in which case it is the last cycle).

To verify this, let S_1 and S_2 be the values of the variable S at the beginning and at the end of a loop cycle respectively, and write $S_1 = \{i_1, \dots, i_m\}$ with $i_1 < \dots < i_m$. Then we have

$$\begin{aligned} &S_2 = \{i_1, \dots, i_m, i_m + 1\} && \text{if } i_m < n \text{ and } A_{S_1} \text{ is positive definite,} \\ &S_2 = \{i_1, \dots, i_{m-1}, i_m + 1\} && \text{if } i_m < n \text{ and } A_{S_1} \text{ is not positive definite,} \\ &S_2 = \{i_1, \dots, i_{m-2}, i_{m-1} + 1\} \text{ or } S_2 = \emptyset && \text{if } i_m = n. \end{aligned} \tag{2.2.1}$$

So we have $S_2 > S_1$ or $S_2 = \emptyset$ in each case, which proves claim (ii). As for claim (i): The algorithm clearly outputs S_1, if A_{S_1} is positive definite. Further, one sees from (2.2.1) that there is no set T with $S_1 < T < S_2$ unless $i_m < n$ and A_{S_1} is not positive definite. In the latter case, all sets T with $S_1 < T < S_2$ are supersets of S_1, and hence none of the corresponding matrices A_T can be positive definite. □

3. Del Pezzo surfaces

Our aim is now to apply Algorithm 2.1 to the Del Pezzo surfaces X_r for $1 \leqslant r \leqslant 8$, which are the blow-ups of $\mathbb{P}^2$ at r general points. To this end, we first need to describe all negative curves on the surfaces X_r. They have been determined by Manin:

Theorem 3.1. *(See Manin [8, Chapter IV].) The negative curves on X_r are*

(1) *the exceptional divisors corresponding to the blown-up points $p_1, \dots, p_r$*

and the proper transforms of the following curves in $\mathbb{P}^2$:

(2) *the lines through pairs of points p_i, p_j;*
(3) *if $r \geqslant 5$, the conics through 5 points from $p_1, \dots, p_r$;*
(4) *if $r \geqslant 7$, the cubics through 7 points from $p_1, \dots, p_r$ with a double point in one of them;*
(5) *if $r = 8$, the quartics through the 8 points $p_1, \dots, p_8$ with double points in 3 of them;*
(6) *if $r = 8$, the quintics through the 8 points $p_1, \dots, p_8$ with double points in 6 of them;*
(7) *if $r = 8$, the sextics through the 8 points $p_1, \dots, p_8$ with double points in 7 of them and a triple point in one of them.*

The proof in [8] works from the more general perspective of root systems. We believe that it can also be useful to have a very quick argument for this basic result in the spirit of [5, Theorem V.4.9], and we provide such an argument below. Since we will at any rate need to describe the classes of the negative curves and their intersection behaviour for our purposes, doing so means little additional effort.

Proof. (i) We start by showing that negative curves as asserted in (2) to (7) exist. An immediate dimension count shows that on $\mathbb{P}^2$ there are in any event effective *divisors* (which may be reducible) having *at least* the indicated multiplicities. Writing $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, $E_i = \pi^{-1}(p_i)$, and $E = E_1 + \dots + E_r$, these divisors on $\mathbb{P}^2$ correspond to effective divisors in the following linear series on X_r:

$$C_{ij}^{(1)} = H - E_i - E_j \qquad 1 \leqslant i < j \leqslant r,$$

$$\begin{aligned}
C^{(2)} &= 2H - E && (\text{if } r = 5),\\
C_i^{(2)} &= 2H - E + E_i && 1 \leqslant i \leqslant 6 \quad (\text{if } r = 6),\\
C_{ij}^{(2)} &= 2H - E + E_i + E_j && 1 \leqslant j < j \leqslant 7 \quad (\text{if } r = 7),\\
C_{ijk}^{(2)} &= 2H - E + E_i + E_j + E_k && 1 \leqslant i < j < k \leqslant 8 \quad (\text{if } r = 8),\\
C_i^{(3)} &= 3H - E - E_i && 1 \leqslant i \leqslant 7 \quad (\text{if } r = 7),\\
C_{ij}^{(3)} &= 3H - E - E_i + E_j && 1 \leqslant i, j \leqslant 8,\ i \neq j \quad (\text{if } r = 8),\\
C_{ijk}^{(4)} &= 4H - E - E_i - E_j - E_k && 1 \leqslant i < j < k \leqslant 8 \quad (\text{if } r = 8),\\
C_{ij}^{(5)} &= 5H - 2E + E_i + E_j && 1 \leqslant i < j \leqslant 8 \quad (\text{if } r = 8),\\
C_i^{(6)} &= 6H - 2E - E_i && 1 \leqslant i \leqslant 8 \quad (\text{if } r = 8). \qquad (3.1.1)
\end{aligned}$$

The point is to show that these divisors are irreducible. To see this, one checks first that if C is any of these divisors, then one has

$$C^2 = -1 \quad \text{and} \quad -K_{X_r} \cdot C = 1. \qquad (3.1.2)$$

As $-K_{X_r}$ is ample, the second equation implies then that C must be irreducible. In particular, its image curve on $\mathbb{P}^2$ has exactly the asserted multiplicities.

(ii) It remains to show that the curves in (1) to (7) are the only negative curves on X_r. So suppose that $C \subset X_r$ is any negative curve that is different from the exceptional curves of the blow-up. Via the adjunction formula it follows from the ampleness of $-K_{X_r}$ that Eqs. (3.1.2) hold for C. One has $C \in |dH - \sum_{i=1}^{r} m_i E_i|$ for suitable integers $d \geqslant 1$ and $m_i \geqslant 0$. We claim that

$$\begin{aligned}
d &\leqslant 2 \quad \text{if } r \leqslant 6,\\
d &\leqslant 3 \quad \text{if } r = 7,\\
d &\leqslant 6 \quad \text{if } r = 8. \qquad (3.1.3)
\end{aligned}$$

To prove (3.1.3), note first that Eqs. (3.1.2) translate to

$$d^2 - \sum m_i^2 = -1 \quad \text{and} \quad 3d - \sum m_i = 1. \qquad (3.1.4)$$

Upon combining these equations with the Cauchy–Schwarz inequality

$$\left(\sum_{i=1}^{r} m_i\right)^2 \leqslant r \sum_{i=1}^{r} m_i^2,$$

we get a quadratic equation for d, which in turn implies $d \leqslant 2$ for $r \leqslant 6$, as well as $d \leqslant 3$ for $r = 7$ and $d \leqslant 7$ for $r = 8$. So the claim (3.1.3) will be established as soon as we can rule out the possibility that $d = 7$ and $r = 8$. In that case we would have equality in the Cauchy–Schwarz inequality, and therefore $m_1 = \cdots = m_8$. But then (3.1.4) would imply $m_i = 5/2$, which is impossible.

To complete the proof, one checks now that Eqs. (3.1.4) have only the solutions corresponding to the classes in (3.1.1). This can be done by trial, since the bounds (3.1.3) on d leave only finitely many possibilities for the integers m_i. □

$$\begin{pmatrix}
-1&0&1&0&1&0&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&1&1&1&1\\
0&-1&1&0&0&1&0&0&1&0&0&0&1&0&0&1&0&0&1&0&0&0&1&0&1&1&1\\
1&1&-1&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&1&1&1&1&0&0&0\\
0&0&0&-1&1&1&0&0&0&1&0&0&0&1&0&1&0&0&0&1&0&0&1&1&0&1&1\\
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0&0&1&1&0&0&1&0&0&-1&0&1&1&0&0&0&0&1&1&0&0&1&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&-1&1&1&1&1&1&0&0&0&0&0&1&1&1&1&1&0\\
1&0&0&0&0&1&0&0&1&1&1&-1&0&0&0&0&0&0&1&1&1&0&1&0&0&0&1\\
0&1&0&0&1&0&0&1&0&1&1&0&-1&0&0&0&0&1&0&1&1&0&0&1&0&0&1\\
0&0&1&1&0&0&0&1&1&0&1&0&0&-1&0&0&0&1&1&0&1&0&0&0&1&0&1\\
0&0&1&0&1&1&1&0&0&0&1&0&0&0&-1&0&0&1&1&1&0&0&0&0&0&1&1\\
1&1&0&1&0&0&1&0&0&0&1&0&0&0&0&-1&0&1&1&1&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&1&1&1&1&1&1&1&1&1&1\\
1&0&0&0&0&1&0&0&1&1&0&0&1&1&1&1&1&-1&0&0&0&0&1&0&0&0&0\\
0&1&0&0&1&0&0&1&0&1&0&1&0&1&1&1&1&0&-1&0&0&0&0&1&0&0&0\\
0&0&1&1&0&0&0&1&1&0&0&1&1&0&1&1&1&0&0&-1&0&0&0&0&1&0&0\\
0&0&1&0&1&1&1&0&0&0&0&1&1&1&0&1&1&0&0&0&-1&0&0&0&0&1&0\\
0&0&1&0&1&1&0&1&1&1&1&0&0&0&0&1&1&0&0&0&0&-1&0&0&0&0&1\\
0&1&1&1&1&0&1&1&0&0&1&1&0&0&0&0&1&1&0&0&0&0&-1&0&0&0&0\\
1&0&1&1&0&1&1&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&-1&0&0&0\\
1&1&0&0&1&1&1&0&0&1&1&0&0&1&0&0&1&0&0&1&0&0&0&0&-1&0&0\\
1&1&0&1&0&0&0&1&1&1&1&0&0&0&1&0&1&0&0&0&1&0&0&0&0&-1&0\\
1&1&0&1&0&0&1&0&0&0&0&1&1&1&0&1&0&0&0&0&1&0&0&0&0&0&-1
\end{pmatrix}$$

Fig. 1. The intersection matrix A_6 of the 27 lines on a smooth cubic surface, obtained as a submatrix of A_8 as described in Section 4.

One sees from (3.1.1) that the number N of negative curves on X_r is given by the following table:

$$\begin{array}{c|cccccccc} r & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline N & 1 & 3 & 6 & 10 & 16 & 27 & 56 & 240 \end{array} \tag{3.1.5}$$

4. Proof of the theorem

We now turn to the proof of the theorem stated in the introduction. We start by determining the intersection products of the negative curves on X_r. Note that it is enough to write down the intersection matrix A_8 of the negative curves on the surface X_8: The intersection matrices A_r for the surfaces X_r, $r < 8$, can then be obtained by taking the principal submatrices corresponding to those curves whose classes are contained in $\mathbb{Z} \cdot [H] \oplus \bigoplus_{i=1}^{r} [E_i]$.

In order to get a compact statement that is suitable for computations, we will use for tuples of integers $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_n)$ the abbreviation

$$(i_1, \ldots, i_m) * (j_1, \ldots, j_n) = \sum_{\substack{\mu=1,\ldots,m \\ \nu=1,\ldots,n}} \operatorname{sign}(i_\mu) \cdot \operatorname{sign}(i_\nu) \cdot \delta_{|i_\mu||j_\nu|},$$

where δ is the Kronecker delta. Keeping the notation for the curves on X_8 and the index ranges as in (3.1.1), we find:

$$E_i \cdot E_\ell = (-i) * (\ell), \qquad C^{(2)}_{ijk} \cdot C^{(3)}_{\ell m} = 1 + (i, j, k) * (\ell, -m),$$

$$E_i \cdot C^{(1)}_{\ell m} = (i) * (\ell, m), \qquad C^{(2)}_{ijk} \cdot C^{(4)}_{\ell mn} = (i, j, k) * (\ell, m, n),$$

$$E_i \cdot C^{(2)}_{\ell mn} = 1 - (i) * (\ell, m, n), \qquad C^{(2)}_{ijk} \cdot C^{(5)}_{\ell m} = 2 - (i, j, k) * (\ell, m),$$

$$E_i \cdot C^{(3)}_{\ell m} = 1 + (i) * (\ell, -m), \qquad C^{(2)}_{ijk} \cdot C^{(6)}_{\ell} = 1 + (i, j, k) * (l),$$

$$\begin{aligned}
&E_i \cdot C^{(4)}_{\ell mn} = 1 + (i) * (\ell, m, n), && C^{(3)}_{ij} \cdot C^{(3)}_{\ell m} = 1 + (-i, j) * (\ell, -m), \\
&E_i \cdot C^{(5)}_{\ell m} = 2 - (i) * (\ell, m), && C^{(3)}_{ij} \cdot C^{(4)}_{\ell mn} = 1 + (-i, j) * (\ell, m, n), \\
&E_i \cdot C^{(6)}_{\ell} = 2 + (i) * (\ell), && C^{(3)}_{ij} \cdot C^{(5)}_{\ell m} = 1 + (i, -j) * (\ell, m), \\
&C^{(1)}_{ij} \cdot C^{(1)}_{\ell m} = 1 - (i, j) * (\ell, m), && C^{(3)}_{ij} \cdot C^{(6)}_{\ell} = 1 + (-i, j) * (\ell), \\
&C^{(1)}_{ij} \cdot C^{(2)}_{\ell mn} = (i, j) * (\ell, m, n), && C^{(4)}_{ijk} \cdot C^{(4)}_{\ell mn} = 2 - (i, j, k) * (\ell, m, n), \\
&C^{(1)}_{ij} \cdot C^{(3)}_{\ell m} = 1 + (i, j) * (-\ell, m), && C^{(4)}_{ijk} \cdot C^{(5)}_{\ell m} = (i, j, k) * (\ell, m), \\
&C^{(1)}_{ij} \cdot C^{(4)}_{\ell mn} = 2 - (i, j) * (\ell, m, n), && C^{(4)}_{ijk} \cdot C^{(6)}_{\ell} = 1 - (i, j, k) * (\ell), \\
&C^{(1)}_{ij} \cdot C^{(5)}_{\ell m} = 1 + (i, j) * (\ell, m), && C^{(5)}_{ij} \cdot C^{(5)}_{\ell m} = 1 - (i, j) * (\ell, m), \\
&C^{(1)}_{ij} \cdot C^{(6)}_{\ell} = 2 - (i, j) * (\ell), && C^{(5)}_{ij} \cdot C^{(6)}_{\ell} = (i, j) * (\ell), \\
&C^{(2)}_{ijk} \cdot C^{(2)}_{\ell mn} = 2 - (i, j, k) * (\ell, m, n), && C^{(6)}_{i} \cdot C^{(6)}_{\ell} = (-i) * (\ell).
\end{aligned}$$

The preceding formulas determine the intersection matrix A_8, which is of dimension 240. As described above, the matrices A_r for $r = 1, \dots, 7$ are obtained as submatrices thereof. They are of dimension 1, 3, 6, 10, 16, 27, and 56 respectively (see (3.1.5)). As an example, we display the matrix A_6 in Fig. 1. Using Algorithm 2.1, applied to the matrix $-A_r$, we obtain the number of negative definite principal submatrices of A_r:

r	1	2	3	4	5	6	7	8
#	1	4	17	75	392	2763	33 644	1 501 680

Proposition 1.5 then gives part (i) of the theorem. With an obvious modification of Algorithm 2.1 we obtain in each case also the maximal cardinality of the positive definite index sets, which shows that for each r there are positive definite principal submatrices of $-A_r$ of dimension r. This proves part (ii) of the theorem.

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References

[1] Th. Bauer, A simple proof for the existence of Zariski decompositions on surfaces, J. Algebraic Geom. 18 (2009) 789–793.
[2] Th. Bauer, A. Küronya, T. Szemberg, Zariski chambers, volumes, and stable base loci, J. Reine Angew. Math. 576 (2004) 209–233.
[3] L. Ein, R. Lazarsfeld, M. Mustata, M. Nakamaye, M. Popa, Asymptotic invariants of line bundles, Pure Appl. Math. Q. 1 (2) (2005) 379–403.
[4] L. Ein, R. Lazarsfeld, M. Mustata, M. Nakamaye, M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier (Grenoble) 56 (6) (2006) 1701–1734.
[5] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math., vol. 52, Springer-Verlag, New York, Berlin, Heidelberg, 1977.
[6] S.J. Kovács, The cone of curves of a K3 surface, Math. Ann. 300 (1994) 681–691.
[7] R. Lazarsfeld, Positivity in Algebraic Geometry II, Springer-Verlag, 2004.
[8] Y.I. Manin, Cubic Forms. Algebra, Geometry, Arithmetic, North-Holland Math. Library, vol. 4, North-Holland, 1974.