ELSEVIER

SCIENCE DIRECT®

PHYSICS LETTERS B

Physics Letters B 564 (2003) 111–114

www.elsevier.com/locate/npe

Tunnelling amplitudes from perturbation expansions

B. Hamprecht, H. Kleinert

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

Received 20 March 2003; received in revised form 27 March 2003; accepted 28 March 2003

Editor: P.V. Landshoff

Abstract

We present a method for extracting tunnelling amplitudes from perturbation expansions which are always divergent and not Borel-summable. We show that they can be evaluated by an analytic continuation of variational perturbation theory. The power of the method is illustrated by calculating the imaginary parts of the partition function of the anharmonic oscillator in zero spacetime dimensions and of the ground state energy of the anharmonic oscillator for all negative values of the coupling constant g and show that they are in excellent agreement with the exactly known values. As a highlight of the theory we recover from the divergent perturbation expansion of the tunnelling amplitude the action of the instanton and the effects of higher loop fluctuations around it.

1. Tunnelling processes govern many important physical phenomena. Their theoretical description requires the calculation of the contribution of critical bubbles to the partition function, including their fluctuation entropy [1]. The latter can be found at most to one-loop order. Higher loop effects are prohibitively complicated [2]. It would be of great advantage tunnelling amplitudes could be derived from ordinary perturbation expansions around the free theory since these can be performed to many loops [3]. The difficulty arising in such a program is that tunnelling amplitudes are described by the analytic continuation of divergent Borel-summable power series expansions in the coupling constant g to negative g where they become non-Borel-summable. None of the currently known resummation schemes [3,4] is able to deal with such expansions. Some time ago it was suggested that a resummation is possible by variational perturbation theory [5]. However, the imaginary parts calculated there gave accurate imaginary parts only in the *sliding regime* of larger negative g and did not invade into the proper tunnelling regime of small g dominated by critical bubbles. A separate variational treatment of an instanton calculation was set up to cover this region [6].

In this Letter we show that the non-Borel-summable series can be evaluated with any desired precision by an appropriate continuation of variational perturbation theory [2,3] in to the complex g-plane.

Variational perturbation theory has a long history [7–10]. It is based on the introduction of a dummy variational parameter Ω on which the full perturbation expansion does not depend, while the truncated expansion does. An optimal Ω is then selected by the principle of minimal sensitivity [11], requiring the quan-

E-mail addresses: bodo.hamprecht@physik.fu-berlin.de (B. Hamprecht), hagen.kleinert@physik.fu-berlin.de (H. Kleinert).

0370-2693
doi:10.1016/S0370-2693(03)00489-1

tity of interest to be stationary as function of the variational parameter. The optimal Ω is usually taken from a zero of the derivative with respect to Ω. If the first derivative has no zero, a zero of the second derivative is chosen. For Borel-summable series, these zeros are always real, in contrast to statements in the literature [12–15] advocating the use of complex zeros. Complex zeros, however, produce in general wrong results for Borel-summable series, as was recently demonstrated in Ref. [16].

The purpose of this Letter is to show that there does exist a wide range of applications of complex zeros in the previously untreatable field of non-Borel-summable series. These arise typically in tunnelling problems, and we shall see that variational perturbation theory provides us with an efficient method for evaluating these series and rendering their real and imaginary parts with any desirable accuracy if only enough perturbation coefficients are available. The choice of the complex zeros is dictated by the requirement to achieve at each order the least oscillating imaginary part when approaching the tip of the cut. We shall call this selection rule the *principle of minimal sensitivity and oscillations*.

2. For an introduction to the method consider the exactly known partition function of an anharmonic oscillator in zero spacetime dimensions, which is a simple integral representation of a modified Bessel function $K_\nu(z)$:

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp\left(-x^2/2 - gx^4/4\right) = e^{1/8g}(4\pi g)^{-1/2} K_{1/4}(1/8g). \tag{2.1}$$

For small $g < 0$, the function $Z(g)$ and its inverse $D(g) \equiv Z^{-1}(g)$ have a divergent non-Borel-summable power series:

$$D(g) = \sum_{l=0}^{\infty} a_l g^l. \tag{2.2}$$

In the strong-coupling regime, there exists a convergent expansion

$$D(g) = g^\alpha \sum_{l=0}^{\infty} b_l g^{-\omega l}, \tag{2.3}$$

with $\alpha = 1/4$ and $\omega = 1/2$. In the context of critical phenomena, the exponent ω coincides with the Wegner exponent [17] of approach to scaling [18]. The Lth variational approximation depending on the variational parameter Ω is given by the truncated series [2,3]

$$D_{\rm var}^{(L)}(g, \Omega) = \Omega^p \sum_{j=0}^{L} \left(\frac{g}{\Omega^q}\right)^j \epsilon_j(\sigma), \tag{2.4}$$

where $q = 2/\omega = 4$, $p = \alpha q = 1$. Introducing the parameter $\sigma = \Omega^{q-2}(\Omega^2 - 1)/g$, the re-expansion coefficients are

$$\epsilon_j(\sigma) = \sum_{l=0}^{j} a_l \binom{(p - lq)/2}{j - l} (-\sigma)^{j-l}. \tag{2.5}$$

Following the principle of minimal sensitivity, we have to find the zeros of the derivative of $\partial_\Omega D_{\rm var}^{(L)}(g, \Omega)$, which happen to coincide with the zeros of a function of the variable σ only, to be denoted by $\zeta^{(L)}(\sigma)$:

$$\zeta^{(L)}(\sigma) = \sum_{l=0}^{L} a_l \binom{(p - lq)/2}{L - l} (p - lq + 2l - 2L) \times (-\sigma)^{L-l}. \tag{2.6}$$

For a proof of this remarkable property see [19], and the textbook [2, p. 291].

As an example, we take the weak-coupling expansion to $L = 16$th order and calculate real and imaginary parts for the non-Borel region $-2 < g < -0.008$ selecting the zero of $\zeta^{(16)}(\sigma)$ according to the principle of minimal sensitivity and oscillations. The result is shown in Fig. 1. In order to point out how well the variational result approximates the essential singularity of the imaginary part $\propto \exp(-1/4g)$ for small $g < 0$, we have removed this factor. The agreement is excellent down to very small $-g$.

3. Let us now turn to the non-trivial problem of summing the instanton region of the anharmonic oscillator for $g < 0$. The divergent weak-coupling perturbation expansion for the ground state energy of the anharmonic oscillator with the potential $V(x) =$

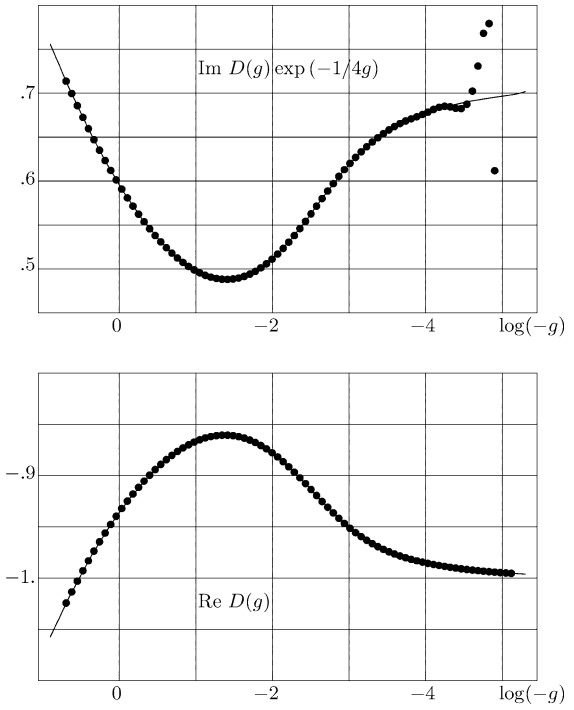

Fig. 1. Imaginary and real parts from variational perturbation theory of 16th order $D^{(16)}_{\rm var}(g)$ as a function of $\log(-g)$ (dots) in comparison with exact curves (curves). In the imaginary part we have removed the leading essential singularity by dividing out a factor $\exp(1/4g)$ to see the amazing accuracy with which this singularity is approximated. For very small g the onset of oscillations in the imaginary part can be seen which moves towards the origin for increasing order L of the variational approximation.

$x^2/2 + gx^4$ is, to order L:

$$E^{(L)}_{0,\rm weak}(g) = \sum_{l=0}^{L} a_l g^l, \tag{3.1}$$

where $a_l = (1/2, 3/4, -21/8, 333/16, -30885/128, \ldots)$. The expansion is obviously not Borel-summable for $g < 0$, but will now be evaluated with our new technique, proceeding in the same way as for the above test function $D(g)$ via Eqs. (2.4)–(2.6). The known strong-coupling growth parameters are $\alpha = 1/3$ and $\omega = 2/3$, so that $p = 1$ and $q = 3$ in Eq. (2.6) which will guarantee the correct scaling properties for $g \to \infty$. To order $L = 64$ we obtain from the optimal zero of $\zeta^{(64)}(\sigma)$ the logarithm of the scaled imaginary part

$$l(g) := \log\bigl[\sqrt{-\pi g/2}\, E^{(64)}_{0,\rm var}(g)\bigr] - 1/3g, \tag{3.2}$$

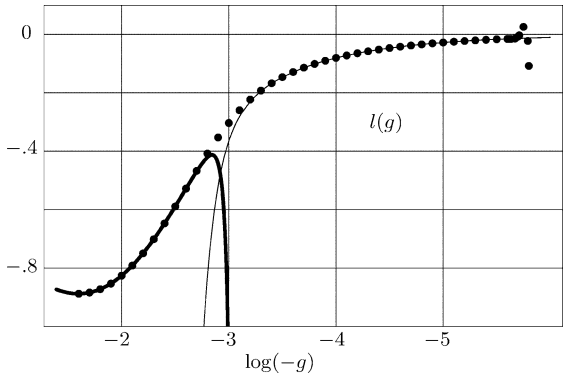

Fig. 2. Logarithm of the imaginary part of the ground state energy of the anharmonic oscillator with the essential singularity factored out for better visualization, $l(g) = \log[\sqrt{-\pi g/2}\, E^{(64)}_{0,\rm var}(g)] - 1/3g$, plotted against small negative values of the coupling constant $-0.2 < g < -0.006$ where the series is non-Borel-summable. The thin curve represents the divergent expansion around an instanton of Ref. [21]. The fat curve is the 22nd order approximation of the strong-coupling expansion, analytically continued to negative g in the sliding regime calculated in Chapter 17 of the textbook [2].

shown in Fig. 2 for $-0.2 < g < -0.006$. The point $g = -0.006$ is the closest approach to the singularity at $g = 0$ for $L = 64$ before the onset of oscillations.

Let us compare our curve with the expansion derived from instanton calculations [21]:

$$f(g) = b_1 g - b_2 g^2 + b_3 g^3 - b_4 g^4 + \cdots, \tag{3.3}$$

with coefficients $b_1 = 3.95833$, $b_2 = 19.344$, $b_3 = 174.21$, $b_4 = 2177$. This expansion is divergent and non-Borel-summable for $g < 0$. Remarkably, we are able to extract this expansion from our data points. Since our result is given by a convergent expansion, the fitting procedure will depend somewhat on the interval chosen over which we fit our curve by a power series. A compromise between a sufficiently long interval and the runaway of the divergent instanton expansion is obtained for a lower limit $g > -0.0229 \pm 0.0003$ and an upper limit $g = -0.006$. Fitting a polynomial to the data, we extract the following first three coefficients:

$$b_1 = 3.9586 \pm 0.0003, \qquad b_2 = 19.4 \pm 0.12,$$
$$b_3 = 135 \pm 18. \tag{3.4}$$

4. The agreement of our curves with the exact results in Fig. 1 and of our expansion coefficients in

(3.4) with the exact ones in (3.3) demonstrates that our method is capable of probing deeply into the instanton region of the coupling constant.

Let us end by remarking that another procedure of summing non-Borel series can be deduced from a development in the first of Refs. [18] (see also Chapter 20 of the textbook [3]). One may derive a strong-coupling expansion of the type (2.3) from the divergent weak-coupling expansion, which can then be continued analytically to negative g by a simple rotation of the power $g^{-\omega l}$ to $e^{-i\pi\omega l}(-g)^{-\omega l}$. This method is applicable only in the sliding regime. In Fig. 2, we have plotted the resulting curve to order $L = 9$. The present work fills the gap between the sliding regime and the tunnelling regime by extending variational perturbation theory to all g arbitrarily close to zero, without the need for a separate treatment of the tunnelling regime.

There exists, of course, a wealth of possible applications of this simple theory, in particular to quantum field theory where variational perturbation theory has so far yielded the most accurate critical exponents from Borel-summable series [3,18,20].

Acknowledgements

This work was partially supported by ESF COSLAB Program and by the Deutsche Forschungsgemeinschaft under Grant Kl-256.

References

[1] J.S. Langer, Ann. Phys. 41 (1967) 108;
S. Coleman, Nucl. Phys. B 298 (1988) 178.
[2] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics and Polymer Physics, World Scientific, Singapore, 1995, http://www.physik.fu-berlin.de/~kleinert/b5.
[3] H. Kleinert, V. Schulte-Frohlinde, Critical Properties of Φ^4-Theories, World Scientific, Singapore, 2001, http://www.physik.fu-berlin.de/~kleinert/b8.
[4] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Clarendon, Oxford, 1989.
[5] R. Karrlein, H. Kleinert, Phys. Lett. A 187 (1994) 133, hep-th/9504048.
[6] H. Kleinert, Phys. Lett. B 300 (1993) 261, http://www.physik.fu-berlin.de/~kleinert/214.
[7] R. Seznec, J. Zinn-Justin, J. Math. Phys. 20 (1979) 1398;
T. Barnes, G.I. Ghandour, Phys. Rev. D 22 (1980) 924;
B.S. Shaverdyan, A.G. Usherveridze, Phys. Lett. B 123 (1983) 316;
K. Yamazaki, J. Phys. A 17 (1984) 345;
H. Mitter, K. Yamazaki, J. Phys. A 17 (1984) 1215;
P.M. Stevenson, Phys. Rev. D 30 (1985) 1712;
P.M. Stevenson, Phys. Rev. D 32 (1985) 1389;
P.M. Stevenson, R. Tarrach, Phys. Lett. B 176 (1986) 436;
A. Okopinska, Phys. Rev. D 35 (1987) 1835;
A. Okopinska, Phys. Rev. D 36 (1987) 2415;
W. Namgung, P.M. Stevenson, J.F. Reed, Z. Phys. C 45 (1989) 47;
U. Ritschel, Phys. Lett. B 227 (1989) 44;
U. Ritschel, Z. Phys. C 51 (1991) 469;
M.H. Thoma, Z. Phys. C 44 (1991) 343;
I. Stancu, P.M. Stevenson, Phys. Rev. D 42 (1991) 2710;
R. Tarrach, Phys. Lett. B 262 (1991) 294;
H. Haugerud, F. Raunda, Phys. Rev. D 43 (1991) 2736;
A.N. Sissakian, I.L. Solovtsov, O.Y. Shevchenko, Phys. Lett. B 313 (1993) 367.
[8] I.R.C. Buckley, A. Duncan, H.F. Jones, Phys. Rev. D 47 (1993) 2554;
C.M. Bender, A. Duncan, H.F. Jones, Phys. Rev. D 49 (1994) 4219.
[9] H. Kleinert, W. Janke, Phys. Lett. A 206 (1995) 283, quant-ph/9509005.
[10] R. Guida, K. Konishi, H. Suzuki, Ann. Phys. 249 (1996) 109, hep-th/9505084.
[11] P.M. Stevenson, Phys. Rev. D 30 (1985) 1712;
P.M. Stevenson, Phys. Rev. D 32 (1985) 1389;
P.M. Stevenson, R. Tarrach, Phys. Lett. B 176 (1986) 436.
[12] B. Bellet, P. Garcia, A. Neveu, Int. J. Mod. Phys. A 11 (1997) 5587.
[13] J.-L. Kneur, D. Reynaud, hep-th/0205133, and Refs. [8–16], quoted therein.
[14] E. Braaten, E. Radescu, cond-mat/0206186.
[15] F.F. de Souza Cruz, M.B. Pinto, R.O. Ramos, Phys. Rev. A 65 (2002) 053613, cond-mat/0112306;
F.F. de Souza Cruz, M.B. Pinto, R.O. Ramos, Phys. Rev. B 64 (2001) 014515;
J.-L. Kneur, M.B. Pinto, R.O. Ramos, Phys. Rev. Lett. 89 (2002) 210403, cond-mat/0207295.
[16] B. Hamprecht, H. Kleinert, Dependence of Variational Perturbation Expansions on Strong-Coupling Behavior. Inapplicability of δ-Expansion to Field Theory.
[17] F.J. Wegner, Phys. Rev. B 5 (1972) 4529;
F.J. Wegner, Phys. Rev. B 6 (1972) 1891.
[18] H. Kleinert, Phys. Rev. D 57 (1998) 2264, cond-mat/9803268;
H. Kleinert, Phys. Rev. D 58 (1998) 107702;
H. Kleinert, Phys. Rev. D 60 (1999) 085001, hep-th/9812197;
H. Kleinert, Phys. Lett. B 434 (1998) 74, cond-mat/9801167.
[19] W. Janke, H. Kleinert, Phys. Lett. A 199 (1995) 287, quant-ph/9502018.
[20] J.A. Lipa, D.R. Swanson, J. Nissen, Z.K. Geng, P.R. Williamson, D.A. Stricker, T.C.P. Chui, U.E. Israelson, M. Larson, Phys. Rev. Lett. 84 (2000) 4894;
H. Kleinert, Phys. Rev. D 60 (1999) 085001, hep-th/9812197;
H. Kleinert, FU-Berlin preprint 1999, cond-mat/9906107.
[21] J. Zinn-Justin, J. Math. Phys. 22 (1981) 511.