OPERATIONS AND COOPERATIONS IN \textit{Im(J)}-\textit{THEORY}

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I. INTRODUCTION

The Steenrod algebra $E^*(E)$ and its dual, the ring of cooperations $E_*(E)$ are important structures which come along with a ring spectrum $E$. The purpose of this paper is to determine $A^*(A)$ and $A_*(A)$ for the spectrum $A$ of connected $\text{Im(J)}$-theory at an odd prime $p$.

Let $l$ be the Adams summand of $p$-local connected $K$-theory, i.e. $l_4(S^0) = \mathbb{Z}_p[v_1]$ with $v_1 \in l_4(S^0)$ and $q := 2p - 2$. We shall always work at a fixed odd prime $p$ and choose $k \in \mathbb{Z}$ as a generator of $(\mathbb{Z}/p')^*$. The stable Adams operation $\psi^k - 1$ on $l$ is divisible by $v_1$. There exists a unique operation $Q : l \to \Sigma l$ with $v_1 \cdot Q = \psi^k - 1$. The spectrum $A$ may be defined as fibre of $Q$. Thus $A$ fits into the cofibre sequence

$$
\xymatrix{
A \ar[r]^Q & l \ar[r] & \Sigma l \ar[r] & \Sigma A \ar[r] & .}
$$

One may choose $k$ to be a prime power, then Quillen's algebraic $K$-theory of the finite field $\mathbb{F}_k$ localized at $p$ is also a model for $A$. Since $\pi_*(A)$ in positive dimensions is isomorphic to the classical image of the $J$-homomorphism, $A_*$ is usually called $\text{Im(J)}$-theory. $\text{Im(J)}$-theory is a nice ring spectrum with many applications to stable homotopy, e.g. the classical $e$-invariant and the $J$-homomorphism are best reformulated using $A_*$.

The simplest cases in which $E_*(E)$ and $E^*(E)$ have been determined are those where $E_*(E)$ is flat or even free over the coefficient ring $E_*$. Then both $E_*(E)$ and $E^*(E)$ possess a rich algebraic structure leading often to powerful applications.

If $E_*(E)$ is free over $E_*$ then $E \wedge E$ splits as a wedge of suspensions of $E$. The next more complicated stage is where $E \wedge E$ splits as an $E$-module spectrum only as

$$
E \wedge E \cong \bigvee_i E \wedge X_i
$$

for a family of finite spectra $X_i$. In this case, $E_*(E)$ is usually not flat over $E_*$, there is less accessible algebraic structure and applications are harder to obtain. A prominent example is connected $K$-theory $bo$ or $l$ where the splittings (2) were first constructed by Mahowald [1] and Kane [2]. The finite spectra $X_i$ are here the integral Brown--Gitler spectra $K(n)$ [2, 3] and (2) reads as

$$
l \wedge l \cong \bigvee_{n \geq 0} l \wedge \Sigma^n K(n).
$$

Also $\text{Im(J)}$-theory falls into this category. Our main result is as follows:

For $n \equiv 0(p)$ we construct maps $F : K(n) \to K(n - 1)$ with cofibre $X_K(n)$ and derive an $A$-module splitting

$$
A \wedge A \cong \bigvee_{n \equiv 0(p)} A \wedge \Sigma^{n-1} X_K(n).
$$
There is also a different, more familiar choice for the splitting pieces $X_i$ yielding only a stable splitting, i.e., we have to add a graded mod $p$ Eilenberg–Mac Lane spectrum to obtain an $A$-module splitting

$$(A \wedge A) \vee (A \wedge T) \simeq \bigvee_{n = 0(p)} A \wedge \Sigma^{q-1} X_0(n).$$

Here $T$ is a spectrum such that $A \wedge T$ is a graded Eilenberg–Mac Lane spectrum and $X_0(n)$ is a finite spectrum constructed out of skeleta $B^m$ of $BS_p$, the classifying space of the symmetric group $\Sigma_p$, the transfer map $tr: BS_p \to S^0$ and certain $v_1$-self-maps of $B^m$.

The relation between a splitting of $E \wedge E$ and the operation ring $E^*(E)$ is as follows. Any $E$-module splitting like (2) yields an isomorphism $E^*(E) \cong \prod_i E^*(X_i)$. Thus (4) also computes $A^*(A)$:

$$A^*(A) \cong \prod_{n = 0(p)} A^*(\Sigma^{q-1} X_k(n)).$$

Since $A$ is relatively close to the sphere spectrum one expects $A^*(A)$ to be relatively small. This is indeed the case. Except for $n = 0$, where $X_0(0) = S^1$ (corresponding to the operations given by multiplication with elements in $A_1$), $A^*(X_k(n))$ is a finite group.

Now for the applications the most powerful operations are not the elements in $A^*(A)$ but the generalized operations

$$A \to A \wedge \Sigma^{q-1} X_k(n)$$

which are obtained from the splitting maps by composition with $\eta_R: S^0 \wedge A \to A \wedge A$. See, for example, the case of $K$-theory in [2].

Among the generalized $A$-operations (7) there is a family which may be used as primary operation to detect the Kervaire invariant-one elements in $\pi_0^s(S^0)$ (Of course, since $p$ is odd, this is by the results of Ravenel, see e.g. [4], a somewhat vacuous problem).

Simpler versions $X(n)$ of $X_k(n)$ are used in [5] to give a splitting of $I \wedge A$. There is also a non-connected version $Ad$ of $Im(J)$-theory. A splitting of $Ad \wedge Ad$ and a computation of $Ad^*(Ad)$ is obtained in [6].

Section 2 fixes notations and collects some preliminaries which are all well known but spread over different sources. In Section 3 we describe the splitting pieces $X_k(n)$, $X_0(n)$ and some other finite spectra $E_0$ relevant to the splitting of $A \wedge A$. Then the main result is stated and proved assuming the existence of maps $p^*_k: \Sigma^{q-1} X_k(n) \to I \wedge A$ with certain properties.

In Section 5 these maps $p^*_k$ are constructed using the results of [3, 7] and Section 6 contains the proof that $p^*_k$ factors through $A \wedge A$.

Throughout the paper $p$ denotes an odd prime, $q := 2p - 2$, $k$ generates $(Z/p)^*$, $k = k^{p-1}$ and $v(n) = v_p(n)$ is the power of $p$ in the prime factorization of $n$. $HF_p$ denotes the mod $p$ Eilenberg–Mac Lane spectrum and $H_*(X)$ mod $p$ homology, other coefficient groups are always indicated. All homology and cohomology theories are taken as reduced. Suspension of spectra and shift suspension of modules are denoted by $\Sigma$.

2. PRELIMINARIES

$$\mathcal{A}(1)_*, E_*, \Gamma_*.$$

Let $\mathcal{A}$ be the mod $p$ Steenrod algebra with dual

$$\mathcal{A}^* \cong F_p[\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_n] \otimes \Lambda(\bar{v}_0, \bar{v}_1, \bar{v}_2, \ldots).$$
Let $Q_0$ denote the mod $p$ Bockstein, $\mathcal{P}^1$ the first Steenrod power and $Q_1 = \mathcal{P}^1 \circ Q_0 - Q_0 \circ \mathcal{P}^1$. We shall need to consider the following sub-Hopf algebras of $\mathcal{A}$:

Let $\Gamma$ be the subalgebra generated by $\mathcal{P}^1$, $E$ the subalgebra generated by $Q_0$ and $Q_1$ and $\mathcal{A}(1)$ the subalgebra generated by $Q_0$ and $\mathcal{P}^1$. Then $\Gamma \cong \mathbb{F}_p[\mathcal{P}^1]/(\mathcal{P}^1)^p$, $E \cong \Lambda(Q_0, Q_1)$ and in $\mathcal{A}(1)$ we have the relations $Q_0 \circ Q_0 = 0$, $Q_1 \circ Q_1 = 0$, $Q_0 \circ Q_1 = - Q_1 \circ Q_0$, $(\mathcal{P}^1)^p = 0$ and $\mathcal{P}^1 \circ Q_1 = Q_1 \circ \mathcal{P}^1$.

Denote by $\mathcal{A}(1)_* = E_*, \Gamma_*$ the dual Hopf algebras to $\mathcal{A}(1)$, $E$ or $\Gamma$. We shall work with comodules over $\mathcal{A}_*$, $\mathcal{B} \in \{\mathcal{A}(1), E, \Gamma\}$ rather than modules over $\mathcal{B}$, but often switch to modules for easier visualization. All of our comodules will be locally finite, so there is no problem in dualizing in the standard way (e.g. see [8] p. 332). We shall call a comodule $M$ over $\mathcal{B}_*$ free, if it is free as a module over $\mathcal{B}$.

We have $\mathcal{A}(1)_* \cong \Lambda(\tau_0, \tau_1) \otimes \mathbb{F}_p[\xi_1]/(\xi_1^p)$, $E_* = \Lambda(\tau_0, \tau_1)$, $\Gamma_* = \mathbb{F}_p[\xi_1]/(\xi_1^p)$, together with canonical maps of coalgebras dual to the inclusions. We shall be met several times with the following standard situation: Let $C, D$ be connected, graded coalgebras over $\mathbb{F}_p$ and $\phi: C \to D$ a map of coalgebras. Then $\phi$ induces a canonical restriction functor $r, \mathcal{C}$ from $C$-comodules to $D$-comodules. Right-adjoint to this functor is the cotensor product $M \otimes_C D$ (e.g. see [4]). We have

$$\text{Hom}_{\mathcal{C}}(N, C \otimes_D M) \cong \text{Hom}_D(r, (N), M)$$

which extends under suitable assumptions to the change-of-rings isomorphism

$$\text{Ext}^{*}_{\mathcal{C}}(N, C \otimes_D M) \cong \text{Ext}^{*}_{\mathcal{D}}(r, (N), M).$$

The following notation will be used: For $f \in \text{Hom}_{\mathcal{C}}(N, C \otimes_D M)$ denote by $\bar{f} = \varepsilon \otimes 1 \circ f$ its image in $\text{Hom}_D(N, M)$, where $\varepsilon: C \to \mathbb{F}_p$ is the counit and $\varepsilon \otimes 1: C \otimes_M \mathbb{F}_p \to \mathbb{F}_p \otimes_M M$ (which is the same as $C \otimes_M D \otimes_M M = M$). For $g \in \text{Hom}_D(N, M)$ its image in $\text{Hom}_{\mathcal{C}}(N, C \otimes_D M)$ is given by

$$\begin{array}{ccc}
N & \xrightarrow{\psi} & C \otimes_D N \\
\downarrow{\phi} & & \downarrow{1 \otimes f} \\
\otimes C & \otimes & C \otimes_M M
\end{array}$$

where $\psi$ is the $C$-coaction on $N$. If $M$ is a $C$-comodule with coaction map $\psi(m) = \sum c_i \otimes m_i \in C \otimes M$ and $C, D$ are now assumed to be Hopf algebras note the following canonical isomorphism (e.g. see [8] p. 338):

$$\Phi: C \otimes_D r_s(M) \to (C \otimes_D \mathbb{F}_p) \otimes A^sB M$$

$$a \otimes m \mapsto \sum a : \bar{c}_i \otimes m_i$$

$$a \otimes c_i \otimes m_i \mapsto a \otimes m.$$  

Here $\bar{c}$ is the conjugate of $c$ and $A \otimes B^s$ denotes the tensor product of $A$ and $B$ with the diagonal $C$-coaction, whereas the coaction on $C \otimes_D r_s(M)$ is induced from the one on $C$. This isomorphism allows us to switch from the diagonal coaction which usually comes up in using the Kunneth formula to the left coaction which is used in the change-of-rings formula.

Comodules over $\Gamma_*, E_*, \mathcal{A}(1)$. (12)

A classification of modules over $\Gamma$ is, of course, trivial by Jordan decomposition: Denote by $V_i$ the indecomposable $\Gamma$-module of $\mathbb{F}_p$-dimension $i$, $1 \leq i \leq p$. Then $V_1$ is irreducible and $V_i$ is free. Jordan decomposition shows that every finite dimensional $\Gamma$-module is a finite direct sum of $V_i$'s.
For the classification of modules over $E$ we refer to [8] or [9]. A classification of modules over $\mathcal{A}(1)$ seems not to be known. We shall need the following comodules:

$$E_i, \quad 1 \leq i \leq p.$$  \hspace{1cm} (13)

Define $E_i$ as the $\mathcal{A}(1)_\ast$-submodule of $\mathcal{A}(1)_\ast$ generated by $\tau_0 \tau_1 \cdots \tau_{i-1}$ (remember $a \in \mathcal{A}(1)$ acts degree decreasing from the left). For example $E_1$ and $E_2$ may be visualized by the following pictures.

Here dots represent $\mathbb{F}_p$-basis elements and the action of $Q_0$, $Q_1$ and $\mathcal{P}^1$ is indicated by lines.

If we regard the $\Gamma$-module $V_i$ as an $\mathcal{A}(1)$-module with $Q_0$ and $Q_1$ acting trivially then clearly $E_i \cong E_1 \otimes^\mathcal{A} V_i$ as $\mathcal{A}(1)$-modules. Since also $E_i \cong \mathcal{A}(1)_\ast \square_{r_1} \mathbb{F}_p$, induced by $E_1 \subset \mathcal{A}(1)_\ast$, we have by (11)

$$E_i \cong E_1 \otimes^\mathcal{A} V_i \cong (\mathcal{A}(1)_\ast \square_{r_1} \mathbb{F}_p) \otimes^\mathcal{A} V_i \cong \mathcal{A}(1)_\ast \square_{r_1} V_i.$$ \hspace{1cm} (14)
We identify \( H_\ast(1) \) with its image, which is
\[
H_\ast(1) = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda(\tau_2, \tau_3, \ldots).
\]
This is the same as
\[
H_\ast(1) = \mathcal{A}_\ast \square_p \mathbb{F}_p.
\]

The action of \( \mathcal{A}(1) \) is given as follows:
\[
Q_0(\xi_i) = \xi_i, \quad Q_0(\xi_i^p) = 0, \quad Q_1(\xi_i) = \xi_i^{p^2 - 1}, \quad Q_1(\xi_i^p) = 0, \quad \mathcal{P}^1(\xi_i) = \xi_i^{p^2 - 1}.
\]
Define a weight function on the monomials in \( H_\ast(1) \) by
\[
wt(\xi_i) = p^{i-1}, \quad wt(a \cdot b) = wt(a) + wt(b)
\]
and let \( W(n) \) denote the \( \mathbb{F}_p \)-vector space generated by all monomials of weight \( n \).

Since \( Q_0 \) and \( Q_1 \) respect the weight filtration, \( H_\ast(l) \cong \bigoplus_n W(n) \) is a decomposition of \( E_\ast \)-comodules. The presence of \( \xi_1 \) implies that \( \mathcal{P}^1 \) respects weight filtration only if \( n = 0(p) \).

For \( n = 0(p) \), \( W(n) \) is thus an \( \mathcal{A}(1)_\ast \)-comodule. The \( \mathcal{A}(1)_\ast \)-comodule \( H_\ast(l) \) then decomposes as \( H_\ast(l) \cong \bigoplus_n W(mp) \otimes V_p \). Information on the \( \mathcal{A}(1)_\ast \)-comodule structure of \( W(mp) \) is obtained in [7].

\[
H_\ast(A).
\]

The description of the mod \( p \) homology of the \( \text{Im}(J) \)-theory spectrum \( A \) is obtained from the exact sequence
\[
0 \rightarrow \mathcal{A}_\ast \rightarrow H_\ast(A) \rightarrow H_\ast(l) \rightarrow 0
\]
induced by (1). If we identify \( H_\ast(l) \) with its image in \( \mathcal{A}_\ast \), then \( Q_\ast = (1 - \bar{k})/p \cdot \mathcal{P}^1_\ast \). Now \( \mathcal{P}^1_\ast \) acts trivially on all generators \( \xi_i, \xi_i^p \), \( i \geq 2 \), except on \( \xi_1 \) where \( \mathcal{P}^1_\ast(\xi_1) = -i^{p^2 - 1} \).

Define \( C := \mathbb{F}_p[\xi_1^p, \xi_2, \xi_3, \ldots] \otimes \Lambda(\tau_2, \tau_3, \ldots) \cong \mathcal{A}_\ast \otimes \mathcal{A}(1)_\ast \mathbb{F}_p \), then \( \ker(\mathcal{P}^1_\ast) = C \) and \( \coker(\mathcal{P}^1_\ast) = C \cdot \Delta_\ast(\xi_1^{p^2 - 1}) \) and we obtain the short exact sequence of \( \mathcal{A}(1)_\ast \)-comodules
\[
0 \rightarrow C \cdot \Delta_\ast(\xi_1^{p^2 - 1}) \rightarrow H_\ast(A) \rightarrow C \rightarrow 0.
\]

Note that \( C = \bigoplus_m W(mp) \).

Watanabe [10] has determined the \( \mathcal{A}_\ast \)-coaction on \( H_\ast(A) \). We shall only need that one can choose preimages \( x_i \in H_\ast(A) \) for \( \xi_i \in C, \beta_i \) for \( \xi_i^p \), \( i \geq 2 \), and \( \beta \) for \( \xi_1^p \) under \( D \), satisfying:
\[
\mathcal{P}^1(\xi_i) = 0, \quad Q_0(\beta_i) = \beta_i, \quad \mathcal{P}^1(\beta_i) = \beta_i^p - 1, \quad \mathcal{P}^1(\beta) = \beta.
\]
Moreover \( Q_0(\beta) = \Delta_\ast(\xi_1^{p^2 - 1}), \quad Q_1(\beta_i) = -\Delta_\ast(\xi_1^{p^2 - 1}) \). Hence (21) does not split as a sequence of \( E_\ast \)-comodules.

Let \( \tilde{C} := \mathbb{F}_p[\beta, \beta_2, \beta_3, \ldots] \otimes \Lambda(\tau_2, \tau_3, \ldots) \) then
\[
H_\ast(A) = \tilde{C} \cdot \Delta_\ast(\xi_1^{p^2 - 1}) \oplus \tilde{C}.
\]

Define a weight function \( wt \) on the monomials in \( H_\ast(A) \) by \( wt(1) = 0, \quad wt(\beta) = wt(\beta_i) = wt(\beta_i^p) = p, \quad wt(x_i) = wt(\beta) = p^{i-1} \) \( (i \geq 2) \) and \( wt(a \cdot b) = wt(a) + wt(b) \).
Let \( W^A(n) \) be the \( \mathbb{F}_p \)-vector space generated by all monomials of weight \( n \). Then \( \mathcal{P}^1, Q_0, Q_1 \) respect weight and \( W^A(n) \) is an \( \mathcal{A}(1)_\ast \)-comodule. The exact sequence (21) decomposes into short exact sequences of \( \mathcal{A}(1)_\ast \)-comodules.

\[
0 \rightarrow \Sigma^{p^2 - 1} W(pm - p) \xrightarrow{\Delta} W^A(pm) \rightarrow W(pm) \rightarrow 0
\]
where \( \Delta_\ast \) is the map \( x \mapsto \Delta_\ast(\xi \cdot \xi_1^{p^2 - 1}) - \xi \cdot \Delta_\ast(\xi_1^{p^2 - 1}) \) with \( D_\ast(\xi) = x \). Since \( \Delta_\ast(\xi_1^{p^2 - 1}) \) is \( \mathcal{A}(1)_\ast \)-primitive, \( \Delta_\ast \) is an \( \mathcal{A}(1)_\ast \)-comodule map.

\[ F\text{-extensions}. \]
Let $F$ be a ring spectrum with unit $i: S^0 \to F$ and $F_Z$ an $F$-module spectrum with multiplication $\mu: F \wedge F_Z \to F_Z$. We shall only need the example $F_Z = F \wedge Z$. The $F$-extension $\overline{f}$ of a map $f: X \to F_Z$ is defined by

$$\overline{f}: F \wedge X \xrightarrow{1 \wedge f} F \wedge F_Z \xrightarrow{\mu} F_Z.$$  

This is clearly an $F$-module map.

If $F_1, F_2$ are $F$-module spectra, write $[F_1, F_2]^m$ for the subgroup of $F$-module maps in $[F_1, F_2]^*$. Sending $f$ to $\overline{f}$ gives then an isomorphism

$$[X, F_Z]^* \cong [F \wedge X, F_Z]^m$$

whose inverse is given by restriction to $S^0 \wedge X$ via $i \wedge \text{id}: S^0 \wedge X \to F \wedge X$.

Let now $F = l$ and $f: X \to l \wedge Y$ be any map. It follows from the change-of-rings formula that

$$\overline{f}_*: H_*(l \wedge X) \to H_*(l \wedge Y)$$

is given by

$$1 \square \overline{f}_*: \mathcal{A}_* \square_{E_*} H_*(X) \to \mathcal{A}_* \square_{E_*} H_*(Y)$$

where we have used the isomorphism $H_*(l \wedge Z) = \mathcal{A}_* \square_{E_*} H_*(Z)$ from (19) and (11). Hence

**Proposition 2.1.** Let $X, Y$ be connective spectra and $f: X \to l \wedge Y$ a map. If the $E_*$-comodule map $\overline{f}_*: H_*(X) \to H_*(Y)$ is an isomorphism, then $f: l \wedge X \to l \wedge Y$ is a $p$-local equivalence.

The corresponding $F$-cohomology statement is as follows. Let $F_Z$ be as above and $f: X \to F \wedge Y$ be any map. Define

$$f_*: [Y, F_Z] \to [X, F_Z]$$

(25)

as the composition

$$[Y, F_Z] \cong [F \wedge Y, F_Z]^m \xrightarrow{f_*} [F \wedge X, F_Z]^m \cong [X, F_Z].$$

Since $\overline{f}$ is an $F$-module map, $f_*$ is well defined and we trivially have

**Proposition 2.2.** If for $f: X \to l \wedge Y$ with $X, Y$ connective the $E_*$-comodule map $\overline{f}_*: H_*(X) \to H_*(Y)$ is an isomorphism, then $f_*: l^p_*(Y) \to l^p_*(X)$ is an isomorphism.

We shall also need that the diagrams (with $l_Z = l \wedge Z$)

$$\begin{array}{ccc}
[Y, l \wedge Z] & \xrightarrow{f_*} & [X, l \wedge Z] \\
\downarrow^h & & \downarrow^h \\
\text{Hom}_{E_*}(H_*(Y), H_*(Z)) & \xrightarrow{(\overline{f}_*, \text{id})_*} & \text{Hom}_{E_*}(H_*(X), H_*(Z)) \\
\end{array}$$

\begin{equation}
(26)
\end{equation}

and

$$\begin{array}{ccc}
[Y, l \wedge Z] & \xrightarrow{f_*} & [X, l \wedge Z] \\
\downarrow^g & & \downarrow^g \\
[Y, G \wedge Z] & \xrightarrow{f_*} & [X, G \wedge Z] \\
\end{array}$$

\begin{equation}
(27)
\end{equation}
commute. Here $h$ is the Hurewicz map followed by the change-of-rings-isomorphism (10) and (19), $g: l \to G$ is the canonical map of $l$ into the periodic Adams summand $G$ of the $p$-local $K$-theory spectrum, i.e. $G = v_1^{-1}l$ and $f': X \to G \wedge Y$ is $g \wedge 1 \circ f$. The proofs are easy diagram chases and are left to the reader.

3. THE SPLITTING MODELS

In this section we shall define various spectra appearing in the splitting of $A \wedge A$, $I \wedge I$, $l \wedge l$.

We begin by recalling the integral Brown–Gitler spectra $K(n)$ and define $X_K(n)$ which is used in the splitting of $A \wedge A$. The spectra $R(n)$, $X(n)$, which are used in [5] for a splitting of $I \wedge I$, appear later on in the construction of the splitting maps $p^n_1: \Sigma^{w-1} X_K(n) \to l \wedge l$. They may be used to construct a second set of splitting pieces $R_\psi(n)$, $X_\psi(n)$. Those only give a stable splitting of $A \wedge A$, i.e. a splitting up to a wedge of mod $p$ Eilenberg–Mac Lane spectra, but have the advantages of a more explicit definition than the spectra $K(n)$ and $X_K(n)$ and giving more insight into the internal structure of $A \wedge A$. In using $K(n)$ and $X_K(n)$ we have to complete all spectra at $p$ whereas in using $R_\psi(n)$, $X_\psi(n)$ we may work $p$-locally.

We shall use the definition of the integral Brown–Gitler spectra given in [3], see also [2]. Since the authors in [3] work in a $p$-complete setting we shall do the same but without indicating this in the notation.

Recall the following discussion and results from [3]: We have

$$H_\ast(\Omega^2 S^3) \cong \mathbb{F}_p[b_1, b_2, \ldots] \otimes \Lambda(a_0, a_1, \ldots)$$

with $|a_i| = 2p^i - 1$, $|b_i| = 2p^i - 2$ and a weight filtration on the monomials in $H_\ast(\Omega^2 S^3)$ given by

$$wt(a_i) = wt(b_i) = p^i \quad \text{and} \quad wt(a_i \cdot b_i) = wt(a_i) + wt(b_i).$$

There is an increasing filtration $F_\mathbb{F} \Omega^2 S^3$ such that $H_\ast(F_\mathbb{F} \Omega^2 S^3)$ is the span of monomials of weight $\leq n$. Let $S^3 \langle 3 \rangle$ denote the 3-connected cover of $S^3$ and

$$\Omega^2 S^3 \langle 3 \rangle \to \Omega^2 S^3 \to S^1$$

the associated homotopy fibration. Definition $A_n$ by the homotopy fibration

$$A_n \to F_{pn+1} \to S^1.$$ 

Then $H_\ast(A_n)$ is the span of monomials of weight $pi$ with $i \leq n$ in $H_\ast(\Omega^2 S^3 \langle 3 \rangle)$. Let $\xi$ denote the spherical fibration over $\Omega^2 S^3$ equivalent to the $p$-complete Eilenberg–Mac Lane spectrum $H \mathbb{F}$. The nth integral Brown–Gitler spectrum at $p$, $B_1(n)$, is then defined to be Thom spectrum $T(\xi | A_n)$ using the map $A_n \xrightarrow{\xi} \Omega^2 S^3 \langle 3 \rangle \to \Omega^2 S^3$.

The map $T(\xi_i): B_1(n) = T(\xi | A_n) \to T(\xi | \Omega^2 S^3 \langle 3 \rangle) \cong H \mathbb{F}$ sends $H_\ast(B_1(n))$ isomorphically onto the span of monomials of weight $\leq pn$ where $H_\ast(H \mathbb{F}) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda(\xi_1, \xi_2, \ldots)$ and $wt(\xi_i) = wt(\xi_i) = p^i$, $wt(a_i \cdot b_i) = wt(a_i) + wt(b_i)$.

There are pairings $B_1(n) \wedge B_1(m) \to B_1(n+m)$ whose homology homomorphism is compatible with the multiplication in $H_\ast(H \mathbb{F})$ (Theorem 1.5 in [3]).

We shall mainly use the notation $K(pn)$ for $B_1(n)$ and define $K(np + i)$ as $K(np)$ for $i \in \{1, \ldots, p-1\}$. This is the notation used in [2] and has the advantage that

$$l \wedge l \cong \bigvee_{n \geq 0} l \wedge \Sigma^n K(n).$$
Our next aim is the construction of a map

\[ F : K(np) \to K(np - 1) \]

as a compression of the multiplication by \( p^{1+v(n)} \) map on \( K(np) \).

**Lemma 3.1.** The cofibre of the map \( i : K(np - 1) \to K(np) \) induced by \( \mathcal{A}_{n-1} \hookrightarrow \mathcal{A}_n \) is equivalent to \( F_{pn}/F_{pn-1} \).

**Proof.** If \( A \xrightarrow{i} X \xrightarrow{j} C_f \) is a cofibration and \( \xi \to C_f \) is a spherical fibration over \( C_f \), then there is the well-known cofibre sequence of Thom spectra

\[ A \to X^\xi \to C_f^\xi \to \Sigma A. \]  

(28)

Here \( \xi \) is taken as a virtual spherical fibration of dimension 0. By [3] (see the proof of 1.3) there is a space \( M_{pn}(\partial I^{pn}) \) and a cofibration

\[ M_{pn}(\partial I^{pn}) \to \mathcal{A}_{n-1} \to \mathcal{A}_n \to F_{pn}/F_{pn-1} \]

with \( \Sigma M_{pn}(\partial I^{pn}) = F_{pn}/F_{pn-1} \). So we may apply (28) with \( X = \mathcal{A}_{n-1} \) and \( C_f = \mathcal{A}_n \).

Let \( C_{n,k} \) be the configuration space of \( k \) distinct ordered points in \( \mathbb{R}^n \) and \( V_k \) the vector bundle

\[ C_{2,k} \times \Sigma_k \mathbb{R}^k \to C_{2,k}/\Sigma_k \]

with Thom space \( t(V_k) \).

Then (up to \( p \)-completion, see e.g. [12]) \( t(V_k) \) is the same as \( F_k/F_{k-1} \) and \( H^*(t(V_k)) \) is determined in [12]. In [12] it is proved that \( H^*(t(V_k)) = 0 \) if \( k \neq 0, 1(p) \), \( t(V_{pk+1}) \simeq \Sigma t(V_{pk}) \) and that \( t(V_{p^{-m}+p}) \) is a model for \( \Sigma^{v(pm+p)} B(m) \) where \( B(m) \) is the \( m \)th mod \( p \) Brown–Gitler spectrum.

We need from [13, 14]

**Lemma 3.2.** The stable order of \( \text{id} : t(V_{pm}) \to t(V_{pn}) \) is \( p^{1+v(n)} \).

**Proof.** This is proved for \( n \neq 0(p) \) in [13] and for general \( n \) in [14]. There is a cofibre sequence

\[ \Sigma t(V_{pm+p}) \xrightarrow{\beta} t(V_{pm}) \xrightarrow{\text{id}} \Sigma^{v(p)} t(V_{pm}) \]  

(29)

which is well known for \( p = 2 \). The map \( \beta \) is defined in [12] and induced by the James–Hopf invariant. A detailed proof that (29) is a cofibre sequence may be found in [14]. Using (29) we get by the obvious induction that \( p^{1+v(n)} \cdot \text{id} \simeq 0 \) on \( t(V_{pm}) \). The cofibre sequence \( l \wedge K(np-1) \to l \wedge K(np) \to l \wedge t(V_{pn}) \) may then be used to show that the order of \( \text{id} : t(V_{pm}) \to t(V_{pn}) \) is not less than \( p^{1+v(n)} \).

**Corollary 3.3.** (a) There is a map \( F : K(np) \to K(np - 1) \) with \( i \circ F : K(np) \to K(np) \) is multiplication by \( (k^{np}-1)/p \) with \( k \) as in Section 1 and \( \overline{k} - k^{np-1} \).

(b) If \( v(n) > 0 \), \( F \) may be chosen to have Adams filtration at least 2.

**Proof.** Since \( v(\overline{k}^{np}-1) = 2 + v(n) \), the map

\[ (k^{np}-1)/p : K(np) \to K(np) \]

is zero on the cofibre of \( i : K(np-1) \to K(np) \) by Lemmas 3.2 and 3.1, hence there is a factorization \( F : K(np-1) \to K(np) \) of \( (k^{np}-1)/p \) through \( i \).
Let now \( n = pm \). We combine the two cofibre sequences (Lemma 3.1 and (29)) to get a commutative diagram

\[
\begin{array}{cccc}
\Sigma^q t(V_{p^m-p}) \\
\downarrow^a & & & \\
K(p^m-1) & \overset{i}{\longrightarrow} & K(p^m) & \overset{j}{\longrightarrow} & t(V_{p^m}) \\
\downarrow & & \downarrow & & \downarrow^\beta \\
Z & \longrightarrow & K(p^m) & \longrightarrow & \Sigma^{q+1} t(V_{p^m})
\end{array}
\]

with \( Z \) the fibre of \( \beta \circ j \). The map \( (k^p - 1)/p^2 : K(p^m) \to K(p^m) \) projects to zero on \( \Sigma^{q+1} t(V_{p^m}) \) hence has a factorization \( F_1 : K(p^m) \to Z \). The map \( p : Z \to Z \) has a factorization \( t_2 : Z \to K(p^m-1) \). Then \( k := t_2 \circ F_1 : K(p^m) \to K(p^m-1) \) is a factorization of \( (k^p - 1)/p^2 : K(p^m) \to K(p^m) \).

Since \( i_o \) is injective and (29) induces a short exact sequence in mod \( p \) homology [14], we find \( F_2 = 0 \) and \( F_1 = 0 \) hence \( AF(F) \geq 2 \).

Choose now for every \( n \) a map \( F : K(pn) \to K(pn-1) \) of maximal Adams filtration \(( \geq 1 + v(n)) \) and define \( X_n(K(pn)) \) to be the cofibre of \( F \).

Denote the weight filtration on \( H_*(H_\ast Z) \) by \( \mathcal{W} \) and let \( H(n) \) denote the span of monomials of weight \( \leq n \). Then

\[
H_*(K(n)) \cong H(n).
\]

Recall the weight filtration \( wt \) on \( H_*(l) = \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda(\bar{\tau}_2, \bar{\tau}_3, \ldots) \) defined by \( wt(\xi_i) = wt(\bar{\tau}_i) = p^{i-1} \) with \( W(n) \) the span of monomials of weight \( n \).

Define a map \( S^\mathcal{W}_n : \Sigma^m H(n) \to W(n) \) as follows: For a monomial \( a = \xi_1^{s_1} \cdots \xi_n^{s_n} \bar{\tau}_1^{r_1} \cdots \bar{\tau}_n^{r_n} \) in \( H(n) \) of weight \( m \) let

\[
S^\mathcal{W}_n(a) := \overline{\xi_1^{n-m} \cdots \xi_n^{n-m} \xi_{n+1}^{r_1} \cdots \bar{\tau}_n^{r_n}}.
\]

Then \( S^\mathcal{W}_n(a) \) is of weight \( n - m + \alpha_1 + \alpha_2 + \cdots + \alpha_n + \beta_1 + \beta_2 + \cdots + \beta_v = n \). Clearly \( S^\mathcal{W}_n \) is an isomorphism of \( \mathbb{F}_p \)-vector spaces which is compatible with the action of \( Q_0, Q_1 \). Moreover importantly, \( S^\mathcal{W}_n \) is compatible with the \( \mathcal{P}_1 \)-action on both modules if \( n \) is divisible by \( p \). The proof for this is a simple computation. Hence

**Proposition 3.4.** The map \( S^\mathcal{W}_n : H_*(\Sigma^m K(np)) \to H_*(l) \) is an injective \( \mathcal{A}l(1)_* \)-comodule map with image \( W(np) \).

Next we recall the models for the splittings of \( l \wedge l \) and \( l \wedge A \) used in [5]. Let \( B \) denote the \( p \)-localization of \( B\Sigma_p \), the classifying space of the symmetric group \( \Sigma_p \), \( B^* \) its \( n \cdot q \)-skeleton and \( tr : B \to S^0 \) the reduced transfer map. We have a cofibre sequence

\[
\rightarrow D^\mathcal{P}_q \overset{tr}{\longrightarrow} S^0 \overset{i}{\longrightarrow} R^\mathcal{P}_q \rightarrow \Sigma D^\mathcal{P}_q \rightarrow
\]

defining the spectrum \( R^\mathcal{P}_q \).

There is a stable map \( F : B^\mathcal{P}_q \to B^\mathcal{P}_q^{-1} \) defined as a compression of the multiplication by \( p \) map on \( B^\mathcal{P}_q \) through \( B^\mathcal{P}_q^{-1} \) (see [15]). The map \( F \) acts as an Adams periodicity or \( v_1 \)-map on the Moore space pieces of \( B \). It is easy to see and shown in [5] that this map fits into the
left-hand commutative square below, defining a map \( R^n \to R^{n-1} \)—also denoted by \( F \)—as a fill in map

\[
\begin{array}{ccc}
B^n & \xrightarrow{\tilde{r}} & S^0 \\
\downarrow F & & \downarrow p \\
B^{n-1} & \xrightarrow{\tilde{r}} & S^0 \\
\end{array}
\]

Let \( X^n \) denote the cofibre of \( F^i = F \circ F \circ \cdots \circ F : R^n \to R^{n-1} \) (\( i \leq n \)). Define

\[
R(n) := R^m, \quad m = v(n!), \quad n > 0
\]

\[
X(n) := X^m, \quad r = v(n), \quad X(0) = S^1
\]

\( = \text{cofibre of } F^*: R(n) \to R(n - 1) \).

Since \( v((k^n - 1)/p) = v(n) \) we can adjust the fill in map \( F^*: R(n) \to R(n - 1) \) in such a way that it has degree \((k^n - 1)/p\) when restricted to \( S^0 \).

There are splittings of \( l \)-module spectra [5]:

\[
l \wedge A \cong \bigvee_{n \geq 0} l \wedge \Sigma^a(R(n) \vee V^{(n)})
\]

\[
l \wedge A \sim \bigvee_{n = 0(p)} l \wedge \Sigma^{a-1}(X(n) \vee Z(n)).
\]

Here \( V^{(n)} \) and \( Z(n) \) are finite wedges of suspensions of the spectrum \( V(1) \), the cofibre of the Adams map \( B : \Sigma^2M \to M \) on the mod \( p \) Moore spectrum \( M \). Note that \( l \wedge V(1) \simeq HF_p \) such that \( l \wedge V^{(n)} \) is a graded Eilenberg–Mac Lane spectrum.

The spectrum \( R^m \) is a geometric realization of the pure lightning flash comodule \( N^m \) of Section 2 (e.g. see [5] Section 1):

\[
H_*(R^m) \cong N^m.
\]

Since \( F_* = 0 \) in \( \text{mod } p \) homology, the cofibre sequence

\[
R^m \xrightarrow{F} R^{m-1} \to X^m \to \Sigma R^m
\]

induces a short exact sequence in homology and therefore a long exact sequence in \( Q_0 \)- or \( Q_1 \)-homology. From this it easily follows that \( H_*(X^m) \) is a free \( E_* \)-comodule. We define the spectra \( E_i, 1 \leq i \leq p, \) by

\[
E_i := X^i = X(ip).
\]

The cofibre sequence (35) then may be used to show

\[
H_*(E_i) \cong E_i \quad (\text{as } \mathcal{O}(1)_* \text{-comodules}).
\]

In particular, \( E_p \) is a geometric realization of \( \mathcal{O}(1)_* \). \( A \wedge E_i \) will be determined in Section 4.

The rest of the section may be skipped, if one is only interested in the \( X_K(n) \)-version of the splitting of \( A \wedge A \).

We define product versions of \( R(n) \) and \( X(n) \) as follows: First consider \( n = sp^s, \) \( 0 < s < p \). In [7] (4.1) it is shown that for the subspace \( W(n) \) of weight filtration \( n \) in \( H_*(l) \) there is an \( A(1)_* \)-isomorphism

\[
s_*^p : \Sigma^a(N(n) \oplus F(n)) \to W(n)
\]
where $N(n) - N^{(n)}$ is the pure lightning flash comodule and $F(n)$ is a finite free $A(1)\ast$-comodule. Now $F(n)$ may be realized as a wedge of suspensions of $E_p$: Define $F(n) := \vee_{i} \Sigma_{n} E_p$ such that $H_\ast(F(n)) = F(n)$ (as an $E_\ast$-comodule $F(n)$ is isomorphic to $H_\ast(V^{(n)})$ in (33)). As a geometric model for $W(n)$, $n = sp^b$, we therefore take

$$R_\ast(n) := R(n) \vee F(n) \quad (R(0) = S^0)$$

and for $n = \sum_{i=0}^{b} n_i p^i$, $a \geq 1$, $0 \leq n_i < p$, let

$$R_\ast(n) := R_\ast(n_a p^a) \wedge R_\ast(n_{a-1} p^{a+1}) \wedge \cdots \wedge R_\ast(n_b p^b). \quad (36)$$

Write $N_\ast(n)$ for $H_\ast(R_\ast(n))$ and define $s_\ast^n$ to be the following composition:

$$s_\ast^n : \Sigma F_\ast N_\ast(n) = \Sigma F_\ast \left( n(n_{i+1}) \right) \rightarrow W(n_{i+1}) \rightarrow W(n)$$

where $s = s_\ast^n \circ \cdots \circ s_\ast^n$ and $m$ is given by multiplication in the ring $H_\ast(I)$.

In order to define a map $F_\ast$ corresponding to $F^{(n)} : R(n) \rightarrow R(n-1)$ we decompose the first factor $R_\ast(n_a p^a)$ in $R(n)$ a bit further. Let $s_a = (n_a - 1)p^a$, $m_a = v(s_a!)$ and define

$$R'_\ast(n_a p^a) := R_{n_a p^a} \wedge R_{n_{a-1} p^{a+1}} \wedge \cdots \wedge R_{n_b p^b} \wedge R_{n_{a-1} p^{a-1}} \wedge R_{n_a p^a} \wedge F(n) \quad (37)$$

With the abbreviations $s_i = (p-1)p^{a-i}$, $p^{a-1} - 1 = v(s_i!)$ we have by definition

$$R_\ast(n_a p^a - 1) = (R_{n_a p^a} \vee F(n_{a-1})) \wedge (R_{n_{a-1} p^{a-1}} \vee F(n_{a-1})) \wedge \cdots \wedge (R_{n_b p^b} \vee F(n_{b-1})).$$

We define $F_\ast : R'_\ast(n_a p^a) \rightarrow R_\ast(n_a p^a - 1)$ by taking on $F(n)$ the constant map and on $R_{n_a p^a} \wedge R_{n_{a-1} p^{a-1}} \wedge \cdots \wedge R_{n_b p^b}$ by taking the smash product of the maps $F : R^j \rightarrow R^{j-1} \vee F(k_j)$, $j = m_a + 1$, $p^{a-1}$, $\ldots$, $p$ and $k_j = s_a, s_1, \ldots, s_{a-1}$. So the map $F_\ast$ is essentially the $a$-fold smash product of $F$ and therefore of Adams filtration $a$.

We extend this definition for general $n = \sum_{i=a}^{b} n_i p^i$, $a \geq 1$, $0 \leq n_i < p$ by taking

$$R_\ast(n) := R_\ast(n_a p^a) \wedge R_\ast(n_{a+1} p^{a+1}) \wedge \cdots \wedge R_\ast(n_b p^b). \quad (38)$$

Let $X_\ast(n)$ be the cofibre of $F_\ast : R'_\ast(n) \rightarrow R_\ast(n-1)$. Observe that if $a = v(n) - 1$, then $F_\ast$ is simply

$$F \wedge 0 : R(n_{a+1} p) \wedge F(n_{a+1} p) \rightarrow R(n_{a+1} p - 1) \wedge F(n_{a+1} p - 1)$$

and therefore $X_\ast(n)$ is the same as

$$X_\ast(n) = X_\ast(n_{a+1} p) \wedge R_\ast(n_{a+2} p^2) \wedge \cdots \wedge R_\ast(n_b p^b). \quad (39)$$

Since for $X(n_{a+1} p) = E_{n_a}$, the homology is free as an $E_\ast$-comodule, it follows that $H_\ast(X_\ast(n))$, $\ast(n) = 1$, is also free as an $E_\ast$-comodule. As an $\ast(1)\ast$-comodule $H_\ast(X_\ast(n))$ consists of a sum of suspensions of $E_i$. The precise decomposition may be computed with the help of (15).

Let $N_\ast(n) = H_\ast(X_\ast(n))$, then $N_\ast(n)$ and $N_\ast(n)$ differ in the first factor. We construct an $\ast(1)\ast$-surjection with finite free kernel

$$\tau : N_\ast(sp^b) \rightarrow N_\ast(sp^b)$$
and define $\omega_n^\otimes : N_\otimes(n) \to W(n)$ by composing $\omega_n^\otimes$ with $\tau \otimes 1$. For the construction of $\tau$ define a map $\mu : N_i \otimes p \to N_i \otimes N^p$ for $i \geq 1$, $k \geq 1$ by

$$
\begin{align*}
\mu(c_{j+p}) &= a_j \otimes b_{p}, & 0 \leq j \leq i \\
\mu(c_{i+p}) &= a_i \otimes b_{p}, & 0 \leq j \leq i \\
\mu(c_{p-j}) &= a_{0} \otimes b_{p-j}, & j \geq 0 \\
\mu(c_{p-j}) &= a_{0} \otimes b_{p-j}, & j \geq 0.
\end{align*}
$$

Here $a_j, b_j, \bar{b}_j, c_j, \bar{c}_j$ denote the standard generators for $N^i$, $N^p$, and $N^{i+k}$ (see (16)).

**Lemma 3.5.** $\mu$ is an $\mathcal{A}(1)_\ast$-monomorphism with finite free cokernel.

**Proof.** That $\mu$ is compatible with the $Q_0, Q_1, \mathcal{P}^1$-action is clear from the definition. Proceeding as in [7] (4,6) one easily shows that $\mu$ induces an isomorphism in $Q_0, Q_1, \mathcal{P}^1$-homology which implies the result.

The splitting (33) of $l \wedge A$ is induced by maps

$$
\rho_n : \Sigma^{m-1} X(n) \to l \wedge A
$$

but these maps do not factor through $A \wedge A \to l \wedge A$, since $Q \wedge 1 \circ \rho_n \neq 0$. In Section 5 we shall modify the splitting of $l \wedge A$ and construct maps

$$
\rho_n^K : \Sigma^{m-1} X^k(n) \to l \wedge A
$$

having the following properties:

$$
Q \wedge 1 \circ \rho_n^K \simeq 0 \quad (40)
$$

$$
\tilde{\rho}_n^K : H_* (\Sigma^{m-1} X_k(n)) \to H_*(A) \quad (41)
$$

the map induced by $\rho_n^K$ and the change-of-rings-theorem, is an injective $\mathcal{A}(1)_\ast$-comodule homomorphism with image $W^A(n)$ (see Section 2). Assuming (40) and (41) we now can derive the splitting of $A \wedge A$.

The map $S = \bigvee_n \rho_n^K : \bigvee_{n = 0(p)} \Sigma^{m-1} X_k(n) \to l \wedge A$ has the properties

$$
Q \wedge 1 \circ S \simeq 0
$$

and $\tilde{S}_* : H_* (\bigvee \Sigma^{m-1} X_k(n)) \to H_*(A)$ is an isomorphism of $E_\ast$-comodules. By Proposition 2.1

$$
\tilde{S} : l \wedge \bigvee_{n = 0(p)} \Sigma^{m-1} X_k(n) \to l \wedge A
$$

is an equivalence. Let now $P : \bigvee_{n = 0(p)} \Sigma^{m-1} X_k(n) \to A \wedge A$ be any map with $D \wedge 1 \circ P \simeq S$. Then
Theorem 4.1. The $A$-extension of $P$

$$\tilde{P} : A \wedge \bigvee_{n \in O(p)} \Sigma^{p-1} X_k(n) \to A \wedge A$$

is an equivalence of $p$-complete $A$-module spectra.

Theorem 4.1 will follow from (42) and the following proposition:

Proposition 4.2. Given a map $f : X \to \mathcal{A}Y$ with $Q \wedge 1 : f \simeq 0$ and $\tilde{f}_* : H_*^A(X) \to H_*^A(Y)$ an $E_\infty$-comodule isomorphism, then any $F : X \to \mathcal{A}Y$ with $D \wedge 1 \circ F \simeq f$ induces an equivalence

$$\tilde{F} : A \wedge X \to A \wedge Y.$$

For $p \neq 2 \text{Im}(J)$-theory $A$ is well known to be a commutative ring spectrum. The multiplication map $\mu_A : A \wedge A \to A$ fits into the commutative diagram of cofibrations

$$\begin{array}{c}
A \wedge A \\
\downarrow \mu_A \\
\Sigma A \wedge A
\end{array} \to
\begin{array}{c}
\Sigma A \wedge A \\
\downarrow \Sigma \mu_A \\
A \wedge A
\end{array}$$

where $\mu = \mu \circ D$. For a proof see Section 4 of [10] or [6].

Proposition 4.3. For a map $F : X \to \mathcal{A}Y$ let $f = D \wedge 1 \circ F : X \to \mathcal{A}Y$. Then the following diagram commutes:

$$\begin{array}{c}
A \wedge X \\
\downarrow 1 \wedge F
\end{array} \to
\begin{array}{c}
\Sigma A \wedge X \\
\downarrow \Sigma \mu_A
\end{array}$$

As already indicated Theorem 4.1 determines the $A$-theory operations.

Corollary 4.4. $A^*(A) \cong \prod_{n \in O(p)} A^*(\Sigma^{p-1} X_k(n))$.

Proof. By Proposition 2.2 $P_1$ is an isomorphism.
Using the models $X_{\phi}(n)$ we only get a stable splitting:

In Section 5 we shall construct maps $\rho_{\phi}^{*}: \Sigma^{n-1}X_{\phi}(n) \to l \land A$ satisfying $Q \land 1 \circ \rho_{\phi}^{*} \simeq 0$ and $(\rho_{\phi}^{*})_{*}: H_{*}(\Sigma^{n-1}X_{\phi}(n)) \to H_{*}(A)$ is an $\mathcal{A}(1)_{\phi}$-comodule map with image $W^{A}(n)$ and kernel $T(n)$, which is a finite free $\mathcal{A}(1)_{\phi}$-comodule.

Since $\mathcal{A}(1)_{\phi}$ is injective (e.g. [9]), we can choose an $\mathcal{A}(1)_{\phi}$-splitting $\tau_{n}: H_{*}(\Sigma^{n-1}X_{\phi}(n)) \to T(n)$. We realize $T(n) = \bigoplus_{i} \Sigma^{n}E_{p}$ as $T(n) := \bigvee_{i} \Sigma^{n}E_{p}$ such that $H_{*}(T(n)) = T(n)$.

Since $l \land T(n)$ is a generalized Eilenberg–Mac Lane spectrum, the Hurewicz map

$$
\text{Hom}_{\mathcal{A}(1)}(H_{*}(\Sigma^{n-1}X_{\phi}(n)), H_{*}(l \land T(n)))
$$

is an isomorphism and there is a map $\tau_{n}: \Sigma^{n-1}X_{\phi}(n) \to l \land T(n)$ inducing $\tau_{n}: H_{*}(\Sigma^{n-1}X_{\phi}(n)) \to T(n)$. The map $\tau_{n}$ is by construction an $\mathcal{A}(1)_{\phi}$-comodule map and from Proposition 6.1 we obtain $Q \land 1 \circ \tau_{n} \simeq 0$.

Let $T := \bigvee_{n=0(p)} T(n)$ then the map

$$S_{\phi}: \bigvee_{n=0(p)} \Sigma^{n-1}X_{\phi}(n) \to l \land (A \lor T)$$

defined by adding up $\rho_{\phi}^{*} \lor \tau_{n}$ induces an equivalence

$$S_{\phi}: \bigvee_{n=0(p)} \Sigma^{n-1}X_{\phi}(n) \to l \land (A \lor T).$$

Since $Q \land 1 \circ S_{\phi} \simeq 0$ there is a map $P_{\phi}: \bigvee_{n=0(p)} \Sigma^{n-1}X_{\phi}(n) \to A \land (A \lor T)$ with $D \land 1 \circ P_{\phi} \simeq S_{\phi}$. The same proof as for Theorem 4.1 shows

**Theorem 4.5.** The $A$-extension of $P_{\phi}$

$$P_{\phi}: \bigvee_{n=0(p)} A \land \Sigma^{n-1}X_{\phi}(n) \to (A \land A) \lor (A \land T)$$

is an equivalence of $p$-local $A$-module spectra.

Since $A \land E_{p} \simeq HF_{p} \lor \Sigma^{n-1}HF_{p}$ (see Proposition 4.6 below) $A \land T$ is a wedge of mod $p$ Eilenberg–Mac Lane spectra and Theorem 4.5 may be called a stable splitting. Dispensing the property “$A$-module splitting” one may obtain an actual splitting of $A \land A$ by splitting off $A \land T$ from the left-hand side in Theorem 4.5.

We now identify $A \land E_{i}$ where $E_{i}$ is defined by the cofibre sequence

$$R^{i} \xrightarrow{\mu} R^{i-1} \to E_{i} \to \Sigma R^{i} \to .$$

The induced sequence in $l$-theory easily gives $l_{*}(E_{i})$ (for $l_{*}(R^{i})$ see e.g. [5] Section 1) and the action of $\mathcal{P}^{i}$ on $H_{*}(E_{i}) = E_{i}$ determines the action of $Q$ on $l_{*}(E_{i})$ (again see e.g. [5] Section 3). This implies that $A \land E_{i}$ has exactly two non-trivial homotopy groups

$$\pi_{0}(A \land E_{i}) = \mathbb{Z}/p$$
$$\pi_{-1}(A \land E_{i}) = \mathbb{Z}/p.$$

Let $HF^{i}$ denote the cofibre spectrum of the $i$th Steenrod power $\mathcal{P}^{i}$, where the dimensions are adjusted by the cofibre sequence

$$\Sigma^{-1}HF_{p} \xrightarrow{\mathcal{P}^{i}} \Sigma^{i-1}HF_{p} \xrightarrow{\Delta} HF^{i} \to HF_{p} \to .$$
Then $H_{p\xi}^i$ has non-trivial homotopy groups in the same dimensions as $A \wedge E_i$. If the Postnikov invariant of $A \wedge E_i$ is non-trivial, then $A \wedge E_i$ must be $H_{p\xi}^i$; otherwise $A \wedge E_i \simeq HF_p \vee \Sigma^{q-1}HF_p$. Computing $A \wedge E_i, H_{p\xi}^i$ and $HF_p \vee \Sigma^{q-1}HF_p$ on some $Y$, with non-trivial $p\xi$-action—for example on $R^i$ or $R^p$—then shows: For $i < p$ the Postnikov invariant of $A \wedge E_i$ is non-zero, whereas the one of $A \wedge E_p$ is trivial. Hence

**Proposition 4.6.** For $i < p$, $A \wedge E_i \simeq H_{p\xi}^i$ and $A \wedge E_p \simeq HF_p \vee \Sigma^{q-1}HF_p$.

**Remark.** Explicit equivalences may be constructed as follows: For $i < p$ consider the commutative diagram

$$
\begin{array}{ccc}
\Sigma^{-1}l \wedge E_i & \xrightarrow{Q \wedge 1} & \Sigma^{q-1}l \wedge E_i \\
\downarrow T_1 & & \downarrow T_2 \\
\Sigma^{-1}HF_p & \xrightarrow{\tau_{p\xi}} & \Sigma^{q-1}HF_p \\
\end{array}
\rightarrow
\begin{array}{c}
A \wedge E_i \\
\downarrow T_3 \\
H_{p\xi}^i
\end{array}
$$

with $T_1 := ch_0^i \wedge 1_{E_i} \in H^0(l \wedge E_i)$ and

$$
T_2 := (-1)^i \sum_{j=1}^{q-1} c_j^i \cdot \chi_{p\xi}^{i-j}(1_{E_i}) \in H^{q(i-1)}(l \wedge E_i).
$$

Here $c_j := (\hat{k}^{i} - 1)/p$, $1_{E_i} \in H^0(E_i)$ is a generator and $ch_0^i \in H^i(l)$ is the mod $p$ reduction of the integral Chern character of Adams (e.g. see [16]). Since $Q^*ch_0^i \wedge 1_{E_i} = c_i \cdot ch_0^i$ [16] and $ch_0^i \in H^*(A)$, the commutativity of the left-hand square above follows. Let $T_3$ be a fill in map, which is unique up to a constant since $(H_{p\xi}^i)(A \wedge E_i) \cong F_p$. It follows that $T_3$ induces an isomorphism in $\pi_0$ and $\pi_{q-1}$.

For $i = p$ we have two cohomology classes

$$
ch_0^p \wedge 1_{E_p} \in H^0(A \wedge E_p), \quad ch_{q-1}^p \wedge 1_{E_p} \in H^{p-1}(A \wedge E_p)
$$

where $ch_0^p \in H^i(A)$ are the generators in the first two dimensions where $H^i(A)$ is non-zero (e.g. see (20)). The natural transformation induced by $ch_0^p$ is called $A$-theory Chern character in [16] and its definition easily implies that

$$
ch_0^p : A_0(E_p) \rightarrow H_0(E_p) \quad \text{and} \quad ch_{q-1}^p : A_{q-1}(E_p) \rightarrow H_0(E_p)
$$

are isomorphisms. From this it follows that

$$
ch_0^p \wedge 1_{E_p} \wedge ch_{q-1}^p \wedge 1_{E_p} : A \wedge E_p \rightarrow HF_p \vee \Sigma^{p-1}HF_p
$$

is an equivalence.

The nature of the operation in $A^*(A)$ is then rather clear: (44)

(a) Firstly, we have Eilenberg–Mac Lane operations, that is $a \in A^*(A)$ factorizes through $A \wedge E_p \simeq HF_p \vee \Sigma^{p-1}HF_p$. These elements may be written as a composition of $ch_0^p, ch_{q-1}^p$, ordinary cohomology operations and $\hat{\beta}_1, \hat{\beta}_2 \in [A \wedge E_p, A]^\pi$ for $\beta_1, \beta_2$ the two generators of $A^*(E_p)$.

(b) The splitting pieces $E_i, i < p$, give $H_{p\xi}^i$-operations, that is maps factorizing through $A \wedge E_i \simeq H_{p\xi}^i$.

(c) The splitting piece $X_K(0) = S^1$ gives the coefficient operations, e.g. maps $\bar{z} : \Sigma^{q-1}A \rightarrow A$ defined as $A$-extensions of $z \in A_{q-1}(S^1)$.

(d) Elements $a$ in $\text{im}(\Delta : l_*(X_K(n)) \rightarrow A^*(X_K(n)))$ will by (4.4) in [5] factorize up to an Eilenberg–Mac Lane operation as $\Delta \wedge \theta \circ D$ where $\theta \in l_*(l)$ is a torsion element (= torsion $l$-operation).
(e) Elements \( a \in A^*(\Sigma^{p^{-1}}X_k(n)) \) \( A^*(A) \) not in \( \text{im}(\Delta) \) are new operations. Again by (4.4) of [5] those operations are factorizations of torsion \( l \)-operations through \( A \), e.g. \( Da = \theta \circ D + z \) where \( z \) is as in (a) and \( \theta \) as in (d).

It remains to compute \( A^*(X_k(n)) \). This will not be done in this paper; the details seem to be too long and tedious and are best postponed until after they are needed for some sensible application. However, in order to get some idea for the size of \( A^*(X_k(n)) \) we shall indicate the main steps for a computation.

For \( n \neq 0(2^2) \) it is shown in Section 5 that \( A \wedge X_k(n) \) splits as a wedge of suspensions of \( A \wedge E_i \), so \( A^*(X_k(n)) \) is given by Proposition 4.6, duality and counting arguments in this case. Let now \( n \equiv 0(p^2) \) and \( n > 0 \). Both \( X_k(r) \) (42) and \( X(r) \lor Z(r) \) (33), \( r \equiv 0(p) \), give splittings of \( I \wedge A \). Hence there are isomorphisms
\[
h_n^*: I^*(X_k(r)) \to I^*(X(r) \lor Z(r))
\]
by Proposition 2.2. But \( I^*(X(n)) \) is easily computed [5] (4.6):
\[
l^{2+qj}(X(n)) = \mathbb{Z}/p^r
\]
with \( m = v(n!) \), \( r_j = \min\{v(n), m + 1 - j\} \), \( j = 1, \ldots, m \). This and the action of \( Q \) follows easily from the defining cofibre sequence for \( X(n) \). This determines \( A^*(X(n)) \), an example is in (45) below:

\[
I^*(Y) \quad \text{and} \quad A^*(Y) \quad \text{for } p = 3 \quad \text{and} \quad Y = \mathbb{S}X^{22}
\]

\[
\begin{array}{ccccccccccccc}
\Delta & \bigcirc & \bigtriangledown & \bigcirc & \bigcirc & \bigtriangledown & \bigcirc & \bigcirc & \bigtriangledown & \cdots & \bigtriangledown & \bigcirc & \bigcirc & \bigtriangledown \\
| & | & | & | & | & | & | & | & | & | & | & |
\end{array}
\]

(45)

\[
j: \quad 1 \quad 3 \quad 6 \quad 9 \quad \ldots \quad 15 \quad 18 \quad 21
\]

\[
\begin{array}{ccccccc}
\bigcirc, \triangle, \bigtriangledown, \bigstar \quad \text{represent a copy of } \mathbb{Z}/p \text{ in } l^{2+qj}(Y) \\
\bigstar \quad \text{represent a copy of } \mathbb{Z}/p \text{ in } A^{3+qj-q}(Y) \text{ (coker } Q) \\
\bigtriangledown \quad \bigstar \quad \text{represent a copy of } \mathbb{Z}/p \text{ in } A^{2+qj}(Y) \text{ (ker } Q) \\
\end{array}
\]

But since in general \( Q \wedge 1 \circ h^n \neq 0 \), \( h^n \) need not commute with \( Q \) and, indeed, it turns out that the action of \( Q \) on both groups is not quite the same. There is some small mixing between some of the \( E_i \) in \( Z(n) \) and \( X(n) \) after applying \( h^n \) (due to the fact that the lightning flash comodule in \( W(n) \) has a tag, see [7]). But the mixing is minimal and \( A^*(X(n) \lor Z(n)) \) is already a good approximation for \( A^*(X_k(n)) \).

The isomorphism \( h^n \) composed with the projection onto \( I^*(X(n)) \) may be induced by a map \( X_k(n) \to I \wedge X(n) \) (see the discussion in Section 5 below). This map has a factorization
may now be used to compute $A_\ast(X_k(n))$ from which $A_\ast(X_k(n))$ follows by duality arguments ($A_\ast(X_k(n))$ is isomorphic to $A_{j-3}(X_k(n))$).

A closer analysis of the cofibre sequence of $Y_k$ gives the following: The spectrum $Y_k(n)$ is a finite two-stage Postnikov system of graded mod $p$ Eilenberg–Mac Lane spectra which—depending on $n$—can be rather large and complicated. Its homotopy may contain non-trivial additive extensions, i.e. cyclic groups of order $p^j$. The exact sequence induced by (46) in homotopy is nearly always short exact. Only in one dimension $q_j$ with $j = \lfloor m/p \rfloor$ and $m = v_p(n - p!)$ the map $(\tilde{f}_k)_\ast$ can have a non-trivial cokernel of order $p$ depending on $v_p(n)$ and $v_p(j)$. There may be non-trivial additive extensions in dimensions $q_j$ and $q_j - 1$ but in dimensions different from $q_j$ and $q_j - 1$ there is a splitting for $(\tilde{f}_k)_\ast$.

5. MODIFICATION OF THE SPLITTING OF $l \wedge A$

The splitting maps $\rho_\ast: \Sigma^{m-1}X(n) \rightarrow l \wedge A$ of [5] do not factor through $D: A \wedge A \rightarrow l \wedge A$. The obstruction for this is a pure Adams-filtration zero phenomenon, there are no obstructions in higher Adams filtration. By changing $\rho_\ast$ and $X(n)$ in “Adams-filtration 0” we shall produce maps

\[ \rho^K_\ast: \Sigma^{m-1}X_k(n) \rightarrow l \wedge A \quad \text{and} \quad \rho^K_\ast: \Sigma^{m-1}X_k(n) \rightarrow l \wedge A \]

which will be shown in Section 6 to factor through $A \wedge A$.

The construction of $\rho^K_\ast$ and $\rho^K_\ast$ is parallel to the one of $\rho_\ast$ given in [5]. Therefore we shall assume that the reader has knowledge of [5] and use the notation and results of Sections 2 and 3 of [5] freely.

The construction is done in several steps. In step 1 we compare $l \wedge K(n)$ and $l \wedge R(n)$. Recall from Section 3 the $E_\ast$-comodule isomorphism

\[ S^K_\ast: H_\ast(K(n)) \rightarrow W(n) \]

From [5] Section 1 we have an $E_\ast$-comodule isomorphism

\[ \Sigma^{m}(L(n) \oplus F(n)) \rightarrow W(n) \]

where $L(n)$ is the lightning flash comodule and $F(n)$ is a finite free $E_\ast$-comodule. Composing $S^K_\ast$ with the inverse of (47) we obtain an $E_\ast$-isomorphism

\[ H_\ast(K(n)) \rightarrow L(n) \oplus F(n). \]

The $E_\ast$-comodule $L(n)$ is realized by $H_\ast(R(n))$ and for $F(n)$ we choose a wedge $F(n)$ of suspensions of $V(1)$'s such that $H_\ast(F(n)) = F(n)$. We now construct a map

\[ f^K_\ast: K(n) \rightarrow l \wedge (R(n) \vee F(n)) \]

such that $f^K_\ast: H_\ast(K(n)) \rightarrow H_\ast(R(n) \vee F(n))$, the map induced by $f^K_\ast$ and the change-of-rings isomorphism, is the $E_\ast$-isomorphism (48) above. With $R = K(n)$, $T = F(n)$ this will follow from
PROPOSITION 5.1. Let \( R, T \) be spectra such that \( g: H_\ast(R) \cong H_\ast(R(n)) \oplus H_\ast(T) \) as \( E_\ast \)-comodules with \( H_\ast(T) \) finite and \( E_\ast \)-free. Then there exists a map \( f: R \to l \wedge (R(n) \vee T) \) such that \( f_*: H_\ast(R) \to H_\ast(R(n) \vee T) \) is the given isomorphism \( g \).

Proof. Consider the Adams spectral sequence for \( [R, l \wedge (R(n) \vee T)]_\ast \). Its \( E_2 \) term consists of the groups (10) and (19)

\[
E_2^{p,q}(H_\ast(R), H_\ast(R(n) \vee T))
\]

For \( s > 0 \) these groups depend only on the stable isomorphism type of \( H_\ast(R) \) and \( H_\ast(R(n) \vee T) \) as \( E_\ast \)-comodules (see [17] or [8]) which is \( L(n) \) in both cases. By [17] (3.8) we have an isomorphism

\[
E_2^{p,q}(L(n), L(n)) \cong Ext_{E_\ast}(L(n), L(n))
\]

for \( s > 0 \) since \( L(n) \) is an invertible \( E_\ast \)-comodule. It is easy to see that this isomorphism is compatible with the homomorphism of spectral sequences induced by multiplication with \( p \). The effect of multiplication by \( p \) on \( Ext_{E_\ast}(F_\ast, F_\ast) \) is well known:

It induces a monomorphism \( Ext_{E_\ast}^{p,q}(F_\ast, F_\ast) \to Ext_{E_\ast}^{p+1, q+1}(F_\ast, F_\ast) \). This forces all differentials in the spectral sequence for \( [R, l \wedge (R(n) \vee T)]_\ast \) to be zero. The argument is exactly the same as for 17.12 in [8]. Hence the Hurewicz map

\[
[R, l \wedge (R(n) \vee T)]_\ast \to Hom_{E_\ast}(H_\ast(R), H_\ast(R(n) \vee T))
\]

is onto and we may realize the isomorphism \( g \) by a map \( f \).

Remark. Alternatively the explicit description of \( R = K(n) \) provides us with a map of Adams filtration \( c = v(n!) \) to \( S^0 \) (use (2.5) or (1.4) in [18]), hence a map \( K(n) \to l[c] \) lifting \( K(n) \to S^0 \to l \) where \( l[c] \) is the \( c \)th term in a minimal Adams resolution of \( l \). Since \( l[c] \cong l \wedge R(n) \) by (1.15) of [5] we have a map \( K(n) \to l \wedge R(n) \) realizing the harder part of the isomorphism \( H_\ast(K(n)) \to H_\ast(R(n) \vee F(n)) \).

COROLLARY 5.2. The \( l \) extension of \( f^*: K(n) \to l \wedge (R(n) \vee F(n)) \) gives an equivalence

\[
\tilde{f}^*: l \wedge K(n) \to l \wedge (R(n) \vee F(n)).
\]

Proof. This follows from Proposition 2.1.

Similarly Proposition 2.2 gives

COROLLARY 5.3. If \( l_Z \) is an \( l \)-module spectrum, then \( f^* \) induces an isomorphism

\[
f^*: l_Z^*(R(n) \vee F(n)) \to l_Z^*(K(n)).
\]

Step 2 (the pull back diagram): Next recall the pull back diagram (2.1) from [5]:

\[
\begin{array}{ccc}
[X, l \wedge I] & \xrightarrow{g} & F^0[X, G \wedge I] \\
\downarrow h & & \downarrow \tau \\
\text{Hom}_{E_\ast}(H_\ast(X), H_\ast(l)) & \xrightarrow{S} & \text{Shom}_{E_\ast}(H_\ast(X), H_\ast(Y))
\end{array}
\]
Here \( h \) and \( g \) are as in (26) and (27), \( F^0 \) is the image of \( g \), \( \text{Shom}_{E_a}(M, N) \) denotes the group of stable \( E_a \)-comodule maps and \( S \) is the canonical map. By (2.2) of \([S]\) there is, for \( X = \Sigma^mR(n) \), a well-defined map \( \tau \) such that (50) is a pull back diagram. We need

**Proposition 5.4.** With \( X = \Sigma^mK(n) \), (50) is a pull back diagram.

**Proof.** From the facts that \( F(n) \) is a wedge of \( V(1) \)'s, that \( V(1) \) is self-dual, \( l \wedge V(1) \cong HF_p \) and from (2.2) of \([S]\) we have that (50) with \( X = \Sigma^m(R(n) \vee F(n)) \) is a pull back diagram. Now the maps \((f^*)_!, (f^*)^*, \text{Hom}_{E_a}(f^*_*, 1), \text{Shom}_{E_a}(f^*_, 1) \) map (50) with \( X = \Sigma^m(R(n) \vee F(n)) \) isomorphically into (50) with \( X = \Sigma^mK(n) \). The resulting cube commutes by (26) and (27).

**Step 3 (construction of \( r^K_a \)):** The map \( r^K_a : \Sigma^kK(n) \to l \wedge l \) is constructed using Proposition 5.4 as follows: Note first that

\[
[\Sigma^kK(n), G \wedge l] \cong [\Sigma^mR(n), G \wedge l]
\]

(since \([V(1), G \wedge l] = 0\)) and

\[
\text{Shom}_{E_a}(H_*(\Sigma^kK(n)), H_*(l)) \cong \text{Shom}_{E_a}(H_*(\Sigma^mR(n)), H_*(l))
\]

(since \( H_*(F(n)) \) is \( E_\ast \)-free).

The element \( r_a : \Sigma^mR(n) \to l \wedge l \) constructed in \([S]\) defines an element \( g(f^*_!(r_a)) \) in \( F^0[\Sigma^mK(n), G \wedge l] \). In \( \text{Hom}_{E_a}(H_*(\Sigma^kK(n)), H_*(l)) \) we take the maps \( S^*_! \) from Section 3, (30). Then by the definition of \( f^*_! \) in (48) we have

\[
S(S^*_!(s_a)) = f^*_!(\tau g(r_a)) = \tau g(f^*_!(r_a))
\]

where \( s_a \) is the first component of (47) which satisfies \( S(s_a) = \tau g(r_a) \) by \([S]\) (2.9). Hence by Proposition 5.4 the pair \( (S^*_!, g(f^*_!(r_a)) \) defines a well-defined element in \([\Sigma^kK(n), l \wedge l] \) which we shall call \( r^K_a \).

**Step 4 (construction of \( r^K_a \)):** We begin with

**Proposition 5.5.** With \( v(n) \geq 1 \) the diagram

\[
\begin{array}{ccc}
\Sigma^kK(n) & \xrightarrow{F} & \Sigma^kK(n-1) \\
l^2 & \downarrow & \Sigma^2l^2 \\
l \wedge l & \xrightarrow{1 \wedge Q} & \Sigma^2l \wedge l
\end{array}
\]

commutes.

**Proof.** The maps \( f_1 := \Sigma^k_0 \circ F \) and \( f_2 := 1 \wedge Q \circ r^K_a \) are elements in \([\Sigma^{k-1}K(n), l \wedge l] \) and by Proposition 5.4 it suffices to show that they have the same images under \( g \) and \( h \). Since \( F \) is of positive Adams filtration we have \( h(f_1) = 0 \). On the other hand, \( h(r^K_a) = S^K_0 \) maps into the subspace \( W(n) \) of \( H_*(l) \) which is annihilated by \( Q_a = c \cdot \mathcal{P}_a \) (see (20)), hence \( h(f_2) = 0 \). To show that \( g(f_1) = g(f_2) \) we may restrict to \([\Sigma^{k-1}K, G \wedge l] \) (see \([S]\), proof of (3.1)) and the only information on \( F \) needed is the degree of \( F \) on the bottom cell. The proof is then the same as for (3.1) in \([S]\). \( \square \)
The commutative square in Proposition 5.5 defines a fill in map \( \rho_\eta^i \) in

\[
\begin{array}{ccccccc}
\Sigma^\eta K(n) & \xrightarrow{F} & \Sigma^\eta K(n-1) & \xrightarrow{1 \Delta^\eta} & \Sigma^\eta X_K(n) & \xrightarrow{1 \Delta^\eta} & \Sigma^\eta + 1 K(n) \\
\downarrow^1 \Delta^\eta & & \downarrow^1 \Delta^\eta & & \downarrow^1 \Delta^\eta & & \downarrow^1 \Delta^\eta \\
l \wedge l & \xrightarrow{1 \wedge Q} & \Sigma l \wedge l & \xrightarrow{1 \wedge \Delta} & \Sigma l \wedge A & \xrightarrow{1 \wedge D} & \Sigma l \wedge l
\end{array}
\] (51)

The map \( \rho_\eta^i \) in (51) is not quite fixed by the property of being a fill in map. We shall use:

**Proposition 5.6.** (a) Two fill in maps \( \rho_1^i, \rho_2^i : \Sigma^\eta - 1 X_K(n) \to l \wedge A \) satisfying \( \rho_1^i = \rho_2^i \) in \( \text{Hom}_{*}(H_*(\Sigma^\eta - 1 X_K(n)), H_*(A)) \) are homotopic.

(b) Every algebraic fill in map \( L \in \text{Hom}_{*}(H_*(\Sigma^\eta - 1 X_K(n)), H_*(A)) \) is realizable.

**Proof.** This is Proposition 3.5 in [5] and the proof is exactly the same except for one detail: In the proof that there is a choice \( \rho_3^i \) of \( \rho_3 \) having finite order one uses the isomorphism

\[
f_i^\eta : [\Sigma^\eta(R(n) \vee F(n)), l] \to [\Sigma^\eta K(n), l]
\]

and the fact that \( \rho_3 \) is of positive Adams filtration to see that \( \rho_3^i \circ \delta_2 = \rho_1^i - \rho_2^i \).

In the last step we describe an algebraic fill in map

\[
L : H_*(\Sigma^\eta - 1 X_K(n)) \to H_*(A)
\]

which we use to fix our choice for \( \rho_3^i \). We first treat the case \( v(n) \geq 2 \). From [7] (5.7) we shall use the result that the short exact sequence of \( \mathcal{A}(1)_* \)-comodules (22)

\[
0 \to \Sigma^{p-1} W(n-p) \xrightarrow{\Delta^s} W^A(n) \xrightarrow{D^s} W(n) \to 0
\]
splits if \( v(n) > 2 \). Let \( \tau : W(n) \to W^A(n) \) be an \( \mathcal{A}(1)_* \)-section of \( D^s \). Since \( F \) is of Adams filtration at least 2 (Corollary 3.3), the short exact sequence

\[
0 \to H_*(K(n-1)) \to H_*(X_K(n)) \to H_*(\Sigma K(n)) \to 0
\]

splits as a sequence of \( \mathcal{A}_* \)-comodules. We combine these two sequences

\[
\begin{array}{ccccccc}
0 & \to & H_*(\Sigma^\eta - 1 K(n-1)) & \to & H_*(\Sigma^\eta - 1 X_K(n)) & \to & H_*(\Sigma^\eta K(n)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Sigma^{p-1} W(n-p) & \xrightarrow{\Delta^s} & W^A(n) & \xrightarrow{\tau} & W(n) & \to & 0
\end{array}
\] (52)

choose a direct sum decomposition of \( H_*(X_K(n)) \) over \( \mathcal{A}_* \) and define \( L \) in the obvious way. Now \( S^K, S^L_{p-1} \) are \( \mathcal{A}(1)_* \)-comodule maps by definition and \( \Delta^s, \) which is \( x \mapsto \Delta^s(x \cdot \Sigma^{p-1}) \), is \( \mathcal{P}^1 \)-compatible since \( \mathcal{P}^1(\Delta^s(\Sigma^{p-1})) = 0 \). Hence \( L \) is an \( \mathcal{A}(1)_* \)-comodule isomorphism. We use this algebraic fill in map to fix \( \rho_3^i \) in this case.

Let now \( v(n) = 1 \) and write \( n = ap + m, 0 < a < p, v(m) > 1 \). The key observation in this case is the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & \Sigma^{p-1} W(ap-p) \otimes W(m) & \xrightarrow{\Delta^s \otimes 1} & W^A(ap) \otimes W(m) & \to & W(ap) \otimes W(m) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Sigma^{p-1} W(n-p) & \xrightarrow{\Delta^s} & W^A(n) & \to & W(n) & \to & 0
\end{array}
\] (53)
Here $m_1, m_2$ are induced by the product in $H_*(A)$ and a section $\tau: W(m) \to W^d(m)$. In [7] (5.8) it is then proved that $m_1, m_2, m_3$ are surjective $\alpha(1)_* \text{-comodule}$ maps with finite free kernels. In low dimensions $ap < p^2$ direct computation (or (5.2) with a verification that there is an $\alpha(1)_* \text{-fill in map } L$) shows $W^d(ap) \cong \Sigma^{\infty} \Sigma^{ap-1} E_\ast$. Hence $W^d(ap) \otimes W(m)$ and therefore $W^d(m)$ is by (15) a finite direct sum of suspensions of the $\alpha(1)_* \text{-comodules } E_i, 1 \leq i \leq p$.

Choose an $\alpha(1)_* \text{-isomorphism}$

$$z_*: \bigoplus_{i,j} \Sigma^{n_0} E_i \cong W^d(n).$$

Instead of $X_*(n)$ we may use the simpler and more explicit model $\bigvee_{i,j} \Sigma^{n_0} E_i$ as a splitting piece for $A \wedge A$: Since each $E_i \wedge l$ is equivalent to a wedge of $V(1) \wedge l$, $V(1)$ is self-dual and $V(1) \wedge l \cong Hlp_p$, the map

$$h: \left[ \bigvee_{i,j} \Sigma^{n_0} E_i, l \wedge A \right] \to \text{Hom}_{*}(H_*(\bigvee_{i,j} \Sigma^{n_0} E_i), H_*(A))$$

is an isomorphism and $z_*$ composed with the inclusion $W^d(n) \subset H_*(A)$ is induced by a map, which we shall call $\rho^F_*: \bigvee_{i,j} \Sigma^{n_0} E_i \to l \wedge A$.

Then $\rho^F_*$ is an $\alpha(1)_* \text{-comodule}$ monomorphism with image $W^d(n)$. The property $Q \wedge l = 1 \wedge F^2 \cong 0$ follows from Proposition 6.1 below, there is nothing to check in higher Adams filtration (since $h$ is an isomorphism). This will be enough for the splitting of $A \wedge A$.

To have a uniform treatment for all $n \equiv 0(p)$ we shall also sketch the argument for using $X_*(n)$ if $\nu(n) = 1$: The pairing between integral Brown–Gitler spectra defines the maps $m_1, m_2$ in the square

$$
\begin{array}{ccc}
K(m) \wedge K(ap) & \overset{1 \wedge F}{\longrightarrow} & K(m) \wedge K(ap-1) \\
\downarrow m_3 & & \downarrow m_1 \\
A \wedge K(n) & \overset{1 \wedge F}{\longrightarrow} & A \wedge K(n-1)
\end{array}
$$

\text{(54)}

\textbf{Lemma 5.7. Diagram (54) commutes.}

\textbf{Proof.} The corresponding square with $A$ replaced by $l$ will commute as an easy application of the relevant version of the pull back diagram (50). Hence $m_1 \circ 1 \wedge F - 1 \wedge F \circ m_3 = \Delta(z)$ for $z \in [K(m) \wedge K(ap), \Sigma^{ap-1} \wedge K(n-1)]$. Consider the commutative diagram with $X = K(m) \wedge K(ap), Y = l \wedge K(n-1)$:

$$
\begin{array}{ccc}
[X, \Sigma^{-1} Y] & \overset{Q}{\longrightarrow} & [X, \Sigma^{-1} Y] \\
\downarrow h_1 & & \downarrow h_2 \\
\text{Hom}_{\alpha}(H_*(X), H_*(\Sigma^{-1} Y)) & \overset{Q}{\longrightarrow} & \text{Hom}_{\alpha}(H_*(X), H_*(\Sigma^{-1} Y))
\end{array}
$$

Using $l \wedge K(n-1) \cong l \wedge (R(n-1) \vee F(n-1))$ and $l^*\!/(X) \cong l^*\!/(R(n) \vee F)$ where $F$ is a wedge of suspensions of $V(1)$'s it follows easily that both Hurewicz maps are isomorphisms (use (1.6) [5]). Since $F_\ast = 0 \text{ mod } p$ homology the same is true for $\Delta(z)$. Hence $h_2(z) = Q_\ast(z_1)$ and $z_1 = h_3(z_2)$. But then $Q(z_2) = z$ and $\Delta(z) = 0$ follows.

From (54) we obtain as a fill in map a map

$$m_2: K(m) \wedge X_K(ap) \to A \wedge X_K(n).$$
Let \( m_i = (D \wedge 1) \cdot m_i \). Then \( Q \wedge 1 \cdot m_2 \simeq 0 \) and this implies that \( m_2 : H_* (K(m) \wedge X_K (ap)) \to H_* (X_K (n)) \) is an \( \mathcal{A}(1)_* \)-comodule map (see the remark following the proof of Proposition 6.1 below).

Using \( l \) instead of \( A \) in (54) and the change-of-rings isomorphism we arrive at the commutative diagram with \( Y = H_* (K(m)) \):

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_* (K(ap - 1)) \otimes Y & \longrightarrow & H_* (X_K (ap)) \otimes Y & \longrightarrow & 0 \\
\downarrow m_1 & & \downarrow m_2 & & \downarrow m_3 & & \\
0 & \longrightarrow & H_* (K(n - 1)) & \longrightarrow & H_* (X_K (n)) & \longrightarrow & H_* (\Sigma K(n)) & \longrightarrow & 0
\end{array}
\]

(55)

Using the maps \( S^K \) we can compare (53) with (55). From the definition of \( S^K \) it follows that these maps are compatible with \( m_i, \tilde{m}_i, \) \( i = 1, 3 \), so that the left- and right-hand sides of (53) and (55) are isomorphic. In the middle we only have the isomorphism \( g_2 := (\rho^K_{\mathcal{A}}) \otimes S^K \). Since \( \tilde{m}_1, \tilde{m}_3 \) are onto with finite free kernels, the same is true for \( \tilde{m}_2 \) and this already implies that \( H_* (X_K (n)) \) has the same decomposition into \( E_*'s \) as \( \Sigma n \) has.

Now \( g_2 : H_* (X_K (ap)) \otimes H_* (K(m)) \to W^4 (ap) \otimes W (m) \) may not map ker(\( m_2 \)) into ker(\( m_2 \)). But it is an easy exercise to show that one can use the deviation from this to define a map \( \eta : H_* (\Sigma K(ap)) \otimes H_* (K(m)) \to \Sigma^{-4} W (ap - p) \otimes W (m) \) and add \( \Delta_* \otimes 1 \cdot \eta \cdot \tilde{\mathcal{J}} \) to \( g_2 \) to produce a new fill in map \( g_2 \) which has the property \( g_2 (\text{ker}(m_2)) \subset \text{ker}(m_2) \). Then \( g_2 \) induces a map \( L : H_* (X_K (n)) \to W^4 (n) \) which is automatically a fill in map (in (52)) and an \( \mathcal{A}(1)_* \)-isomorphism. This is the required algebraic fill in map to fix \( \rho^K_n \) for \( \nu (n) = 1 \).

Alternatively we may use \( \rho^K_n \).

We have proved

**Theorem 5.8.** For \( n \equiv 0 (p) \) there is a commutative diagram

\[
\begin{array}{cccccc}
\Sigma^{\nu K} (n) & \stackrel{F}{\longrightarrow} & \Sigma^{\nu K} (n - 1) & \longrightarrow & \Sigma^{\nu K} (X_K (n)) & \longrightarrow & \Sigma^{\nu + 1} K(n) \\
\downarrow \nu_{\Sigma} & & \downarrow \nu_{\Sigma^{\nu - 1}} & & \downarrow \nu_{\Sigma^{\nu - 1}} & & \downarrow \nu_{\Sigma^{\nu - 1}} \\
l \wedge l & \longrightarrow & \Sigma l \wedge l & \longrightarrow & \Sigma l \wedge l & \longrightarrow & \Sigma l \wedge l
\end{array}
\]

such that \( \rho^K_{\mathcal{A}} : H_* (\Sigma^{\nu - 1} X_K (n)) \to H_* (A) \) is an \( \mathcal{A}(1)_* \)-monomorphism with image \( W^4 (n) \).

**Remark.** The construction of \( \rho^K_{\mathcal{A}} : \Sigma^{\nu - 1} X_K (n) \to l \wedge A \) is almost identical to the one of \( \rho^K_n \). From [7] (4.6) we know that \( g_2 \) is a surjective map with a finite free \( E_* \)-comodule \( F^1 (n) \) as kernel. Instead of (48) one uses the composition

\[
H_* (R_{\mathcal{A}} (n)) \xrightarrow{s_{\mathcal{A}}} \Sigma^{-\nu} W (n) \otimes F^1 (n) \to L (n) \oplus F (n) \oplus F^1 (n)
\]

and replaces \( F(n) \) by \( F(n) \oplus F^1 (n) \) throughout. The necessary algebraic fill in maps to fix \( \rho^K_n \) are described in [7].

6. THE PROOF OF \( Q \wedge 1 \cdot \rho^K_n \simeq 0 \)

We start by studying the homology obstructions for a map \( f : X \to l \wedge Y \) to factor through \( D \wedge 1 : A \wedge Y \to l \wedge Y \). We need conditions on \( f_* : H_* (X) \to H_* (Y) \) which imply that \( Q \wedge 1 \cdot f \) is zero in mod \( p \) homology. Note that \( Q \) operates on the factor \( H_* (l) \) which disappears under the change-of-rings isomorphism. The condition we get is quite simple:
Proposition 6.1. Let \( f: X \to l \wedge Y \) be a map and \( f_*: H_*^e(X) \to H_*^e(Y) \) be the \( E_* \) comodule map defined by \( f_* \) and the change-of-rings isomorphism. If \( f_* \) is an \( A(1)_* \)-comodule map (i.e. \( f_* \) is in addition compatible with the \( P^1 \) action on \( H_*^e(X) \) and \( H_*^e(Y) \)), then \((Q \wedge 1 \circ f)_* = 0\) in mod \( p \) homology.

Up to a non-trivial mod \( p \) factor the action of \( Q \wedge 1 \) on \( H_*^e(l \wedge Y) \) is given by \( \Phi_* \otimes 1 \) where \( \Phi_*: A_* \to A_* \) is the map induced by \( P^1: HF_p \to \Sigma^* HF_p \). \( \Phi_* = c \circ P^1 \circ c \) with \( c \) conjugation in \( A_* \). Let \( N = H_*^e(X), M = H_*^e(Y) \). By the change-of-rings isomorphism we get back the homomorphism \( f_*: N \to H_*^e(l) \otimes M \) from \( f_* \) as composition (see Section 2):

\[
N \xrightarrow{\psi} A_* \otimes E_* N \xrightarrow{\Delta E_* f_*} A_* \otimes E_* M \xrightarrow{\Phi} (A_* \otimes E_* F_p) \otimes^N M_* .
\]

Lemma 6.2. Under the isomorphism \( \Phi: A_* \otimes E_* N \to (A_* \otimes E_* F_p) \otimes^N M, P^1 \otimes 1 + 1 \otimes P^1 \) corresponds to \( P^1 \otimes 1 \).

Proof: If we identify \( H_*^e(HF_p \wedge X) \) with \( A_* \otimes H_*^e(X) \) by using the Künneth isomorphism then the map induced by

\[
(i \wedge 1 \wedge 1): H_*^e(X) = \pi_*(S^0 \wedge HF_p \wedge X) \to \pi_*(HF_p \wedge HF_p \wedge X) = H_*^e(HF_p \wedge X)
\]

is \( c \otimes 1 \circ \psi \) (by definition of \( \psi \), e.g. see [8]). But \((i \wedge 1 \wedge 1) \circ P^1 = P^1 \circ (i \wedge 1 \wedge 1) \) which gives \( c \otimes 1 \circ \psi \) \( (P^1 X) \). Let now \( \psi(x) = \sum_i x_i^t \otimes x_i^r \) and \( x_i^t \). Then

\[
(\Phi \otimes 1) \Phi(a \otimes x) = \Phi \otimes 1 \left( \sum_i a \cdot x_i^t \otimes x_i^r \right) \]

by the Cartan formula

\[
= \Phi(\Phi(a) \otimes x) + a \cdot P^1 \otimes 1 \circ 1 \circ \psi(x)
\]

by the observation above

\[
= \Phi(a \otimes x) + \Phi(a \otimes P^1(x)).
\]

Lemma 6.3. In \( A_* \otimes E_* N \) we have \( (P^1 \otimes 1 + 1 \otimes P^1) \circ \psi = 0 \).

Proof. The assertion is equivalent to

\[
N \xrightarrow{\psi} A_* \otimes E_* N \xrightarrow{\Phi} (A_* \otimes E_* F_p) \otimes^N N \xrightarrow{\Delta N \otimes 1} (A_* \otimes E_* F_p) \otimes^N N
\]

is zero. Now \( g = i \wedge 1: S^0 \wedge X \to l \wedge X \) has \( g_* = id \). Hence \( \Phi \circ \psi \) is simply \( g_* \). But \( g = i \wedge 1 \) trivially factorizes through \( D \wedge 1: A \wedge X \to l \wedge X \). Hence \( Q \wedge 1 \circ g \simeq 0 \) which implies \( P^1 \otimes 1 \circ g_* = 0 \).

Proof of Proposition 6.1.

\[
P^1 \otimes 1 \circ f_* = P^1 \otimes 1 \circ \Phi \circ 1 \circ f_* \circ \psi
\]

(by Lemma 6.2)

\[
= \Phi \circ (P^1 \otimes 1 + 1 \otimes P^1) \circ 1 \circ f_* \circ \psi
\]

(by Lemma 6.3)
Remark. Using Lemmas 6.2 and 6.3 it is also easy to see that if \( f: X \to I \wedge Y \) satisfies \( Q \wedge l \circ f \simeq 0 \), then \( f \) is necessarily compatible with \( pH^1 \), i.e. is an \( \mathcal{A}(1)_* \)-comodule map.

**Corollary 6.4.** \( Q \wedge l \circ p_*^K \) induces the zero map in mod \( p \) homology.

**Proof.** This follows from Proposition 6.1 and Theorem 5.8.

We now turn to the obstructions in higher Adams filtration. The main step is

**Proposition 6.5.** The composition

\[
\Sigma^{q_0} K(n - 1) \overset{r_1^*}{\to} \Sigma^q I \wedge I \overset{1 \wedge \Delta}{\to} I \wedge \Sigma A \overset{Q \wedge 1}{\to} \Sigma^{q+1} I \wedge A
\]

is zero.

The following three facts prepare the proof:

1. Recall the splittings (33) of \( l \wedge l \) and \( l \wedge A \). Denote the composition of the inverse of the splitting map and the projection onto the wedge summand \( l \wedge \Sigma^{q-1} X(n) \) by \( p_{n^1}^{l \wedge A} \). Similarly define maps \( p_{n^1}^{l \wedge A} : I \wedge l \to I \wedge \Sigma^{q_0} R(n) \).

From (3.3) of [5] we get a commutative diagram

\[
\begin{array}{ccc}
I \wedge l & \overset{1 \wedge Q}{\longrightarrow} & \Sigma^q I \wedge I \\
\downarrow p_*^{l \wedge i} & & \downarrow \Sigma^q p_*^{l \wedge i} \\
I \wedge \Sigma^q R(i) & \overset{1 \wedge F_{\infty}}{\longrightarrow} & I \wedge \Sigma^q R(i - 1) \\
\end{array}
\]

(56)

2. The “inverse” of \( p_{l \wedge i}^{l \wedge A} \) is the \( l \)-extension of the map \( r_m : \Sigma^{q_0} R(m) \to I \wedge l \). Under the inclusion \( i : S^{q_0} \to \Sigma^{q_0} R(m) \), \( r_m \) restricts to the element \( t_{m^1}^{l \wedge A} \) which is defined by

\[
t_{m^1}^{l \wedge A} := \frac{u - v}{p} \cdot \frac{u - k v}{p} \cdot \frac{u - k^2 v}{p} \cdots \frac{u - k^{m-1} v}{p}
\]

with \( k = k^{m-1} \), \( v = \eta_L(v_1) \), \( u = \eta_R(v_1) \). That the splitting of \( l \wedge l \) is defined by using \( r_m : I \wedge \Sigma^{q_0} R(m) \to I \wedge l \) immediately implies:

**Lemma 6.6.** \( p_{l \wedge i}^{l \wedge A}(t_{m^1}^{l \wedge A}) = \delta_{a, j} \gamma(0) \).

Here \( \gamma(0) \) denotes a generator of \( \pi_0(l \wedge R(j)) \). The computation of \( Q \wedge l \circ r_m \) will be reduced to

**Lemma 6.7.** \((Q \wedge l)(t_{m^1}^{l \wedge A}) = (1 - k^e/p) t_{m^1}^{l \wedge A}/t_{l}^{l \wedge A} \).

**Proof.** Using \( v \cdot Q = \psi^k - 1 \) and \( \psi^k(v) = k v \) the definition of \( t_{m^1}^{l \wedge A} \) implies the statement up to an element \( z \) with \( v \cdot z = 0 \). But by [8] those elements are detected by the Hurewicz map \( \pi_*(l \wedge l) \to H_*(l \wedge l) \) and an easy homology calculation finishes the proof.

**Remark.** For our application of Lemma 6.7 the statement up to \( v_1 \)-torsion will be sufficient.

**Lemma 6.8.** \( t_{m^1}^{l \wedge A}/t_{l}^{l \wedge A} = t_{m^1}^{l \wedge A - 1} + ((1 - k^{e-1}/p) v \cdot t_{m^1}^{l \wedge A - 2}/t_{l}^{l \wedge A} \).

\[
\text{Remark.}
\]
Proof. This follows directly from the definition of $t'_{1u}$. Here, of course, $t'_{1u}/t'_{1}$ means $t'_{1u}$ without the $(u - v/p)$-factor.

**Corollary 6.9.**

$$Q \wedge 1(t'_{1u}) = \frac{k_{u}}{p} t'_{u-1} + \frac{1}{p} \left( \frac{k_{u}}{p} \right) (1 - \frac{k_{u-1}}{p}) t'_{u-2} \cdot v + \ldots$$

3. Consider the following commutative diagram in which all groups are isomorphic to $\mathbb{Z}_{(p)}$:

$$\begin{array}{ccc}
\left[ R^{a+b}, l \wedge R^{a} \right] & \xrightarrow{F_{*}} & \left[ R^{a+b}, l \wedge S^{0} \right] \\

\downarrow i^{*} & & \downarrow i^{*} \\

\left[ R^{a}, l \wedge R^{a} \right] & \xrightarrow{F_{*}} & \left[ R^{a}, l \wedge S^{0} \right] \\

\downarrow i^{*} & & \downarrow i^{*} \\

\left[ S^{0}, l \wedge R^{a} \right] & \xrightarrow{F_{*}} & \left[ S^{0}, l \wedge S^{0} \right] \\
\end{array}$$

The facts:

(i) $I^{0}(B \Sigma^{\infty}_{+}) \cong \mathbb{Z}/p^{*}$,

(ii) $tr(1)$ is a generator of $I^{0}(B \Sigma^{\infty}_{p})$,

(iii) $F^{*}$ is of degree $p^{*}$ on the bottom cell

imply that $i^{*}: I^{0}(R^{a}) \rightarrow I^{0}(S^{0})$ is of degree $p^{*}$ and

$$(F^{*})^{*}(1) = F_{*}(\text{id}_{R^{a}} \wedge 1_{1}) \in I^{0}(F^{*})$$

is a generator. This immediately gives all the degrees as indicated in (57) above. All maps appearing in the proofs of Proposition 6.5 and Corollary 6.4 are subsumed in the commutative diagram

$$\begin{array}{ccc}
\Sigma^{a}K(n-1) & \xrightarrow{\delta} & \Sigma^{a}X_{K}(n) \\

\downarrow \Sigma^{a+1} & & \downarrow j \\

\Sigma^{a}l \wedge l & \xrightarrow{\Sigma^{a}l \wedge A} & \Sigma^{a+1}l \wedge l \\

\downarrow Q \wedge l & & \downarrow Q \wedge A \\

\Sigma^{a}l \wedge l & \xrightarrow{\Sigma^{a+1}l \wedge A} & \Sigma^{a+1}l \wedge l \\

\downarrow \Sigma^{a+1} & & \downarrow j \\

\Sigma^{a}l \wedge R(i-1) & \xrightarrow{\Sigma^{a}l \wedge X(i)} & \Sigma^{a}l \wedge X(i) \\

\downarrow \Sigma^{a+1} & & \downarrow j \\

\Sigma^{a}(i+1)l \wedge R(i-1) & \xrightarrow{\Sigma^{a}(i+1)l \wedge X(i)} & \Sigma^{a}(i+1)l \wedge X(i) \\
\end{array}$$

**Proof of Proposition 6.5.** Since $(1 \wedge A) \circ r_{n-1}^{a} \circ \rho_{n}^{a}$ factors through $\rho_{n}^{a}$ and $(Q \wedge 1 \circ \rho_{n}^{a})_{*} = 0$ in mod $p$ homology by Corollary 6.4 we have

$$((1 \wedge A) \circ (Q \wedge 1) \circ r_{n-1}^{a})_{*} = 0.$$
Since \(1 \wedge \partial ^{(i)}(p_i^{\wedge i} \wedge Q \wedge 1 \circ r_{K-1}^n) - p_i^{\wedge i} \wedge (1 \wedge \Delta) \wedge (Q \wedge 1) \circ r_{K-1}^n\) is of positive Adams filtration and \(1 \wedge \partial \) is injective in mod \(p\) homology (since \(v(i) > 0\)) we know that \(q_i := p_i^{\wedge i} \circ Q \wedge 1 \circ r_{K-1}^n\) is zero in homology.

Now \(g_i\) is in \([\Sigma^{q(n-i-1)} K(n-1), l \wedge R(i-1)]\) and this group is isomorphic to \([\Sigma^{q(n-i-1)}(R(n-1) \vee F(n-1)), l \wedge R(i-1)]\) via \((f_i^{n-1})^{-1}\) (Corollary 5.3). From \(g_i = 0\) it follows that the \(F(n-1)\)-component of \((f_i^{n-1})^{-1}(g_i)\) is zero. Since
\[
g_q : [\Sigma^{q(n-i-1)} R(n-1), l \wedge R(i-1)] \to [\Sigma^{q(n-i-1)} R(n-1), G \wedge R(i-1)]
\]
is injective we find that the \(R(n-1)\)-component of \((f_i^{n-1})^{-1}(g_i)\) is—by the definition of \(r_{K-1}^n\)—the element
\[
h_i := p_i^{\wedge i} \circ Q \wedge 1 \circ r_{K-1}^n
\]
and we have reduced to show \(1 \wedge \partial \circ h_i = 0\). This is done by showing \(m := q(n-i-1)\)
\[
h_i \in \text{im}(1 \wedge F^{v(n)} : [\Sigma^m R(n-1), l \wedge R(i)] \to [\Sigma^m R(n-1), l \wedge R(i-1)]).
\]
Both groups are isomorphic to \(Z_{(p)}\) or 0 (for \(n-1 < i\)) ((1.6) in [5]) and (59) can be checked by degree considerations:

The map \(i : S^0 \to R(n-1)\) induces a monomorphism and by Corollary 6.9 and Lemma 6.6 we have
\[
i^*(h_i) = p_i^{\wedge i} \circ Q \wedge 1 \circ i^*(r_{n-1}^i)
\]
\[
= p_i^{\wedge i} \left( \sum_{j=1}^{n-i} \prod_{j'=1}^{i} \frac{1 - k^{n-j}}{p} \cdot t_{n-1}^{v^{n-j}} \right)
\]
\[
= \left( \prod_{j=1}^{n-i} \frac{1 - k^{n-j}}{p} \right) \cdot v_i^{n-1-i}
\]
\[
= v_i^{n-1-i} \chi
\]
with \(\gamma^{(0)} \in n_0(l \wedge R(i-1))\) a generator.

In the commutative diagram
\[
[R(n-1), l \wedge R(i-1)] \xrightarrow{i^*} [S^0, l \wedge R(i-1)]
\]
\[
\xrightarrow{\tau^*} [S^0, l \wedge R(i-1)]
\]
we know the degrees by (57). This shows that
\[
x = \prod_{j=1}^{n-i} \frac{1 - k^{n-j}}{p} \cdot \gamma^{(0)}
\]
comes from \([R(n-1), l \wedge R(i)]\), hence \(i^*(h_i) = v_i^{n-1-i} \cdot x\) comes from \([\Sigma^{q(n-i-1)} R(n-1), l \wedge R(i)]\) and (59) is proved. \(\square\)

Now we are ready to prove:

**Theorem 6.10.** For \(n \equiv 0(p): Q \wedge 1 \circ p_{n-k}^\infty \simeq 0\).

**Proof.** We shall work with diagram (58). We know \((Q \wedge 1 \circ p_{n-k}^\infty) = 0\), arguing as above we see that it is enough to show
\[
h_i := p_i^{\wedge i} \circ Q \wedge 1 \circ p_{n-k}^\infty \simeq 0.
\]
By Proposition 6.5 we have $Q \land 1 \circ \rho^K \circ \delta = 1 \land \Delta \cdot Q \land 1 \circ \Sigma^q \rho^K_{n-1} \approx 0$, hence there is a factorization $\rho' : \Sigma^{q+1} K(n) \to \Sigma^{q+1} \land A$ of $Q \land 1 \circ \rho^K$ through $j$. As in the proof of (3.5) [5] we may suppose that $\rho'$ is of finite order. Since $j_* : H_*(X_k(n)) \to H_*(\Sigma K(n))$ is onto, $(Q \land 1 \circ \rho^K_{n-1} \approx 0$ implies $\rho'_* = 0$. Now $(1 \land j)_* \circ \rho^{l \land A} \circ \rho'$ in

$$[\Sigma^{(n-i-1)} K(n), l \land R(i)] \cong [\Sigma^{(n-i-1)}(R(n) \vee F(n)), l \land R(i)]$$

is of finite order, the elements of finite order in this group are detected by their homology homomorphisms and $\rho'_* = 0$ implies $1 \land j \circ \rho^{l \land A} \circ \rho' = 0$ and $p_{l \land A} \circ \rho'$ is in

$$im(\delta : \Sigma^{(i+1)} l \land R(i-1) \to \Sigma^{(i+1)} l \land X(i)).$$

Hence there is a map

$$\rho^* : \Sigma^{(n-i-1)+1} K(n) \to l \land R(l-1)$$

with $1 \land \delta \circ \rho^* = \rho^{l \land A} \circ \rho'$. Now $1 \land \delta$ is injective, so $\rho'_* = 0$ implies $\rho^* = 0$. But the group $[\Sigma^{(n-i-1)+1} K(n), l \land R(l-1)]$ consists of elements of Adams filtration 0 ((1.6)b in [5] and Corollary 5.3) so $\rho^* = 0$ implies $\rho^* = 0$, finishing the proof.

Remark. The proof that $Q \land 1 \circ \rho^F_0 \approx 0$ for $\rho^F_0 : \Sigma^{m-1} X_0(n) \to l \land A$ is exactly the same as for $\rho^K_n$.

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