NORTH-HOLLAND

Recursive Solution of Löwner-Vandermonde Systems of Equations. I*

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ABSTRACT

An $O(n^2)$ algorithm for the solution of a linear system the $n \times n$ coefficient matrix of which consists of a Vandermonde and a Löwner part is presented. In the first part of the algorithm the corresponding Vandermonde system is solved. In the second part the last rows of the Vandermonde matrix are successively replaced by rows of the Löwner structure. The algorithm is convenient if the Vandermonde part is dominant. The main tool is the connection of such systems with rational interpolation problems.

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1. INTRODUCTION

Throughout this paper we associate two given complex vectors $[w_i]_{i=-K+1}^{N}$ and $[z_i]_{i=-K+1}^{N}$ ($z_i \neq z_j$ for all $i \neq j$) with the $K \times N$ Löwner matrix

$$\mathbf{L}_K(w, z) := \left[\frac{w_{-i} - w_j}{z_{-i} - z_j}\right]_{i=0\ j=1}^{K-1\ N}.$$

Moreover, we denote by $\mathbf{V}_L(z^N)$, $z^N := [z_i]_{i=1}^{N}$ the $L \times N$ Vandermonde matrix

$$\mathbf{V}_L(z^N) := \left[z_j^i\right]_{i=0\ j=1}^{L-1\ N}.$$

The $(l+k) \times n$ matrices

$$\mathbf{A}_n^{lk} := \begin{bmatrix} \mathbf{V}_l(z^n) \\ \mathbf{L}_k(w, z) \end{bmatrix} \qquad (0 \leqslant l \leqslant N,\quad 0 \leqslant k \leqslant K,\quad 0 < n \leqslant N) \tag{1.1}$$

(as well as their transposes) will be called Löwner-Vandermonde matrices (with respect to the parameters introduced above). The main aim of this paper is to develop recursive formulas for the solution of systems of linear equations

$$\mathbf{A}y = b, \tag{1.2}$$

where the matrix

$$\mathbf{A} = \mathbf{A}_N^{LK} := \begin{bmatrix} \mathbf{V}_L(z^N) \\ \mathbf{L}_K(w, z) \end{bmatrix} \qquad (L + K = N) \tag{1.3}$$

is assumed to be nonsingular, and

$$b = \begin{bmatrix} b^L \\ b^{-K} \end{bmatrix}, \qquad b^L = [b_i]_{i=1}^{L}, \quad b^{-K} = [b_{-i}]_{i=0}^{K-1}. \tag{1.4}$$

(In case $L = 0$, $\mathbf{A}$ is a Löwner matrix; in case $K = 0$, $\mathbf{A}$ is a Vandermonde matrix.)

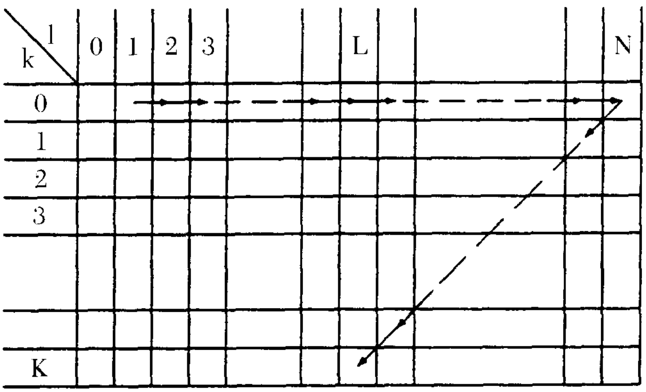
Löwner-Vandermonde systems (1.2) occur when solving certain interpolation problems involving two polynomials of degree less than N. This will be explained in more detail in Section 2. In the special case $w_i = 0$ for all $i < 0$

and $w_i = -1$ for $i \geqslant 0$, the Löwner part has the structure of a Cauchy matrix, i.e.

$$\mathbf{L}_K(w, z) = \left[\frac{1}{z_{-i} - z_j}\right]_{i=0\ j=1}^{K-1\ N},$$

and $\mathbf{A}$ is a Cauchy-Vandermonde matrix. Cauchy-Vandermonde matrices were considered for the first time in [4]. Linear systems with such coefficient matrices were also the subject of [3] and [6].

To compute the solution of (1.2) recursively, we consider a sequence of systems of linear equations with coefficient matrices of the form (1.1), where the couples (l, k) proceed in a sequence $\mathscr{A} \subset \mathscr{L} \times \mathscr{K}$, $\mathscr{L} = \{1, \dots, N\}$, $\mathscr{K} = \{1, \dots, K\}$ as illustrated in the following picture:



(1.5)

In particular we consider the square case $n = k + l$ and the case $n = k + l + 1$. In these cases we write briefly

$$\mathbf{A}^{lk} := \mathbf{A}^{lk}_{l+k}, \qquad \tilde{\mathbf{A}}^{lk} := \mathbf{A}^{lk}_{l+k+1}. \tag{1.6}$$

Throughout this paper we assume that every matrix $\mathbf{A}^{lk}$ for $(l, k) \in \mathscr{A}$ is nonsingular. We show that the recursion for the solutions of (1.2) can be easily found if nontrivial elements of the kernels of $\tilde{\mathbf{A}}^{lk}$ are used. Thus the main problem is to find recursions for the nontrivial solutions of the homogeneous systems

$$\tilde{\mathbf{A}}^{lk} x^{lk} = 0.$$

The rest of the paper is organized as follows. In Section 3 we recall a recursion formula for the solution $y^{l,0}$ of the nested Vandermonde systems the coefficient matrices of which are

$$\mathbf{V}_l(z^l) = \left[z_j^i\right]_{i=0}^{l-1}{}_{j=n}^{l} \qquad (l = 1, 2, \dots, N)$$

(see [6]). This recursion is illustrated by the horizontal line in the picture (1.5). Moreover, we describe the step from $(N-1, 0), (N, 0)$ to $(N-1, 1)$ (that is, the "break point" of our path in the picture). Section 4 is devoted to a three-term recursion

$$(l+1, k-1), (l, k) \to (l-1, k+1),$$

which is the path along the diagonal in the picture. The main tool utilized here is the connection between Löwner-Vandermonde systems and certain interpolation problems discussed in Section 2.

For the solution of Löwner systems an analogous idea is exploited in [5]. The connections of Löwner systems with rational interpolation problems have been already examined by many authors; see e.g. [1, 2, 7].

Finally let us note that further possibilities of recursions, including the systems with a transposed Löwner-Vandermonde coefficient matrix, are in preparation.

2. CONNECTION WITH INTERPOLATION PROBLEMS

We are going to formulate an interpolation problem corresponding to the equation

$$\mathbf{A}^{lk} y^{lk} = b^{lk} \qquad (l + k = n). \tag{2.1}$$

where $\mathbf{A}^{lk}$ is defined in (1.6), $y^{lk} = [y_i^{lk}]_{i=1}^n$, and

$$b^{lk} = \begin{bmatrix} b^l \\ b^{-k} \end{bmatrix}, \qquad b^l = [b_i]_{i=1}^l, \quad b^{-k} = [b_{-i}]_{i=0}^{k-1}. \tag{2.2}$$

Denote by $g_n(\lambda)$ the following monic polynomial of degree n:

$$g_n(\lambda) = \prod_{i=1}^n (\lambda - z_i).$$

(Here the explicit notation with indices k, l, n is not necessary, but it will be important in Sections 3 and 4.)

THEOREM 2.1. *The vector* $y^{lk} \in \mathbf{C}^n$ *is a solution of* (2.1) *if and only if the polynomial* $q(\lambda)$ *defined by*

$$q(\lambda) = \sum_{j=1}^{n} \frac{g_n(\lambda) y_j^{lk}}{\lambda - z_j} \tag{2.3}$$

satisfies the following conditions:

(1) *The entries of the vector* b^l *are the first* l *coefficients of the Laurent series expansion at infinity of*

$$\frac{q(\lambda)}{g_n(\lambda)}.$$

(2) *There exists a polynomial* $p(\lambda)$, deg $p < n$, *such that the pair* (q, p) *meets the following interpolation conditions*:

$$w_i q(z_i) - p(z_i) = \begin{cases} b_i g_n(z_i), & 1 - k \leqslant i \leqslant 0, \\ 0, & 1 \leqslant i \leqslant n. \end{cases} \tag{2.4}$$

Proof. Let y^{lk} be a solution of (2.1). Clearly,

$$\frac{q(\lambda)}{g_n(\lambda)} = \sum_{j=1}^{n} \frac{y_j^{lk}}{\lambda - z_j}.$$

Taking into account that the Laurent series expansion of $1/(\lambda - z_j)$ at infinity is

$$\frac{1}{\lambda - z_j} = \sum_{m=0}^{\infty} z_j^m \lambda^{-m-1},$$

the first l equations of the system (2.1) are equivalent to condition (1). Putting

$$p(\lambda) = \sum_{j=1}^{n} \frac{g_n(\lambda) y_j^{lk}}{\lambda - z_j} w_j,$$

the last k equations of the system (2.1) can be written in the form

$$\frac{w_{-i} q(z_{-i}) - p(z_{-i})}{g_n(z_{-i})} = b_{-i} \qquad (i = 0, 1, \dots, k-1),$$

which proves the first part of (2.4). The second part of (2.4) follows from the equalities

$$p(z_i) = g_n'(z_i) y_i^{lk} w_i \quad \text{and} \quad w_i q(z_i) = w_i g_n'(z_i) y_i^{lk}$$

for $0 < i \leqslant n$. Thus, the proof of the necessity part is complete.

To prove the sufficiency part we assume that condition (2) is satisfied. In particular, the conditions $\deg p < n$ and $p(z_i) = w_i q(z_i)$ for $i = 1, \dots, n$ imply that

$$p(\lambda) = \sum_{j=1}^{n} w_j q(z_j) \frac{g_n(\lambda)}{(\lambda - z_j) g_n'(z_j)}$$

(by the Lagrange interpolation formula). Taking (2.3) into account, the expression takes the form

$$p(\lambda) = \sum_{j=1}^{n} w_j y_j^{lk} \frac{g_n(\lambda)}{\lambda - z_j}.$$

Now, the first part of (2.4) and condition (1) mean that y^{lk} is a solution of (2.1). ∎

In the following we need a homogeneous variant of Theorem 2.1 with respect to the matrices $\mathbf{A}^{lk}$ defined in (1.6), $l + k = n$.

THEOREM 2.2. *The vector* $x^{lk} = [x_i^{lk}]_{i=1}^{n+1}$ *belongs to the kernel of* $\tilde{\mathbf{A}}^{lk}$ *if and only if the polynomial*

$$Q^{lk}(\lambda) := \sum_{j=1}^{n+1} \frac{g_{n+1}(\lambda) x_j^{lk}}{\lambda - z_j} \tag{2.5}$$

satisfies the following conditions:

(1) *The degree of* $Q^{lk}(\lambda)$ *is at most* k $(= n - l)$.

(2) *There is a polynomial* $P^{lk}(\lambda)$, $\deg P^{lk} \leqslant n$, *such that the pair of polynomials* (Q^{lk}, P^{lk}) *meets the interpolation conditions.*

$$w_i Q^{lk}(z_i) - P^{lk}(z_i) = 0 \qquad (1 - k \leqslant i \leqslant n + 1). \tag{2.6}$$

Proof. The equation $\tilde{\mathbf{A}}^{lk}x^{lk} = 0$, written entrywise, looks as follows:

$$\sum_{j=1}^{n+1} z_j^i x_j^{lk} = 0 \qquad (i = 0, \ldots, l-1). \tag{2.7}$$

$$\sum_{j=1}^{n+1} \frac{w_{-i} - w_j}{z_{-i} - z_j} x_j^{lk} = 0 \qquad (i = 0, \ldots, k-1). \tag{2.8}$$

We know from Theorem 2.1 that (2.7) is satisfied if and only if the first l Laurent coefficients of

$$\frac{Q^{lk}(\lambda)}{g_{n+1}(\lambda)}$$

are equal to zero. This is, obviously, equivalent to $\deg Q^{lk} \leqslant k$. Putting

$$P^{lk}(\lambda) = \sum_{j=1}^{n+1} \frac{g_{n+1}(\lambda) x_j^{lk} w_j}{\lambda - z_j}, \tag{2.9}$$

we obtain that (2.6) holds for $i = 1, \ldots, n+1$ by this very definition, and the system (2.8) implies that (2.6) holds also for $i = 1-k, \ldots, 0$.

Conversely, let (1) and (2) hold. Due to the validity of the equations (2.6) for $i = 1, \ldots, n+1$, the polynomial $P^{lk}(\lambda)$ must have the form (2.9). Then the validity of (2.8) follows from (2.6) for $i = 1-k, \ldots, 0$, and the validity of (1) is ensured by (2.7). ■

Since the formula (2.5) establishes a one-to-one correspondence between polynomials of degree at most n and $(n+1)$-component vectors, we will briefly say $Q^{lk}(\lambda)$ corresponds to x^{lk} or (vice versa) in such a situation.

Finally, let us generalize Theorem 2.2 to present necessary and sufficient conditions for a vector being in the kernel of $\mathbf{A}_m^{lk}$ for any $m > l + k$.

THEOREM 2.3. *Let m, l, k be arbitrary nonnegative integers, $m > l + k$. The vector $x^{lkm} = [x_i^{lkm}]_{i=1}^m$ belongs to the kernel of $\mathbf{A}_m^{lk}$ if and only if the polynomial*

$$Q_m^{lk}(\lambda) = \sum_{j=1}^{m} \frac{g_m(\lambda) x_j^{lkm}}{\lambda - z_j}$$

satisfies the following conditions:

(1) *The degree of* $Q_m^{lk}(\lambda)$ *is at most* $m - 1 - l$.

(2) *There is a polynomial* $P_m^{lk}(\lambda)$, $\deg P_m^{lk}(\lambda) \leqslant m - 1$, *such that the pair of polynomials* (Q_m^{lk}, P_m^{lk}) *meets the interpolation conditions*

$$w_i Q_m^{lk}(z_i) - P_m^{lk}(z_i) = 0 \qquad (1 - k \leqslant i \leqslant m).$$

3. RECURSION OF TYPE $(l-1,0) \to (l,0)$ AND $(N-2,0),(N-1,0) \to (N-2,1)$

Introduce the vectors y^{lk}, x^{lk}, b^{lk} as in Section 2, where b_i, $-K < i \leqslant L$, are taken from (1.4) and b_i for $i > L$ may be arbitrarily chosen. Moreover, in all what follows we denote by u^{ij}, v^{ij} the row vectors of the first j components of the ith rows of $\mathbf{L}_k(w, z), \mathbf{V}_l(z^N)$, respectively.

Now we consider the case $k = 0$. In this case the matrices $\mathbf{A}^{lk}$ and $\tilde{\mathbf{A}}^{lk}$ are pure Vandermonde matrices of full rank.

$$\mathbf{A}^{l0} = \mathbf{V}_l(z^l), \qquad \tilde{\mathbf{A}}^{l0} = \mathbf{V}_l(z^{l+1}) = \left[\mathbf{V}_l(z^l)a^{l0}\right], \tag{3.1}$$

where $a^{l0} =: [z_{l+1}^i]_{i=0}^{l-1}, l < N$.

THEOREM 3.1. *The following recursion formulas are true*:

$$y^{l+1,0} = \begin{bmatrix} y^{l0} \\ 0 \end{bmatrix} + \alpha_l x^{l0} \qquad (l = 1, \dots, N-1). \tag{3.2}$$

where $\alpha_l := (b_{l+1} - v^{l+1,l} y^{l0})/\beta_{l+1}$, *with* $\beta_l := v^{l,l} x^{l-1,0}$ *always nonzero*;

$$x_i^{l0} = \frac{x_i^{l-1,0}}{z_i - z_{l+1}} \quad (i = 1, \dots, l) \quad \textit{and} \quad x_{l+1}^{l0} = -\sum_{i=1}^{l} x_i^{l0}. \tag{3.3}$$

Proof. Clearly $\beta_{l+1} \neq 0$; otherwise x^{l0} would be a nonzero element of the kernel of $\mathbf{A}^{l+1,0}$, which contradicts the nonsingularity of $\mathbf{A}^{l+1,0}$ $(l = 1, \dots, N-1)$. Taking into account that

$$\mathbf{A}^{l+1,0} = \begin{bmatrix} \mathbf{A}^{l0} & a^{l0} \\ v^{l+1,l+1} & \end{bmatrix},$$

the formula (3.2) follows immediately on evaluating

$$\mathbf{A}^{l+1,0}\begin{bmatrix} y^{l0} \\ 0 \end{bmatrix} + \alpha_l \mathbf{A}^{l+1,0} x^{l0}.$$

The recursion (3.3) is a consequence of

$$\frac{c}{g_l(\lambda)} = \sum_{i=1}^{l} \frac{x_i^{l-1,0}}{\lambda - z_i} \quad (c = \text{const} \neq 0) \quad \text{and}$$

$$g_{l+1}(\lambda) = g_l(\lambda)(\lambda - z_{l+1}).$$

This completes the proof. ■

Let us introduce the numbers

$$\phi_{lk} := u^{k+1,n+1} x^{lk}, \tag{3.4}$$

which will be needed here for $k = 0$, and in the next section also for other subscripts. Since x^{lk} is a nontrivial element of $\ker \tilde{\mathbf{A}}^{lk}$, we have

$$\mathbf{A}^{l,k+1} x^{lk} = \begin{bmatrix} 0 \\ \phi_{lk} \end{bmatrix},$$

which leads, in view of the nonsingularity of $\mathbf{A}^{l,k+1}$, to

$$\phi_{lk} \neq 0 \qquad [(l, k+1) \in \mathscr{A}]. \tag{3.5}$$

Now we present the recursion formula which is necessary to start the second part of the algorithm. It corresponds to the "break point" of the path drawn in the picture (1.5).

THEOREM 3.2. *We have*

$$y^{N-1,1} = \begin{bmatrix} y^{N-1,0} \\ 0 \end{bmatrix} + \delta x^{N-1,0}, \tag{3.6}$$

where $\delta := \phi_{N-1,0}^{-1}(b_0 - u^{1,N-1} y^{N-1,0})$, *and*

$$x^{N-2,1} = \alpha \begin{bmatrix} x^{N-2,0} \\ 0 \end{bmatrix} - \beta x^{N-1,0}, \tag{3.7}$$

where $\alpha = 1$, $\beta = 0$ *in case* $\phi_{N-2,0} = 0$, *but* $\alpha = \phi_{N-1,0}/\phi_{N-2,0}$, $\beta = 1$ *in case* $\phi_{N-2,0} \neq 0$.

Proof. We establish that

$$\dim \operatorname{span}\left\{ \begin{bmatrix} x^{N-2,0} \\ 0 \end{bmatrix}, x^{N-1,0} \right\} = 2.$$

Otherwise, since $x^{N-1,0} \in \ker \mathbf{V}_{N-1}(z^N)$, we get that $x^{N-2,0}$ belongs to $\ker \mathbf{V}_{N-1}(z^{N-1})$ which is a contradiction. Thus, the vector $x^{N-2,1}$ defined by (3.7) is nonzero and satisfies the equation

$$\tilde{A}^{N-2,1} x^{N-2,1} = \begin{bmatrix} \mathbf{V}_{N-2}(z^N) \\ u^{1N} \end{bmatrix} x^{N-2,1} = \alpha \begin{bmatrix} 0 \\ \phi_{N-2,0} \end{bmatrix} - \beta \begin{bmatrix} 0 \\ \phi_{N-1,0} \end{bmatrix},$$

which is, obviously, equal to zero for the choice of α and β in both cases. Analogously, the formula (3.6) is shown. ■

4. RECURSION OF TYPE $(l+1, k-1), (l, k) \to (l-1, k+1)$

This section is devoted to recursions which correspond to the diagonal path in the picture (1.5). First of all let us recall that the matrix $\mathbf{A}^{l-1,k+1}$ is obtained from $\mathbf{A}^{lk}$ by canceling the lth row, which is v^{lN}, and appending the row $u^{k+1,N}$ as the last row. Analogously $b^{l-1,k+1}$ is obtained by omitting the lth component of b^{lk} [introduced in (2.2)] and extending the remaining vector by b_{-k} as the last component. The recursions continue until the step $(L+2, K-2), (L+1, K-1) \to (L, K)$, in which the required solution of the system (1.1) ($A = A^{LK}$) is reached.

Besides the assumption that the matrices $\mathbf{A}^{lk}$ are nonsingular, let the following condition be satisfied for $k = 1, \dots, K$, $l + k = N$:

$$\ker \tilde{\mathbf{A}}^{l,k-1} \cap \ker [w_1 \quad w_2 \quad \cdots \quad w_N] = \{0\}. \tag{4.1}$$

[An equivalent condition to (4.1) will be presented in Proposition 4.1.]

From now on let $k + l = N - 1$. Using (3.6), the following recursion (the initial vectors $y^{N-1,1}$, $x^{N-2,1}$ of which have been determined in Theorem 3.2) is easily shown.

THEOREM 4.1. *We have*

$$y^{l,k+1} = y^{l+1,k} + \nu_{lk} x^{lk}, \tag{4.2}$$

where

$$\nu_{lk} := \frac{1}{\phi_{lk}}\left(b_{-k} - u^{k+1,N} y^{l+1,k}\right)$$

$$(k = 1, 2, \dots, K-1, \quad k + l = N - 1).$$

REMARK 4.1. In case $k = 0$, $l = N - 1$ one can also use the formula (4.2) instead of (3.6).

In what follows we use the so-called Hadamard product of two vectors $u = [u_i]_{i=1}^N$ and $v = [v_i]_{i=1}^N$ defined by

$$u \circ v := [u_i v_i]_{i=1}^N,$$

and we introduce the vectors

$$c^r := [z_j - z_{-r}]_{j=1}^N \qquad (0 \leqslant r < K).$$

To prove recursion formulas expressing $x^{l-1,k+1}$ by means of $x^{l+1,k-1}$ and x^{lk} we need the following assertions.

LEMMA 4.1.

$$\dim \operatorname{span}\{x^{l+1,k-1}, x^{lk}\} = 2.$$

Proof. Since $x^{lk} \in \ker \tilde{\mathbf{A}}^{lk}$ and $x^{l+1,k-1} \in \tilde{\mathbf{A}}^{l+1,k-1}$, the linear dependence of x^{lk} and $x^{l+1,k-1}$ would lead to

$$x^{lk} \in \ker \begin{bmatrix} \mathbf{V}_{l+1}(z^N) \\ \mathbf{L}_k(w, z) \end{bmatrix},$$

which contradicts the nonsingularity of $\mathbf{A}^{l+1,k}$. ■

Let us recall that the equation (2.5) in Theorem 2.2 establishes a one-to-one correspondence between $(n+1)$-component vectors and polynomials of degree at most n. We are going to use this correspondence for $n = N - 1$ several times. Lemma 4.1 gives a direct corollary in this way:

COROLLARY 4.1. *The polynomials $Q^{l+1,k-1}(\lambda)$ and $Q^{lk}(\lambda)$ are linearly independent.*

Besides ϕ_{lk} [see (3.4)] we need here the following numbers:

$$\xi_{lk} := \sum_{j=1}^{N} w_j x_j^{lk} \qquad (k = 0, \dots, K-1, \quad l + k = N - 1).$$

By the assumption (4.1) it is clear that $\xi_{lk} \neq 0$.

It is important to note that the coefficient at λ^{N-1} in the polynomial $P^{lk}(\lambda)$ defined by (2.9) is just ξ_{lk}. Moreover, it is easily established that

$$\xi_{lk} x^{l+1,k-1} - \xi_{l+1,k-1} x^{lk} \in \ker[w_1 \quad w_2 \quad \cdots \quad w_N]. \tag{4.3}$$

LEMMA 4.2. *Let $Q^{lk}(\lambda)$ be the polynomial corresponding to x^{lk} [see (2.5)], $l \geqslant 1$. Then the polynomial $Q^{lk}(\lambda)(\lambda - z_{-k})$ is of the form*

$$Q^{lk}(\lambda)(\lambda - z_{-k}) = \sum_{j=1}^{N} x_j^{lk}(z_j - z_{-k}) \frac{g_N(\lambda)}{\lambda - z_j} \tag{4.4}$$

and corresponds to the vector $x^{lk} \circ c^k$.

Proof. Obviously,

$$Q^{lk}(\lambda)(\lambda - z_{-k}) = g_N(\lambda) \sum_{j=1}^{N} x_j^{lk} \left(1 + \frac{z_j - z_{-k}}{\lambda - z_j} \right).$$

From $x^{lk} \in \ker \tilde{\mathbf{A}}^{lk}$ we get, in particular, $\sum_{j=1}^{N} x_j^{lk} = 0$, and (4.4) becomes clear. ■

THEOREM 4.2. *The vector $x^{l-1,k+1}$ can be determined recurrently from $x^{l+1,k-1}$ and x^{lk} by the following recursion:*

$$x^{l-1,k+1} = \xi_{lk} x^{l+1,k-1} \circ c^{k-1} - \xi_{l+1,k-1} x^{lk} \circ v^{1N} - \psi_{lk} x^{lk}, \tag{4.5}$$

where

$$\psi_{lk} := u^{k+1,N}\left(\xi_{lk} x^{l+1,k-1} \circ c^{k-1} - \xi_{l+1,k-1} x^{lk} \circ v^{1N} \right) / \phi_{lk}$$

$$(k = 1, \dots, K-1, \quad l + k = N - 1).$$

Proof. Theorem 2.2 enables us to handle polynomials instead of vectors, which offers certain advantages in this proof. We consider the polynomial $Q(\lambda)$ corresponding to the right-hand side of (4.5), which is, due to Lemma 4.2, of the form

$$Q(\lambda) := \xi_{lk} Q^{l+1,k-1}(\lambda)(\lambda - z_{-k+1}) \\ - \xi_{l+1,k-1} Q^{lk}(\lambda)\lambda - \psi_{lk} Q^{lk}(\lambda), \qquad (4.6)$$

and we show that $Q(\lambda)$ satisfies conditions (1), (2) of Theorem 2.2 after replacing l by $l-1$ and k by $k+1$. Since, clearly, $\deg Q(\lambda) \leqslant k+1$ condition (1) is satisfied. Introducing

$$P(\lambda) := \xi_{lk} P^{l+1,k-1}(\lambda)(\lambda - z_{-k+1}) \\ - \xi_{l+l,k-1} P^{lk}(\lambda) - \psi_{lk} P^{lk}(\lambda), \qquad (4.7)$$

we are going to show that $\deg P(\lambda) \leqslant N-1$ and

$$w_i Q(z_i) - P(z_i) = 0 \qquad (-k \leqslant i \leqslant N), \qquad (4.8)$$

which means that condition (2) is satisfied. From the definition of $P(\lambda)$ it is clear that $\deg P(\lambda) \leqslant N$. But, taking (4.3) into account, we establish that the coefficient at λ^N of $P(\lambda)$ vanishes. It is evident that $w_i Q(z_i) - P(z_i) = 0$ for $1-k \leqslant i \leqslant N$. It remains to show that $w_{-k} Q(z_{-k}) - P(z_{-k}) = 0$. Denoting by $\tilde{x}^{lk}$ the vector $\xi_{lk} x^{l+1,k-1} \circ c^{k-1} - \xi_{l+1,k-1} x^{lk} \circ v^{1N}$, we have, due to the definition of ψ_{lk} and $\phi_{lk}(= u^{k+1,N} x^{lk})$,

$$u^{k+1,N}\left(\tilde{x}^{lk} - \psi_{lk} x^{lk}\right) = 0.$$

Thus (4.8) is shown. Consequently, $x^{l-1,k+1}$ defined in (4.5) is an element of $\ker \tilde{\mathbf{A}}^{l-1,k+1}$.

Finally we show that it is nontrivial. Indeed, if it were zero, then

$$\xi_{lk} Q^{l+1,k-1}(\lambda)(\lambda - z_{-k+1}) = Q^{lk}(\lambda)[\xi_{l+1,k-1}\lambda + \psi_{lk}].$$

would hold. Since, due to Corollary 4.1, the polynomials $Q^{l+1,k-1}(\lambda)$ and $Q^{lk}(\lambda)$ are linearly independent, this would mean that $Q^{lk}(z_{-k+1}) = 0$ and $P^{lk}(z_{-k+1}) = 0$. Consequently, the following polynomials would satisfy conditions (1) and (2) of Theorem 2.3 for the matrix $\mathbf{A}_N^{l,k-1}$:

$$Q^{l,k-1}(\lambda) := \frac{Q^{lk}(\lambda)}{\lambda - z_{-k+1}}, \qquad P^{lk}(\lambda) := \frac{P^{lk}(\lambda)}{\lambda - z_{-k+1}}.$$

Since $\deg P^{l,k-1} < N-1$, the vector $x^{l,k-1} = [x_j^{l,k-1}]_1^N$ (corresponding to $Q^{l,k-1}$) would have the property

$$\sum_{j=1}^{N} w_j x_j^{l,k-1} = 0,$$

which would be a contradiction to (4.1), and the proof is complete. ■

Finally let us note that the condition (4.1) on $\ker \tilde{\mathbf{A}}^{l,k-1}$ can be replaced by a condition on $\ker \tilde{\mathbf{A}}^{l-1,k}$:

PROPOSITION 4.1. *For each $k = 1, 2, \ldots, K$, $l + k = N$, the following conditions are equivalent*:

$$\ker \tilde{\mathbf{A}}^{l,k-1} \cap \ker[w_1 \quad w_2 \quad \cdots \quad w_N] = \{0\}, \tag{4.9}$$

$$\ker \tilde{\mathbf{A}}^{l-1,k} \cap \ker\left[\frac{1}{z_{-k+1} - z_1} \quad \cdots \quad \frac{1}{z_{-k+1} - z_N}\right] = \{0\}. \tag{4.10}$$

Proof. Firstly, we show if (4.9) is violated one can construct a nontrivial element of the intersection of (4.10). Let $x^{l,k-1} \in \ker \tilde{\mathbf{A}}^{l,k-1}$ be nonzero and $[w_1 \ \cdots \ w_N]x^{l,k-1} = 0$. Using the correspondence of polynomials and vectors in the notation of Section 2, we get, in view of Lemma 4.2, that

$$Q(\lambda) := (\lambda - z_{-k+1})Q^{l,k-1}(\lambda)$$

corresponds to the (nonzero) vector $x = [x_j]_1^N := x^{l,k-1} \circ c^{k-1}$ and satisfies together with $P(\lambda) := (\lambda - z_{-k+1})P^{l,k-1}(\lambda)$ the conditions

$$w_i Q(z_i) - P(z_i) = 0 \qquad \text{for} \quad 1 - k \leqslant i \leqslant N.$$

Since $[w_1 \ \cdots \ w_N]x^{l,k-1}$ is just the highest-order coefficient of $P(\lambda)$, which is, per assumption, zero, we get $\deg P(\lambda) \leqslant N - 1$. Taking Theorem 2.2 into account, this leads to $x \in \ker \tilde{\mathbf{A}}^{l-1,k}$. Moreover,

$$0 = Q(z_{-k+1}) = g_N(z_{-k+1}) \sum_{j=1}^{N} \frac{x_j}{z_{-k+1} - z_j}.$$

The latter shows that x is also an element of $\ker[1/(z_{-k+1} - z_j)]_{j=1}^{N}$.

Secondly, if there exists a nontrivial element $x^{l-1,k}$ of $\ker \tilde{\mathbf{A}}^{l-1,k}$, which satisfies

$$\left[\frac{1}{z_{-k+1} - z_1} \quad \cdots \quad \frac{1}{z_{-k+1} - z_N} \right] x^{l-1,k} = 0,$$

we get, with the arguments above but in reversed order, that

$$Q^{l-1,k}(z_{-k+1}) = 0$$

and thus $P^{l-1,k}(z_{-k+1}) = w_{-k+1} Q^{l-1,k}(z_{-k+1}) = 0$. Consequently, the vector x corresponding to $Q^{l-1,k}(\lambda)(\lambda - z_{-k+1})^{-1}$ is a nontrivial element of $\ker \tilde{\mathbf{A}}^{l,k-1}$, which satisfies $[w_1, \dots, w_N]x = 0$ in view of

$$\deg \frac{P^{l-1,k}(\lambda)}{\lambda - z_{-k+1}} < N - 1.$$

■

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