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Energy Decay Rates and the Dynamical von Karman Equations

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Abstract—We find uniform rates of decay of the solutions of the dynamical von Karman equations in the presence of dissipative effects. Our proof is elementary and uses ideas of a recent technique due to E. Zuazua while studying nonlinear dissipative wave equations [1].

Keywords—Decay rates, von Karman equations, Dissipation.

1. INTRODUCTION

We consider global strong solutions of the dynamical von Karman equations in the presence of a dissipation. More precisely, let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Consider the system

$$u_{tt} + \Delta^2 u + \lambda u_{xx} + \beta(-\Delta)^\alpha u_t, = [v, u], \qquad \text{in } \Omega \times \mathbb{R}, \tag{1.1}$$

$$\Delta^2 v = -\frac{1}{2}\,[u, u], \qquad \text{in } \Omega \times \mathbb{R}, \tag{1.2}$$

with boundary conditions

$$\begin{aligned} u &= 0, \quad \frac{\partial u}{\partial \eta} = 0, \qquad \text{on } \partial\Omega, \times\mathbb{R}, \\ v &= 0, \quad \frac{\partial v}{\partial \eta} = 0, \qquad \text{on } \partial\Omega, \times\mathbb{R}, \end{aligned} \tag{1.3}$$

and initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \qquad \text{for } x \in \Omega. \tag{1.4}$$

Here, the bracket [,] means

$$[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx},$$

where subscripts denote partial differentiation. In (1.1), β and λ are positive constants, Δ^2 denotes the biharmonic operator. Since $-\Delta$ is positive and self-adjoint, then $(-\Delta)^\alpha$ denotes the

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX fractional power operator with domain obtained as an interpolation space between $L^2(\Omega)$ and the domain of Δ^2 with Dirichlet boundary conditions. Also, $0 \le \alpha \le 2$, and $\frac{\partial}{\partial\eta}$ denotes the normal derivative.

In (1.1)–(1.2), $u = u(x,y,t)$ describes the transversal displacement of the plate and $v = v(x,y,t)$ is the Airy-stress function of the vibrating plate. The boundary conditions (1.3) mean that the boundary is clamped in transversal direction, but free in the horizontal direction. In this paper, we shall prove that the global strong solution-pair $\{u,v\}$ of (1.1)–(1.4) decays uniformly as $t \to +\infty$ provided that λ is small enough. More precisely, there exist positive constants C and δ such that

$$E(t) = \left|\left|u_t(\cdot,t)\right|\right|_{L^2}^2 + \left|\left|\Delta u(\cdot,t)\right|\right|_{L^2}^2 + \left|\left|\Delta v(\cdot,t)\right|\right|_{L^2}^2 \le Ce^{-\delta t},$$

as $t \to +\infty$, if $0 \le \lambda < \lambda_1$ where λ_1 is the first eigenvalue of $-\Delta$ in Ω. In a recent paper on the subject, P. Aviles and J. Sandefur [2] proved the asymptotic stability as $t \to +\infty$, provided λ is small enough. No uniform rates were obtained in [2]. Our strategy is based on differential inequalities and a technique due to E. Zuazua who first used it to treat similar problems for nonlinear dissipative wave equations [1]. Our proof is elementary and is based essentially in a convenient choice of a Liapunov functional for problem (1.1)–(1.4).

The existence and uniqueness of global strong solutions for problems (1.1)–(1.4) can be obtained in a similar way as in [2] or [3] with suitable modifications, therefore, we will not repeat the calculations here. If $\beta = 0$ (that is, there is no dissipation), then we only know local existence results due to Von Wahl [3] and more recently to A. Stahel [4]. Weak solutions, with a dissipation of the form $-\Delta u_{tt}$ instead of $(-\Delta)^\alpha u_t$ in (1.1), were studied by J.-L. Lions in [5] (see also [6]).

We shall use standard notation: we denote the $L^p(\Omega)$ norm by $\|\ \|_{L^p}$, $1 \le p \le +\infty$. The Sobolev space $H^m(\Omega)$ norm by $\|\ \|_{H^m}$ and $H_0^m(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Here $C_0^\infty(\Omega)$ denotes the space of C^∞ functions with compact support contained in Ω.

2. SOME TECHNICAL LEMMAS

LEMMA 1. *Let f, g, $h \in H_0^2(\Omega)$, then*

$$\int_\Omega f[g,h]\,dA = \int_\Omega g[f,h]\,dA. \tag{2.1}$$

PROOF. Whenever f, g and h are smooth functions, say in $C_0^\infty(\Omega)$ then, the following identity holds:

$$\begin{aligned} f[g,h] - g[f,h] &= \left(fgh_{yy}\right)_{xx} - 2\left(fgh_{xy}\right)_{xy} + \left(fgh_{xx}\right)_{yy} \\ &\quad + 2\left(f_y g h_{xy} - f_x g h_{yy}\right)_x + 2\left(f_x g h_{xy} - f_y g h_{xx}\right)_y, \end{aligned} \tag{2.2}$$

as can easily be verified. Integration of identity (2.2) in Ω proves (2.1). The general situation is obtained by usual approximation procedure.

LEMMA 2. *Let $\{u,v\}$ be the solution pair of system (1.1), (1.2) with boundary conditions (1.4) and initial conditions $\varphi \in H^4(\Omega) \cap H_0^2(\Omega)$, $\psi \in H_0^2(\Omega)$. Then,*

(1) $\frac{d}{dt}\left[\int_\Omega \left(u_t^2 + (\Delta u)^2 - \lambda u_x^2\right) dA\right] + 2\beta \int_\Omega u_t(-\Delta)^\alpha u_t dA = 2\int_\Omega u_t[v,u]\,dA,$

(2) $\frac{d}{dt}\int_\Omega (\Delta v)^2 dA = -\int_\Omega v_t[u,u]\,dA,$

(3) $\int_\Omega v[u_t,u]\,dA = \int_\Omega u_t[v,u]\,dA,$

(4) $\int_\Omega u_t[v,u]\,dA = -\frac{d}{dt}\int_\Omega (\Delta v)^2 dA - \frac{1}{2}\int_\Omega v_t[u,u]\,dA.$

PROOF.

(1) Multiply equation (1.1) by u_t, integrate in Ω and use Green's formula to obtain the result.
(2) Multiply (1.2) by v_t and integrate over Ω.
(3) Use Lemma 1 with $f = u_t$, $g = v$ and $h = u$.
(4) We have the identities

$$\int_\Omega v[u_t, u]\, dA = \frac{1}{2}\frac{d}{dt}\int_\Omega v[u,u]\, dA - \frac{1}{2}\int_\Omega v_t[u,u]\, dA. \tag{2.3}$$

Using equation (1.1), we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v[u,u]\, dA = -\frac{d}{dt}\int_\Omega v\Delta^2 v dA = -\frac{d}{dt}\int_\Omega (\Delta v)^2 dA,$$

which together with (2.3) proves item (4).

Next, let us define the functionals $E(t)$ and $F(t)$ given by

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2 + (\Delta u)^2 - \lambda u_x^2 + (\Delta v)^2\right) dA, \tag{2.4}$$

$$F(t) = \int_\Omega u u_t dA + \frac{1}{2}\,\beta \int_\Omega u(-\Delta)^\alpha u\, dA, \tag{2.5}$$

where $\{u, v\}$ is the solution pair of (1.1)–(1.4). Let $\varepsilon > 0$ and consider $G_\varepsilon(t)$ given by $G_\varepsilon(t) = E(t) + \varepsilon F(t)$.

LEMMA 3. *There exist positive constants c_1, c_2 and c_3 such that*

(a) $G_\varepsilon(t) \geq c_1 \int_\Omega (\Delta u)^2 dA + c_2 \int_\Omega \left(u_t^2 + (\Delta v)^2\right) dA$,

(b) $G_\varepsilon(t) \leq c_3 \int_\Omega \left(u_t^2 + (\Delta u)^2 + (\Delta v)^2\right) dA$,

for all t, provided that λ and ε are sufficiently small.

PROOF. Using Poincare's inequality, we know that $-\lambda \int_\Omega u_x^2 dA \geq -\frac{\lambda}{\lambda_1}\int_\Omega |\Delta u|^2$, $(\lambda > 0)$ and λ_1 denotes the first eigenvalue of $-\Delta$ in Ω. Also, by Hölder's inequality $\int_\Omega u u_t dA \leq \frac{1}{2}\int_\Omega u^2 dA + \frac{1}{2}\int_\Omega u_t^2 dA$. Thus, we obtain a lower bound for $G_\varepsilon(t)$:

$$\begin{aligned} G_\varepsilon(t) \geq \left(\frac{1}{2} - \frac{\lambda}{2\lambda_1}\right)\int_\Omega (\Delta u)^2 dA + \frac{1}{2}\int_\Omega \left(u_t^2 + (\Delta v)^2\right) dA \\ - \frac{\varepsilon}{2}\int_\Omega \left(u_t^2 + u^2\right) Da - \frac{\varepsilon}{2}\beta\int_\Omega |(-\Delta)^{\alpha/2} u|^2 dA. \end{aligned} \tag{2.6}$$

Using Poincare's inequality once more in (2.6), we deduce that

$$\begin{aligned} G_\varepsilon(t) \geq \left(\frac{1}{2} - \frac{\lambda}{2\lambda_1} - \frac{\varepsilon}{2\lambda_1^2} - \frac{\varepsilon\beta}{2\lambda_1^{2-\alpha}}\right)\int_\Omega (\Delta u)^2 dA \\ + \frac{(1-\varepsilon)}{2}\int_\Omega u_t^2 dA + \frac{1}{2}\int_\Omega (\Delta v)^2 dA, \end{aligned}$$

which proves item (a) as long as ε and λ are sufficiently small. To prove (b), we again use Poincare's inequality to obtain

$$\begin{aligned} G_\varepsilon(t) &\leq \frac{1}{2}\int_\Omega \left(u_t^2 + (\Delta u)^2 - \lambda u_x^2 + (\Delta v)^2\right) dA \\ &\quad + \frac{\varepsilon}{2}\left\{\int_\Omega \left(u^2 + u_t^2 + \beta u(-\Delta)^\alpha u\right) dA\right\} \\ &\leq \left(\frac{1}{2} + \frac{\varepsilon}{2\lambda_1^2} + \frac{\varepsilon\beta}{2\lambda_1^{2-\alpha}}\right)\int_\Omega (\Delta u)^2 dA + \frac{1+\varepsilon}{2}\int_\Omega u_t^2 dA + \frac{1}{2}\int_\Omega (\Delta v)^2 dA, \end{aligned}$$

which proves item (b).

3. THE FINAL RESULT

THEOREM 1. *Let $\{u, v\}$ be the global solution-pair of problem (1.1)-(1.2) with boundary conditions (1.3) and initial conditions (1.4) with $\varphi \in H^4(\Omega) \cap H_0^2(\Omega)$ and $\psi \in H_0^2(\Omega)$, then, there exist positive constants $C > 0$ and $\delta > 0$ such that*

$$\int_\Omega \left(u_t^2 + (\Delta u)^2 + (\Delta v)^2\right) dA \leq ce^{-\delta t},$$

for all t, provided that $\lambda > 0$ is sufficiently small.

PROOF.

Consider the derivative of $G_\varepsilon(t)$. An easy calculation shows that

$$\frac{d}{dt} G_\varepsilon(t) = -\beta \int_\Omega \left|(-\Delta)^{\alpha/2} u_t\right|^2 dA + \varepsilon \left\{ \int_\Omega u_t^2 dA - \int_\Omega (\Delta u)^2 dA - 2 \int_\Omega (\Delta v)^2 dA + \lambda \int_\Omega u_x^2 dA \right\}. \tag{3.1}$$

Using Poincare's inequality in (3.1) as we did in the proof of Lemma 3, we obtain

$$\begin{aligned} \frac{d}{dt} G_\varepsilon(t) &\leq (-\beta\lambda_1^\alpha + \varepsilon) \int_\Omega u_t^2 dA - 2\varepsilon \int_\Omega (\Delta v)^2 dA - \varepsilon \left(1 - \frac{\lambda}{\lambda_1}\right) \int_\Omega (\Delta u)^2 dA \\ &\leq -c_4 \int_\Omega \left(u_t^2 + (\Delta u)^2 + (\Delta v)^2\right) dA, \end{aligned} \tag{3.2}$$

where $c_4 > 0$ as long as $\varepsilon > 0$ and $\lambda > 0$ are taken sufficiently small. From (3.2) and item (b) of Lemma 3, we deduce that

$$c_3 \frac{d}{dt} G_\varepsilon(t) + c_4 G_\varepsilon(t) \leq 0,$$

for all t, consequently $G_\varepsilon(t) \leq G_\varepsilon(0) e^{-(c_3/c_4)\, t}$ which proves Theorem 1, because by item (a) of Lemma 3, we know that $G_\varepsilon(0)$ is bounded which together with item (a) completes our claim.

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