Note on the Gaussian Channel with Feedback and a Power Constraint*

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1. INTRODUCTION

This note is a comment on the beautiful coding schemes for a Gaussian channel with feedback due to Schalkwijk and Kailath (1966) and Schalkwijk (1966), particularly on the second of these. Very important and basic work from a different point of view has been done by Elias (1961, 1966). The coding scheme is simply described in Section 2. The new result of this paper is the strong converse in Section 3, whose proof is a simple modification of the proof of Theorem 9.2.2 of Wolfowitz (1961, 1964). Section 4 contains a few comments on the Gaussian channel with feedback and a power constraint (channel GFP) which seem not to have been made elsewhere. Section 5 is a digression on the time-continuous Gaussian channel which contains a very simple proof of the strong converse for that channel.

A simple description of the Gaussian channel without feedback and with a power constraint is given in Section 9.2 of Wolfowitz (1961, 1964). We now describe channel GFP, which differs from the former channel only in having feedback. A message is transmitted over channel GFP by \( n \) signals. The first signal is a function of the message; suppose the signal is \( a \). Let \( z_1, \ldots, z_n \) be independent, normal chance variables, with means zero and variance \( \sigma^2 \). The first signal received is \( y_1 = a + z_1 \). The next signal sent is a function \( \varphi_2(y_1) \) of \( y_1 \) and the message. The \( i \)th signal sent is a function \( \varphi_i(y_1, \ldots, y_{i-1}) \) of the message and the variables exhibited, etc. The \( i \)th signal received is \( y_i = \varphi_i + z_i \), etc. The functions \( \varphi \) depend upon the particular coding scheme used.

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2. DESCRIPTION OF THE SCHALKWIJK CODING SCHEME

Let $C$, to be determined later, be the capacity of channel $GFP$. Let $n$ be, as already described, the number of signals used in transmitting each message; for the $k$th message the signals sent will be called

\[ m(k, 1), m(k, 2), \ldots, m(k, n). \tag{2.1} \]

We suppose that there are $2^{nR}$ messages to transmit, where $0 < R < C$.

Let $z_1, z_2, \ldots$ be independent chance variables, each normally distributed with mean zero and variance $\sigma^2$. Let $y(k, l)$ be the $l$th signal received when the $k$th message is being transmitted.

We now describe the Schalkwijk scheme. First choose

\[ y(k, 1) = m(k, 1) + z_1, \tag{2.3} \]

and similar expressions. While this is not strictly correct its meaning is obvious and no error or even ambiguity will be caused by this space saving notation. Because of the feedback the quantity $z_1$ is known to the sender. Write $g = (\alpha^2 - 1)^{1/2}$. We choose

\[ y(k, 2) = m(k, 2) = -gz_1. \tag{2.4} \]

Then

\[ y(k, 2) = -gz_1 + z_2, \tag{2.5} \]

so that $z_2$ is known to the sender. In general, for $3 \leq l \leq n$, define

\[ w(k, l) = \frac{1}{2} - \frac{y(k, 1)}{\alpha} = -\frac{g \sum_{i=2}^{l-1} y(k, i)}{\alpha^i} \tag{2.6} \]

and choose

\[ m(k, l) = g\alpha^{l-1}[w(k, l) - k \cdot 2^{-nR}]. \tag{2.7} \]

Of course

\[ y(k, l) = m(k, l) + z_l. \tag{2.8} \]

We now verify, by induction or otherwise, that, for $l \geq 2$,

\[ w(k, l + 1) = k \cdot 2^{-nR} - \frac{z_1}{\alpha^{2i-1}} - \frac{g \sum_{i=2}^{l} z_i}{\alpha^{2i-1}}. \tag{2.9} \]
Hence \( w = w(k, n + 1) \) is distributed normally with mean \( k \cdot 2^{-nR} \) and variance \( \sigma^2 / \alpha^2 n \). Because of (2.6) \( w \) can be computed by the receiver. Suppose the receiver decodes, as the serial number of the message sent, that multiple of \( 2^{-nR} \) which is nearest to \( w \); in case of a tie between two multiples of \( 2^{-nR} \) he may choose either at pleasure. Then, if

\[ R < \log_2 \alpha, \quad (2.10) \]

the probability of incorrect decoding approaches zero as \( n \to \infty \), uniformly in the message sent.

The average power \( L \) of the signals required to send the \( k \)th message is defined to be

\[ L = \frac{1}{n} \left[ \sum_{i=1}^{n} (m(k, i))^2 \right]. \quad (2.11) \]

As \( n \to \infty \) this expression converges with probability one, uniformly in \( k \), to \( g^2 \sigma^2 \). This last quantity, the limit of the average power, is required to be a given constant \( L^* \), say, so that \( \alpha \) is determined by

\[ (\alpha^2 - 1) \sigma^2 = L^*, \quad \text{say}. \quad (2.12) \]

The previous remarks prove a coding theorem under the condition (2.10); this proof is in Schalkwijk and Kailath (1966) and Schalkwijk (1966). In the next section we shall prove a strong converse, which will complete the proof that

\[ C = \log_2 \alpha = \frac{1}{2} \log_2 \left( 1 + \frac{L^*}{\sigma^2} \right). \quad (2.13) \]

3. PROOF OF THE STRONG CONVERSE

This section is devoted to the proof of the following:

**Theorem.** Consider channel GFP, with the requirement\(^2\) that the average power for each message converge stochastically to a given constant \( L^* \), (say). Let \( \epsilon > 0 \) and \( \lambda < 1 \) be arbitrary. For all \( n \) sufficiently large a code with maximum probability of error \( \lambda \) cannot have a length greater than \( \alpha^{n(1+\epsilon)} \).

The proof, to which we proceed at once, will require a familiarity with Section 9.2 of Wolfowitz (1961, 1964), and in fact will be a simple modification of the proof of its Theorem 9.2.2. We assume at first that always

\(^2\) This requirement can easily be replaced by the following requirement: Let \( \epsilon, 0 < \epsilon < 1 \), be arbitrary. There exists a function of \( \epsilon \), say \( n_0(\epsilon) \), with the following property. If \( n > n_0(\epsilon) \) then, whatever be the message, the probability exceeds \( 1 - \epsilon \) that the average power should be less than \( L^* + \epsilon \).
Of course $g$, and hence $a$, are determined by $L^* = g^2 \sigma^2$. In the notation of Theorem 9.2.2 where there is no feedback, if the signal sequence

$$u_0 = (x_1, \cdots, x_n)$$

is sent, the chance signal sequence received is

$$v(u_0) = (Y_1(u_0), \cdots, Y_n(u_0)).$$

Let $A_0$ be the region used to decode this transmitted signal sequence $u_0$, so that

$$P\{v(u_0) \in A_0\} \geq 1 - \lambda.$$  \hspace{1cm} (3.2)

The left member of (3.2) is, of course,

$$([2\pi]^{1/2} \sigma)^{-n} \int \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 \right\} dy_1 \cdots dy_n,$$  \hspace{1cm} (3.3)

the integral extended over the translate of $A_0$ by the vector $(-x_1, -x_2, \cdots, -x_n)$. It is proved, in the course of the proof of Theorem 9.2.2, that our (3.2) implies that the volume of $A_0$ is, for all $n$ sufficiently large, at least

$$\frac{\pi^{n/2}}{\Gamma \left( \frac{n+2}{2} \right)} \left( \sigma \left( 1 - \frac{\epsilon}{2} \right) \right)^n.$$  \hspace{1cm} (3.4)

From this and (9.2.18) of Wolfowitz (1961, 1964) the conclusion of Theorem 9.2.2 followed immediately. We will now show that (3.4) holds also when feedback is employed. This will similarly prove the present theorem under the additional assumption (3.1), since the latter assures the validity of (9.2.18) of Wolfowitz (1961, 1964) for the present situation.

Consider a specific message which is sent over channel GFP as follows: The first signal sent is $a = \varphi_1$. If the first signal received is $y_1$, the next signal sent is $\varphi_2(y_1)$. If the $i$th signal received is $y_i$, the $(i + 1)^{st}$ signal sent is $\varphi_{i+1}(y_1, \cdots, y_i)$ [Assumption (3.1) holds], etc.

Let $B_0$ be the (measurable) set in the space of $y_1, \cdots, y_n$ which is used to decode the message of the last paragraph. The probability of a correct decision when this message is sent is
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\[
([2\pi]^{1/2})^{-n} \int \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \varphi_i)^2 \right\} dy_1 \cdots dy_n, \quad (3.5)
\]

the integral taken over the set \(B_0\). The volume of \(B_0\) is, of course,

\[
\int_{B_0} dy_1 \cdots dy_n. \quad (3.6)
\]

Performing the transformation

\[
s_1 = y_1 - a
\]

\[
s_2 = y_2 - \varphi_2(y_1)
\]

\[
\vdots
\]

\[
s_n = y_n - \varphi_n(y_1, \ldots, y_{n-1}),
\]

which is one-to-one and whose Jacobian is 1, we let \(B_{00}\) be the set into
which \(B_0\) is carried by this transformation. The volume of \(B_{00}\) is therefore
the same as that of \(B_0\). The integral (3.5) is carried over into the inte-
gral (3.3) over the set \(B_{00}\). From the error inequality [analog of (3.2)]
it therefore follows that the volume of \(B_0\) is at least (3.4) for all \(n\) suffi-
ciently large. This proves the theorem under the assumption (3.1).

We now remove assumption (3.1). Let \(n\) be so large that

\[
P\left\{ L > L^* + \frac{\epsilon}{2} \right\} < \frac{1 - \lambda}{2}. \quad (3.8)
\]

It follows that, conditional upon \(L \leq L^* + \epsilon/2\), the probability of the
decoding region is at least \((1 - \lambda)/2\). The previous argument did not
depend on the size of \(\lambda\) (although it may now be necessary to enlarge the
lower bound on \(n\)), and is unchanged in the conditional case, since the
condition that \(L \leq L^* + \epsilon/2\) is essentially equivalent to (3.1). Hence
the theorem follows.

4. OPTIMALITY OF THE SCHALKWIJK SCHEME

The results of Sections 2 and 3 imply that the Schalkwijk coding
scheme achieves capacity, i.e. that, essentially, the maximum possible
number of messages which can be sent by any scheme, is the number
which can actually be sent by the Schalkwijk scheme (for large \(n\)).
It was pointed out in Schalkwijk and Kailath (1966) and Schalkwijk
(1966) that, for any fixed rate \(R < C\), the maximum probability of error
under the Schalkwijk scheme goes to zero doubly exponentially; i.e.,
there exist positive functions $c_1, c_2, \text{ and } c_3$ of $R$ such that the probability of error is less than

$$c_1 \exp \{-c_2 e^{n \varepsilon_2}\}. \quad (4.1)$$

This is an immediate consequence of the remarks made in Section 2 after (2.9). According to these remarks, if $\theta$ is the message sent and $w$ is the estimator of $\theta$ after $n$ signals,

$$E(w - \theta)^2 = \frac{\sigma^2}{\alpha^{2n}}. \quad (4.2)$$

(An "estimator" of $\theta$ is any measurable function of the received signals.)

Suppose that there existed a coding scheme for the GFP channel and an estimator $v$ of $\theta$ (after $n$ signals) such that, for all $n$ sufficiently large and all $\theta$, $0 \leq \theta \leq 1$, say,

$$E(v - \theta)^2 < \frac{c}{\beta^{2n}}, \quad (4.3)$$

where $c > 0$ is a constant and $\beta > \alpha$. From (4.3) it would at once follow that the capacity of the GFP channel is at least $\log_2 \beta > \log_2 \alpha$, which contradicts the theorem of Section 3 (or, for that matter, the weak converse, which can be proved by an almost literal repetition of Shannon’s argument (Shannon, 1956)). We immediately conclude that the Schalkwijk scheme is, at the least, the asymptotic solution of the problem posed and solved by Omura (1966) [especially pp. 4, 5, and 44]. (Omura’s problem is precisely to give a coding scheme and an estimator $v$ of the message point $\theta$ such that $v$ is linear in the $z$'s and minimizes $E(v - \theta)^2$. In Omura’s solution $v$ is normal, with mean $\theta(1 - \alpha^{-2n-2})$ and variance equal a constant multiplied by $\alpha^{-2n-2}(1 - \alpha^{-2n-2})$.) Indeed, we can say something more: Even if Omura’s problem were generalized by canceling the seemingly unnecessarily restrictive limitation to estimators linear in the $z$'s, no essential improvement could be achieved for large $n$.

The estimator $w$ is linear in the $z$'s [Eq. (2.9)]. Suppose now the channel GFP to be modified in the following respect, that the $z$'s are no longer necessarily identically and normally distributed. (They remain independent chance variables with zero means and variances $\sigma^2$. We can even weaken the requirement of independence for the $z$'s; it is enough if they are uncorrelated.) Then there exists no estimator $v$, linear in the $z$'s, for which (4.3) holds. For, if there were such an estimator, we could use
the same estimator for the GFP channel, in violation of the first part of the preceding paragraph. This does not, however, imply that the capacity of the modified channel is \( \log_2 \alpha \), because of the limitation to linear estimators. The capacity of the modified channel is at least \( \log_2 \alpha \). If one is willing to transmit at a rate less than \( \log_2 \alpha \), the estimator \( w = w(k, n + 1) \) of (2.9) can be used in the case of the modified channel.

A bound (4.1) on the probability of error can always be achieved if the messages to be sent can be coded, as happened in Section 2, into points of a fixed, bounded set. It should also be noted that the coding schemes of Schalkwijk and Kailath (1966) and Schalkwijk (1966) may require the sending of signals which are outside of this set, through the operation of chance errors.

A discussion, from another point of view, of the optimality of the Schalkwijk coding scheme, is given by Kailath (1967).

5. THE TIME-CONTINUOUS GAUSSIAN CHANNEL

A clear description of this channel is given in, for example, Ash (1965), Section 8.5. The weak converse for this channel was proved by Ash (1964) and the strong converse by Yoshihara (1964). Both these authors proceed by replacing the decoding sets by cylinder sets in a finite-dimensional space (e.g., Ash, 1965, bottom p. 253). At this point the argument and conclusion of Theorem 9.2.2 of Wolfowitz (1961, 1964), cited in Section 3 above, apply in toto, so that a proof of the strong converse for the time-continuous Gaussian channel follows at once.

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