Uniform convexity of $\psi$-direct sums of Banach spaces

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Abstract
Let $X$ and $Y$ be Banach spaces and $\psi$ a continuous convex function on the unit interval $[0, 1]$ satisfying certain conditions. Let $X \oplus_\psi Y$ be the direct sum of $X$ and $Y$ equipped with the associated norm with $\psi$. We show that $X \oplus_\psi Y$ is uniformly convex if and only if $X, Y$ are uniformly convex and $\psi$ is strictly convex. As a corollary we obtain that the $\ell_{p,q}$-direct sum $X \oplus_{p,q} Y$, $1 \leq q \leq p \leq \infty$ (not $p = q = 1$ nor $\infty$), is uniformly convex if and only if $X, Y$ are, where $\ell_{p,q}$ is the Lorentz sequence space. These results extend the well-known fact for the $\ell_p$-sum $X \oplus_p Y$, $1 < p < \infty$. Some other examples are also presented.

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1. Introduction

For every continuous convex function $\psi$ on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1 - t, t\} \leq \psi(t) \leq 1$ ($0 \leq t \leq 1$) there corresponds a unique absolute normalized norm $\|\cdot\|$ on $C^2$ (that is, $\|(z, w)\| = \|(\xi, |w|)\|$ and $\|(1, 0)\| = \|(0, 1)\| = 1$) such that $\psi(t) = \|(1 - t, t)\|$ ($0 \leq t \leq 1$) (see Bonsall and Duncan [4]; cf. also the next section of the present paper and [3]). Owing to this correspondence we have plenty of
non $\ell_p$ type concrete norms on $\mathbb{C}^2$. Recently some geometric properties of these norms were discussed by means of the corresponding functions $\psi$ in [6,7]. In [6] Saito et al. determined and estimated their von Neumann–Jordan constant and as a corollary they showed that all absolute normalized norms are uniformly non-square except the $\ell_1$- and $\ell_\infty$-norms, which are the largest and smallest such norms respectively. In [7] they showed that an absolute normalized norm $\|\cdot\|_\psi$ is strictly convex if and only if $\psi$ is. They also introduced the $\psi$-direct sum $X \oplus_\psi Y$ of Banach spaces $X$ and $Y$ equipped with the norm $\|(x,y)\|_\psi = \|\|(x, y)\|\|_\psi (x \in X, y \in Y)$, and proved that $X \oplus_\psi Y$ is strictly convex if and only if $X, Y$ and $\psi$ are strictly convex. The $\psi$-direct sum extends the notion of the $\ell_p$-sum $X \oplus_p Y$ and provides many interesting examples.

The aim of this paper is to characterize the uniform convexity of $X \oplus_\psi Y$. In Section 2 we shall recall some fundamental facts on the $\psi$-direct sum of Banach spaces and present several examples. In Section 3 we shall prove that $X \oplus_\psi Y$ is uniformly convex if and only if $X, Y$ are uniformly convex and $\psi$ is strictly convex. As a corollary we obtain that the $\ell_{p,q}$-direct sum $X \oplus_{p,q} Y$, $1 \leq q \leq p \leq \infty$ (not $p = q = 1$ nor $\infty$), is uniformly convex if and only if $X$ and $Y$ are, where $\ell_{p,q}$ is the Lorentz sequence space. These results extend the well-known fact for the $\ell_p$-sum $X \oplus_p Y$, $1 < p < \infty$. Some other examples are also given.

2. $\psi$-Direct sum $X \oplus_\psi Y$

In this section we shall recall the definition of $\psi$-direct sum Banach spaces $X \oplus_\psi Y$, and also present several examples. A norm $\|\cdot\|$ on $\mathbb{C}^2$ is called \textit{absolute} if $\|(z, w)\| = \|(|z|, |w|)\|$ for all $(z, w) \in \mathbb{C}^2$ and \textit{normalized} if $\|(1, 0)\| = \|(0, 1)\| = 1$. The set of all absolute normalized norms on $\mathbb{C}^2$ is denoted by $Na$. The $\ell_p$-norms $\|\cdot\|_p$ are such examples and for any $\|\cdot\| \in Na$,

$$\|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$ (2.1)

Let $\Psi$ be the set of continuous convex functions $\psi$ on $[0, 1]$ satisfying

$$\psi(0) = \psi(1) = 1$$ (2.2)

and

$$\max\{1 - t, t\} \leq \psi(t) \leq 1 \quad (0 \leq t \leq 1).$$ (2.3)

For each $\|\cdot\| \in Na$ the function $\psi$ defined by

$$\psi(t) = \|(1 - t, t)\| \quad (0 \leq t \leq 1)$$ (2.4)

belongs to $\Psi$. Conversely, for each $\psi \in \Psi$ let

$$\|(z, w)\|_\psi = \begin{cases} (|z| + |w|) \psi\left(\frac{|w|}{|z| + |w|}\right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases}$$ (2.5)
Then \( \| \cdot \| \in N_a \) and \( \| \cdot \| \) satisfies (2.4). Thus the norms in \( N_a \) correspond with the convex functions in \( \Psi \) in a one-to-one manner (see Bonsall–Duncan [4], also [6]). The functions which correspond with the \( \ell_p \)-norms are

\[
\psi_p(t) := \begin{cases} 
(1 - t)^p + t^p \\
\max\{1 - t, t\}
\end{cases}
\frac{1}{p} \quad \text{if } 1 \leq p < \infty, \\
\max\{1 - t, t\} \quad \text{if } p = \infty.
\]

(2.6)

In the following let \( X \) and \( Y \) be Banach spaces. We denote by \( X \oplus_\psi Y \) the direct sum \( X \oplus Y \) equipped with the norm

\[
\| (x, y) \|_\psi = \| (\| x \|, \| y \|) \|_\psi \quad \text{for } (x, y) \in X \oplus Y.
\]

(2.7)

For completeness we see that \( X \oplus_\psi Y \) is a Banach space.

**Proposition 1.** Let \( X, Y \) be Banach spaces and \( \psi \in \Psi \). Then \( X \oplus_\psi Y \) is a Banach space.

**Proof.** By Lemma 1 we see the triangle inequality as

\[
\| (x_1 + x_2, y_1 + y_2) \|_\psi \leq \| (x_1, y_1) \|_\psi + \| (x_2, y_2) \|_\psi.
\]

Let \( \{(x_n, y_n)\} \) be a Cauchy sequence in \( X \oplus_\psi Y \). Then since \( 1/2 \leq \psi \leq 1 \), we have

\[
\frac{1}{2} \left( \| x_n - x_m \| + \| y_n - y_m \| \right)
\leq \left( \| x_n - x_m \| + \| y_n - y_m \| \right) \psi \left( \frac{\| y_n - y_m \|}{\| x_n - x_m \| + \| y_n - y_m \|} \right)
= \| (x_n, y_n) - (x_m, y_m) \|_\psi \to 0,
\]

whence \( \| x_n - x_m \| + \| y_n - y_m \| \to 0 \) as \( n, m \to \infty \). So \( \{x_n\} \) and \( \{y_n\} \) are Cauchy in \( X \) and \( Y \), respectively. Let \( x_n \to x \) in \( X \) and \( y_n \to y \) in \( Y \). Then

\[
\| (x_n, y_n) - (x, y) \|_\psi = \left( \| x_n - x \| + \| y_n - y \| \right) \psi \left( \frac{\| y_n - y \|}{\| x_n - x \| + \| y_n - y \|} \right)
\leq \| x_n - x \| + \| y_n - y \| \to 0 \quad \text{as } n \to \infty.
\]

Thus \( X \oplus_\psi Y \) is complete. \( \diamondsuit \)

**Example 1.** Let \( 1 \leq p \leq \infty \) and let \( \psi_p \) as in (2.6). Then \( \psi_p \)-direct sum \( X \oplus_{\psi_p} Y \) is just the usual \( \ell_p \)-sum \( X \oplus_{\ell_p} Y \); namely,

\[
\| (x, y) \|_{\psi_p} = \| (x, y) \|_p = \begin{cases} 
\left\{ \| x \|^p + \| y \|^p \right\}^{1/p} \\
\max\{\| x \|, \| y \|\}
\end{cases}
\quad \text{if } 1 \leq p < \infty, \\
\max\{\| x \|, \| y \|\} \quad \text{if } p = \infty
\]

(2.8)

for \( (x, y) \in X \oplus_{\psi_p} Y \).
Proposition 2. Let $X$, $Y$ be Banach spaces and $\psi \in \Psi$. Then

$$\| (x, y) \|_{\infty} \leq \| (x, y) \|_{\psi} \leq \| (x, y) \|_1$$

for all $(x, y) \in X \oplus Y$.

Example 2. Let $1 \leq q < p < \infty$ and $2^{1/p-1} < \lambda < 1$. Let $\psi_{p,q,\lambda} = \max\{\psi_p, \lambda \psi_q\}$, where $\psi_p$ is as in (2.6). Then $\psi_{p,q,\lambda} \in \Psi$ and the norm of $X \oplus \psi_{p,q,\lambda} Y$ is given by

$$\| (x, y) \|_{\psi_{p,q,\lambda}} = \max\{\| (x, y) \|_p, \lambda \| (x, y) \|_q\}.$$  

(2.9)

Indeed, $\psi_p$ and $\lambda \psi_q$ meet in $(0, 1)$ (note that $\psi_p < \psi_q$, and $\psi_p$ and $\psi_q$ have their minimums $2^{1/p-1}$ and $2^{1/q-1}$, respectively), and $\psi_{p,q,\lambda}$ is convex, so $\psi_{p,q,\lambda} \in \Psi$. Then $\psi_{p,q,\lambda}(t) = \max\{\psi_p(t), \lambda \psi_q(t)\} = \max\{\|(1-t,t)\|_p, \lambda \|(1-t,t)\|_q\}$.

Example 3 (cf. [6]). For $1/2 \leq \alpha \leq 1$ let

$$\psi_{\alpha}(t) = \begin{cases} \frac{2}{\alpha} t + 1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases}$$  

(2.10)

Then $\psi_{\alpha} \in \Psi$, and the norm of $X \oplus \psi_{\alpha} Y$ is given by

$$\| (x, y) \|_{\psi_{\alpha}} = \max\{\| x \| + (2 - \frac{1}{\alpha}) \| y \|, \| y \|\}.$$  

(2.11)

In particular,

$$\| (x, y) \|_{\psi_{\alpha}} = \begin{cases} \| x \| + \| y \| & \text{if } \alpha = 1, \\ \max\{\| x \|, \| y \|\} & \text{if } \alpha = 1/2. \end{cases}$$  

(2.12)

Thus $\| \cdot \|_{\psi_{\alpha}}$ are non-$\ell_p$ and $\ell_\infty$ type norms combining the $\ell_1$- and $\ell_\infty$-sum norms as $\alpha$ varies $1$ through $1/2$.

In fact,

$$\| (x, y) \|_{\psi_{\alpha}} = \begin{cases} \| x \| + \| y \| \left[\frac{2}{\alpha} \| x \| + \| y \| + 1 \right] & \text{if } \frac{1}{\| x \| + \| y \|} \leq \alpha, \\ \| x \| + \| y \| \left[\frac{1}{\| x \| + \| y \|} \right] & \text{if } \frac{1}{\| x \| + \| y \|} \geq \alpha, \\ \| y \| & \text{if } \frac{1}{\| x \| + \| y \|} \leq \alpha, \\ \| x \| + (2 - \frac{1}{\alpha}) \| y \| & \text{if } \frac{1}{\| x \| + \| y \|} \geq \alpha. \end{cases}$$

Noting that $\| x \| + (2 - 1/\alpha) \| y \| \geq \| y \|$ if and only if $\| y \|/(\| x \| + \| y \|) \leq \alpha$, we have (2.11).

Example 4. Let $1 \leq q \leq p \leq \infty$. Let $\| \cdot \|_{p,q}$ be the (Lorentz) $\ell_{p,q}$-norm:

$$\| (z, w) \|_{p,q} = \left\{ z^{*q} + 2^{(q/p)-1} w^{*q} \right\}^{1/q},$$

where $\{z^*, w^*\}$ is the non-increasing rearrangement of $\{|z|, |w|\}$, that is, $z^* \geq w^*$. Note that in case of $1 \leq p < q \leq \infty$, $\| \cdot \|_{p,q}$ is not a norm but a quasi-norm (cf. [5, Proposition 1],
such that, whenever \[ \text{Theorem A} \] showed the following. Let \[ \psi \] be a function provided for any \( x, y \in X \). Then the absolute norm \( \| \cdot \| \in \mathcal{N}_a \) and the corresponding convex function is given by

\[
\psi_{p,q}(t) = \begin{cases} 
(1 - t)^q + 2q/p - 1 t^q 
& \text{if } 0 \leq t \leq 1/2, \\
(1 - t)^q + 2q/p - 1 t^q 
& \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]  

(2.13)

Thus \( \psi_{p,q} \) yields the \( \ell_{p,q} \) direct sum \( X \oplus_{p,q} Y \):

\[
\|(x, y)\|_{p,q} = \|(x, y)\|_{p,q} = \left\{ \|x\|^q + 2q/p - 1 y^q \right\}^{1/q},
\]  

(2.14)

where \( \|x\|^* \geq \|y\|^* \).

3. Uniform convexity of \( X \oplus_{\psi} Y \)

We recall some definitions \[1\] and recent results in \[7\]. A Banach space \( X \) is called strictly convex provided for any \( x, y \in X \), \( \|x\| = \|y\| = 1 \), \( x \neq y \), we have \( \|(x + y)/2\| < 1 \). \( X \) is called uniformly convex provided for any \( \varepsilon > 0 \) there is a \( \delta (0 < \delta < 1) \) such that, whenever \( \|x - y\| \geq \varepsilon \), \( \|x\| \leq 1 \), \( \|y\| \leq 1 \), we have \( \|(x + y)/2\| < 1 - \delta \). A function \( \psi \) on \([0, 1] \) is called strictly convex if, for any \( s, t \in [0, 1], s \neq t \), and for any \( c (0 < c < 1) \), \( \psi((1 - c)s + ct) < (1 - c)\psi(s) + c\psi(t) \). Recently Takahashi et al. \[7\] showed the following.

**Theorem A** [7, Theorem 5]. Let \( \psi \in \Psi \). Then the absolute norm \( \| \cdot \|_{\psi} \) on \( C^2 \) is strictly convex if and only if \( \psi \) is.

**Theorem B** [7, Theorem 6]. Let \( X \) and \( Y \) be Banach spaces and let \( \psi \in \Psi \). Then \( X \oplus_{\psi} Y \) is strictly convex if and only if \( X \) and \( Y \) are strictly convex and \( \psi \) is strictly convex.

The strict and uniform convexity are equivalent for finite dimensional spaces, and a fortiori for absolute normalized norms on \( C^2 \), but not for \( X \oplus_{\psi} Y \) in general. We are now going to characterize the uniform convexity of \( X \oplus_{\psi} Y \).

**Lemma 1** [4, Lemma 2]. Let \( \| \cdot \| \in \mathcal{N}_a \). Let \( |p| \leq |r| \) and \( |q| \leq |s| \). Then \( \|(p, q)\| \leq \|(r, s)\| \). Furthermore, if \( |p| < |r| \) and \( |q| < |s| \), then \( \|(p, q)\| < \|(r, s)\| \).

One should note that in the latter assertion of Lemma 1 the condition \( |p| < |r| \) or \( |q| < |s| \) is not enough to imply that \( \|(p, q)\| < \|(r, s)\| \). Indeed consider the \( \ell_\infty \)-norm. We need the following more precise fact about the monotonicity property of absolute norms.

**Lemma 2** (Takahashi et al. [7]). Let \( \psi \in \Psi \). Then the following are equivalent:

(i) If \( |z| \leq |u| \) and \( |w| \leq |v| \), or \( |z| < |u| \) and \( |w| < |v| \), then \( \|(z, w)\|_{\psi} < \|(u, v)\|_{\psi} \).

(ii) \( \psi(t) > \psi_\infty(t) = \max\{1 - t, t\} \) for all \( t \in (0, 1) \).

(iii) \( \psi(t)/t \) is strictly decreasing for all \( t \in (0, 1) \), and \( \psi(t)/(1 - t) \) is strictly increasing for all \( t \in [0, 1) \).

In particular, if \( \psi \) is strictly convex all these assertions are valid.
Now we obtain the following.

**Theorem 1.** Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi$. Then $X \oplus_\psi Y$ is uniformly convex if and only if $X$ and $Y$ are uniformly convex and $\psi$ is strictly convex.

**Proof.** If $X \oplus_\psi Y$ is uniformly convex, so are $X$ and $Y$ because $X, Y$ are isometrically embedded into $X \oplus_\psi Y$. We have the strict convexity of $\psi$ by Theorem B.

Assume that $X$ and $Y$ are uniformly convex and $\psi$ is strictly convex. Let $\epsilon > 0$ be arbitrary. Take arbitrary $(x_1, y_1), (x_2, y_2) \in X \oplus_\psi Y$ so that
\[
\| (x_1, y_1) - (x_2, y_2) \|_\psi \geq \epsilon, \quad \| (x_1, y_1) \|_\psi = \| (x_2, y_2) \|_\psi = 1.
\] (3.1)

Then there are $\delta_X$ and $\delta_Y$ ($0 < \delta_X, \delta_Y < 1$) such that
\[
\| u_1 - u_2 \| \geq \frac{\epsilon}{2}, \quad \| u_1 \| \leq 1, \quad \| u_2 \| \leq 1 \quad (u_1, u_2 \in X)
\] (3.2)

and
\[
\| v_1 - v_2 \| \geq \frac{\epsilon}{2}, \quad \| v_1 \| \leq 1, \quad \| v_2 \| \leq 1 \quad (v_1, v_2 \in Y)
\] (3.3)

respectively. Put
\[
t := \frac{\| y_1 - y_2 \|}{\| x_1 - x_2 \| + \| y_1 - y_2 \|}.
\]

Then since
\[
\| y_1 - y_2 \| = \frac{t}{1 - t} \| x_1 - x_2 \|
\]
for $t \neq 1$, we have
\[
\epsilon \leq \| (x_1, y_1) - (x_2, y_2) \|_\psi = \left( \| x_1 - x_2 \| + \| y_1 - y_2 \| \right) \psi(t) = \left( \| x_1 - x_2 \| + \frac{t}{1 - t} \| x_1 - x_2 \| \right) \psi(t) = \frac{\psi(t)}{1 - t} \| x_1 - x_2 \|.
\]

from which it follows that
\[
\| x_1 - x_2 \| \geq \frac{1 - t}{\psi(t)} \epsilon
\] (3.4)

and
\[
\| y_1 - y_2 \| \geq \frac{t}{\psi(t)} \epsilon.
\] (3.5)

Now put
\[
s_1 = \frac{\| y_1 \|}{\| x_1 \| + \| y_1 \|}, \quad s_2 = \frac{\| y_2 \|}{\| x_2 \| + \| y_2 \|}.
\]
Then
\[ \|x_i\| = \frac{1-s_i}{\psi(s_i)}, \quad \|y_i\| = \frac{s_i}{\psi(s_i)} \quad (i = 1, 2). \] (3.6)

We assume that \( s_1 \leq s_2 \) without loss of generality. Since \( \psi \) is strictly convex, the function \( \psi(s)/(1-s) \) is strictly increasing and \( \psi(s)/s \) is strictly decreasing by Lemma 2. Therefore by (3.6) we have
\[ \|x_1\| \geq \|x_2\| \] (3.7)
and
\[ \|y_1\| \leq \|y_2\|. \] (3.8)

**Case 1.** Let \( 0 \leq t \leq 1/2 \). In this case, by (3.4) we have
\[ \|x_1 - x_2\| \geq \frac{1-t}{\psi(t)} \geq \frac{1-1/2}{\psi(1/2)} \geq \frac{\epsilon}{2}. \] (3.9)
Hence
\[ \|x_1\| \geq \frac{\epsilon}{4}, \]
as \( \|x_1\| \geq \|x_2\| \) and \( \|x_1\| + \|x_2\| \geq \|x_1 - x_2\| > \epsilon/2. \) Thus \( (1-s_1)/\psi(s_1) = \|x_1\| > \epsilon/4. \)
Since \( (1-s)/\psi(s) \) is strictly decreasing and \( \psi(1) = 0, \) there exists \( a (s_1 < a < 1) \) such that
\[ \frac{1-a}{\psi(a)} = \frac{\epsilon}{4}. \]
Now since \( \psi \) is uniformly continuous on \([0, (a+1)/2], \) there exists \( \rho (0 < \rho < (1-a)/2) \) so that, whenever \( 0 \leq s_2 - s_1 \leq \rho, s_1 \in [0, a], s_2 \in [0, (a+1)/2], \)
\[ \frac{1-s_1}{\psi(s_1)} - \frac{1-s_2}{\psi(s_2)} \leq \frac{\delta X}{2(1-\delta X)} \cdot \frac{1-(a+1)/2}{\psi((a+1)/2)}. \]
Then we have
\[ \frac{1-s_1}{\psi(s_1)} - \frac{1-s_2}{\psi(s_2)} \leq \frac{\delta X}{2(1-\delta X)} \cdot \frac{1-s_2}{\psi(s_2)}, \]
from which it follows that
\[ \frac{1-s_1}{\psi(s_1)} \leq \frac{1-\delta X/2}{1-\delta X} \cdot \frac{1-s_2}{\psi(s_2)}, \]
or
\[ \|x_1\| \leq \frac{1-\delta X/2}{1-\delta X}\|x_2\|. \] (3.10)

Now let us consider the case \( 0 \leq s_2 - s_1 \leq \rho. \) We first note that
\[ \|x_1 + x_2\| \leq 2(1-\delta X)\|x_1\|. \] (3.11)
Indeed, as
\[ \frac{x_1 - x_2}{\|x_1\|} = \frac{x_1 - x_2}{\|x_1\|} \geq \frac{e}{2} \]
by (3.9), we have by (3.2)
\[ \frac{x_1 + x_2}{\|x_1\|} < 2(1 - \delta_X), \]
or (3.11). In this case we have \( s_2 \in [0, (a + 1)/2] \) (note that \( s_1 \in [0, a] \)). Therefore according to (3.11) and (3.10),

\[ \|(x_1, y_1) + (x_2, y_2)\|_\psi = \|(\|x_1 + x_2\|, \|y_1 + y_2\|)\|_\psi \leq \|(2(1 - \delta_X)\|x_1\|, \|y_1\| + \|y_2\|)\|_\psi \]

\[ \leq \|(2(2 - \delta_X)\|x_2\|, 2\|y_2\|)\|_\psi < 2\|(\|x_2\|, \|y_2\|)\|_\psi = 2. \]

Here put
\[ f(s_2) = \|(2 - \delta_X)\|x_2\|, 2\|y_2\|)\|_\psi = \|(2 - \delta_X) \frac{1 - s_2}{\psi(s_2)} \cdot \frac{2s_2}{\psi(s_2)} \|_\psi. \]

Then \( f \) is continuous on \([0, (a + 1)/2]\) and \( 0 < f(s_2) < 2 \). So if we let \( M_1 := \max\{f(s_2): 0 \leq s_2 \leq (a + 1)/2\} \),

then \( 0 < M_1 < 2 \). Therefore we obtain
\[ \|(x_1, y_1) + (x_2, y_2)\|_\psi \leq M_1 < 2, \]
as desired.

Next we consider the case \( s_2 - s_1 \geq \rho \). In this case we have
\[ \|(x_1, y_1) + (x_2, y_2)\|_\psi = \|(\|x_1 + x_2\|, \|y_1 + y_2\|)\|_\psi \leq \|(\|s_1\|, \|y_1\| + \|y_2\|)\|_\psi \]

\[ \leq \left\| \left( \frac{1 - s_1}{\psi(s_1)} + \frac{1 - s_2}{\psi(s_2)} \right) \cdot \left( \frac{s_1}{\psi(s_1)} + \frac{s_2}{\psi(s_2)} \right) \right\|_\psi \]

\[ = \frac{1}{\psi(s_1)\psi(s_2)} \left\| \left( (1 - s_1)\psi(s_2) + (1 - s_2)\psi(s_1), s_1\psi(s_2) + s_2\psi(s_1) \right) \right\|_\psi \]

\[ = \frac{1}{\psi(s_1)\psi(s_2)} \left( \psi(s_1) + \psi(s_2) \right) \left( \psi(s_1) + \psi(s_2) \right) \left( \psi(s_1) + \psi(s_2) \right) \]

\[ \leq \frac{\psi(s_1) + \psi(s_2)}{\psi(s_1)\psi(s_2)} \left( \psi(s_1) + \psi(s_2) \right) \left( \psi(s_1) + \psi(s_2) \right) \]

\[ = 2. \]

The function
\[ g(s_1, s_2) = \psi(s_1) + \psi(s_2) \psi(s_1) + \psi(s_2) \]

is
\[ (s_1\psi(s_2) + s_2\psi(s_1)) \psi(s_1) + \psi(s_2) \psi(s_1) + \psi(s_2) \]

(3.14)
is continuous on the set
\[ \Omega = \{(s_1, s_2): 0 \leq s_1 \leq a, \ 0 \leq s_2 \leq 1, \ s_2 - s_1 \geq \rho \} \]
and has there the maximum \( M_2 < 2 \). Consequently we obtain
\[ \|(x_1, y_1) + (x_2, y_2)\|_\psi \leq M_2 < 2. \quad (3.15) \]

**Case 2.** Let \( 1/2 \leq t \leq 1 \). By (3.5) we have
\[ \|y_1 - y_2\| \geq \frac{t}{\psi(t)} \epsilon \geq \frac{1/2}{\psi(1/2)} \epsilon > \frac{\epsilon}{2}. \quad (3.16) \]
Hence
\[ \|y_2\| \geq \frac{\epsilon}{4}. \]
The function \( s / \psi(s) \) is strictly increasing and \( \psi(0) = 0 \), and so there exists \( b (0 < b < s_2) \) such that
\[ \frac{b}{\psi(b)} = \frac{\epsilon}{4}. \]
As before, take \( \rho \) \((0 < \rho < b/2)\) so that, whenever \( 0 \leq s_2 - s_1 \leq \rho \), \( s_1 \in [b/2, 1] \), \( s_2 \in [b, 1] \), we have
\[ \frac{s_2}{\psi(s_2)} - \frac{s_1}{\psi(s_1)} \leq \frac{\delta_Y}{2(1 - \delta_Y)} \frac{b/2}{\psi(b/2)}. \]
Then since
\[ \frac{s_1}{\psi(s_1)} \leq \frac{1 - \delta_Y/2}{1 - \delta_Y} \|y_1\|. \]
we have
\[ \frac{s_2}{\psi(s_2)} \leq \frac{1 - \delta_Y/2}{1 - \delta_Y} \|y_1\|. \]
or
\[ \|y_2\| \leq \frac{1 - \delta_Y/2}{1 - \delta_Y} \|y_1\|. \quad (3.17) \]
In the case \( 0 \leq s_2 - s_1 \leq \rho \) since
\[ \|y_1 + y_2\| \leq 2(1 - \delta_Y) \|y_2\|, \]
with (3.17), we have
\[ \|(x_1, y_1) + (x_2, y_2)\|_\psi = \|(\|x_1 + x_2\|, \|y_1 + y_2\|)\|_\psi \leq \|(\|x_1\| + \|x_2\|, 2(1 - \delta_Y) \|y_2\|)\|_\psi \leq \|(2\|x_1\|, (2 - \delta_Y) \|y_1\|)\|_\psi < 2\|(\|x_1\|, \|y_1\|)\|_\psi = 2. \]
Therefore, the maximum $M_3$ of the function
\[ f(s_1) = \left\| \left( 2 \frac{1 - s_1}{\psi(s_1)}, (2 - \delta_1) \frac{s_1}{\psi(s_1)} \right) \right\|_\psi \text{ on } \left[ \frac{b}{2}, 1 \right] \]
is less than 2. Consequently we have
\[ \left\| (x_1, y_1) + (x_2, y_2) \right\|_\psi \leq M_3 < 2. \]  \hfill (3.18)
Let next $s_2 - s_1 \geq \rho$. Then according to (3.13)
\[ \left\| (x_1, y_1) + (x_2, y_2) \right\|_\psi \leq \frac{\psi(s_1) + \psi(s_2)}{\psi(s_1) + \psi(s_2)} \left( \frac{\psi(s_2)}{\psi(s_1) + \psi(s_2)} s_1 + \frac{\psi(s_1)}{\psi(s_1) + \psi(s_2)} s_2 \right) < 2. \]
So the function $g(s_1, s_2)$ in (3.14) takes the maximum $M_4 < 2$ on the set
\[ \Omega_0 = \{(s_1, s_2): 0 \leq s_1 \leq 1, b \leq s_2 \leq 1, s_2 - s_1 \geq \rho \}, \]
from which it follows that
\[ \left\| (x_1, y_1) + (x_2, y_2) \right\|_\psi \leq M_4 < 2. \]  \hfill (3.19)
Consequently, letting $M = \max\{M_1, M_2, M_3, M_4\}$, we obtain that
\[ \left\| (x_1, y_1) + (x_2, y_2) \right\|_\psi \leq M < 2, \]
which proves that $X \oplus \psi Y$ is uniformly convex. This completes the proof. \hfill \square

By putting $\psi = \psi_{p,q}$ we have the following:

**Corollary 1.** Let $X$ and $Y$ be Banach spaces and let $1 < q < p \leq \infty$, not $p = q = 1$ nor $\infty$. Then $X \oplus \psi_{p,q} Y$ is uniformly convex if and only if $X$ and $Y$ are uniformly convex.

In particular we have the well-known fact:

**Corollary 2.** Let $1 < p < \infty$. Then $X \oplus_p Y$ is uniformly convex if and only if $X$ and $Y$ are uniformly convex.

**Corollary 3.** Let $X$ and $Y$ be Banach spaces. Let $1 < q < p < \infty$ and $2^{1/p - 1/q} < \lambda < 1$. Let $\psi_{p,q,\lambda} = \max\{\psi_p, \lambda \psi_q\} \in \Psi$. Then $X \oplus \psi_{p,q,\lambda} Y$ is uniformly convex if and only if $X, Y$ are uniformly convex.

**Example 5.** Let $X$ and $Y$ be uniformly convex. Let $\psi_\alpha (1/2 \leq \alpha \leq 1)$ be as in Example 3. Then $X \oplus \psi_\alpha Y$ is not uniformly convex.

**References**