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The Erdös–Ko–Rado Theorem for Vector Spaces

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Let V be an n-dimensional vector space over $GF(q)$ and for integers $k \geqslant t > 0$ let $m_q(n, k, t)$ denote the maximum possible number of subspaces in a t-intersecting family $\mathscr{F}$ of k-dimensional subspaces of V, i.e., $\dim F \cap F' \geqslant t$ holds for all $F, F' \in \mathscr{F}$. It is shown that $m_q(n, k, t) = \max\{\left[\begin{smallmatrix} n-t \\ k-t \end{smallmatrix}\right], \left[\begin{smallmatrix} 2k-t \\ k \end{smallmatrix}\right]\}$ for $n \geqslant 2k - t$ while for $n \leqslant 2k - t$ trivially $m_q(n, k, t) = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ holds. © 1986 Academic Press, Inc.

1. INTRODUCTION

Suppose X is an n-element set, $n \geqslant k \geqslant t > 0$. A family of k-subsets of X, i.e., $\mathscr{F} \subset \binom{X}{k}$ is called t-intersecting if $|F \cap F'| \geqslant t$ holds for all $F, F' \in \mathscr{F}$. The maximum size of a t-intersecting family was determined by Erdös, Ko, and Rado [2] for $n > n_0(k, t)$.

ERDÖS–KO–RADO THEOREM. *Suppose $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ is t-intersecting. Then for $n \geqslant n_0(k, t)$,*

$$|\mathscr{F}| \leqslant \binom{n-t}{k-t} \qquad \textit{holds.} \tag{1.1}$$

It was shown by the present authors [3, 6] that $n_0(k, t) = (k - t + 1)(t + 1)$, i.e., (1.1) if and only if $n \geqslant (k - t + 1)(t + 1)$. Moreover, for $n > n_0(k, t)$ the only family achieving equality in (1.1) is obtained by taking all k-subsets of X containing a fixed t-set.

However, very little is known for $n < n_0(k, t)$. Denote by $m(n, k, t)$ the maximum size of a t-intersecting family $\mathscr{F} \subset \binom{X}{k}$. For $0 \leqslant i \leqslant k - t$ and

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$Y_i \in \binom{X}{t+2i}$ define $\mathscr{F}_i = \{F \in \binom{X}{k}: |F \cap Y_i| \geqslant t+i\}$. Clearly, $\mathscr{F}_i$ is t-intersecting. Let us mention the following conjecture.

Conjecture 1 [3]. $m(n, k, t) = \max_i |\mathscr{F}_i|$.

This problem has an obvious extension to t-intersecting families of k-subspaces of a n-dimensional vector space V over $GF(q)$. Let $m_q(n, k, t)$ denote the corresponding analog of $m(n, k, t)$, i.e., $m_q(n, k, t) = \max\{|\mathscr{F}|: \mathscr{F} \subset \left[{V \atop k}\right], \dim F \cap F' \geqslant t$ holds for all $F, F' \in \mathscr{F}\}$. If $n \leqslant 2k - t$ then $\binom{V}{k}$ is t-intersecting. Therefore trivially $m_q(n, k, t) = \left[{n \atop k}\right]$ holds. Here, and in the sequel $\left[{a \atop b}\right]_q$ is the Gaussian coefficient, i.e., $\left[{a \atop b}\right]_q = \prod_{0 \leqslant i < b} ((q^a - q^i)/(q^b - q^i))$. If it causes no confusion, we shall omit the subscript q.

Hsieh [5] proved that $m_q(n, k, t) = \left[{n-t \atop k-t}\right]$ holds for $n \geqslant 2k+1$, $q \geqslant 3$ and for $n \geqslant 2k+2$, $q = 2$. Hsieh's proof is entirely combinatorial but it involves lengthy computations. Greene and Kleitman [4] gave a short proof for the case $t = 1$, $n \geqslant 2k$, k divides n. Using the case $n = 2k$ as the base step, in [1] a short, inductive argument is given for the $t = 1$ case.

Checking the families in Conjecture 1, one sees that among them $\mathscr{F}_0 = \{F \in \left[{V \atop k}\right], Y_0 \subset F\}$ has the largest size if $n \geqslant 2k$ ($Y_0 \in \left[{V \atop t}\right]$), and $\mathscr{F}_{k-t} = \{F \in \left[{Y_{k-t} \atop k}\right]\}$, $Y_{k-t} \in \left[{V \atop 2k-t}\right]$, has the largest size if $2k \geqslant n \geqslant 2k - t$, in particular, for $n = 2k$ their sizes are equal.

The aim of this paper is to show that, in fact, $m_q(n, k, t) = \max\{|\mathscr{F}_0|, |\mathscr{F}_{k-t}|\}$ holds for all $n \geqslant 2k - t$.

THEOREM 1. *Suppose $n \geqslant 2k - t$, $\mathscr{F} \subset \left[{V \atop k}\right]$ is t-intersecting then*

$$|\mathscr{F}| \leqslant \max\left\{\left[{n-t \atop k-t}\right]_q, \left[{2k-t \atop k}\right]_q\right\}. \tag{1.2}$$

The proof relies on the ideas of [6], however, the actual computation is done differently, in a shorter way, using the fast growth of the q-nomial coefficients.

Let us also mention that (1.2) and the methods of [1] easily imply the uniqueness of the optimal families for $n \geqslant 2k+1$ (and hence by Section 3, for $2k - t < n < 2k$).

It appears likely that for $n = 2k$ there are only two non-isomorphic optimal families but we could not prove this for $t \geqslant 2$. In Section 2 the outline of the proof is given for the case $n \geqslant 2k$; the detailed argument is left for Sections 4 and 5. In Section 3 we derive the case $2k \geqslant n \geqslant 2k - t$ from the case $n \geqslant 2k$.

2. OUTLINE OF THE PROOF FOR $n \geqslant 2k$

Suppose $n \geqslant 2k$ and $\mathscr{F} \subset \left[{V \atop k}\right]$ is t-intersecting. Let φ be the characteristic vector of $\mathscr{F}$, i.e., φ is a (0, 1)-vector of length $\left[{n \atop k}\right]_q$, with coordinates indexed by the k-subspaces $S \in \left[{V \atop k}\right]$, the entry indexed by S is 1 if and only if $S \in \mathscr{F}$.

Let c be a positive scalar, A a real symmetric matrix of order $\left[{n \atop k}\right]$ (with rows and columns indexed by the k-subspaces of V), $I(J)$ is the identity matrix (all 1 matrix) of order $\left[{n \atop k}\right]$, respectively. Suppose further that (2.1), (2.2) hold.

$$\text{The entry in row } S \text{ and column } T \text{ of } A \text{ is } 0 \text{ whenever } \dim S \cap T \geqslant t. \tag{2.1}$$

$$A + I - c^{-1}J \qquad \text{is positive semi-definite.} \tag{2.2}$$

Since $\mathscr{F}$ is t-intersecting (2.1) implies $\varphi A \varphi^T = 0$. Now (2.2) yields

$$0 \leqslant \varphi(A + I - c^{-1}J)\,\varphi^T = \varphi\varphi^T - c^{-1}\varphi J \varphi^T = |\mathscr{F}| - c^{-1}|\mathscr{F}|^2, \tag{2.3}$$

or equivalently, $|\mathscr{F}| \leqslant c$.

In order to prove (1.2) for $n \geqslant 2k$ one needs to find a matrix A satisfying (2.1), (2.2), with $c = \left[{n-t \atop k-t}\right]$.

To define A let us first define the matrices $W_{j,k}$ ($\bar{W}_{j,k}$) of size $\left[{n \atop j}\right] \times \left[{n \atop k}\right]$ with rows indexed by the j-subspaces $P \in \left[{V \atop j}\right]$, columns indexed by the k-subspaces $S \in \left[{V \atop k}\right]$, and whose (P, S) entry is 1 if $P \leqslant S$ (resp. if $\dim P \cap S = 0$) and is 0 otherwise, $0 \leqslant j \leqslant k$.

Now we can define A.

$$A = q^{-k^2 + k + \binom{t}{2}} \sum_{i=0}^{t-1} (-1)^{t-1-i} \times q^{(k-t)i + \binom{i}{2}} \begin{bmatrix} k-1-i \\ k-t \end{bmatrix} \begin{bmatrix} n-k-t+i \\ k-t \end{bmatrix}^{-1} \bar{W}^T_{k-i,k} W_{k-i,k}. \tag{2.4}$$

Let us set $B_i = \bar{W}^T_{i,k} W_{i,k}$. Then the general entry $b(S, T)$ of B_i is the number i-dimensional subspaces of V contained in T and intersecting S only in the zero vector. Thus $b(S, T)$ depends only on $\dim S \cap T$ and $b(S, T) = 0$ if $\dim S \cap T \geqslant t$ (here we used $i > k - t$).

This shows that A fulfills (2.1). In order to show that A fulfills (2.2) as well, we show that the $\left[{n \atop k}\right]$-dimensional Euclidean space E (with coordinates indexed by $S \in \left[{V \atop k}\right]$) has an orthogonal decomposition $E = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ satisfying

$$U_i = V_0 \oplus \cdots \oplus V_i \text{ is the row space of } W_{i,k},\ 0 \leqslant i \leqslant k. \tag{2.5}$$

$$V_i \text{ is an eigenspace of } B_j \text{ with corresponding eigenvalue } (-1)^i \begin{bmatrix} k-i \\ j-i \end{bmatrix} \begin{bmatrix} n-i-j \\ k-i \end{bmatrix} q^{j(k-i)} q^{\binom{i}{2}}. \tag{2.6}$$

3. The Case $2k > n > 2k - t$

Assume that Theorem 1 is proved for all (n, k) with $n > 2k$. Suppose $\mathscr{F} \subset \left[{V \atop k}\right]$ is t-intersecting. Let $f(u, v)$ be a non-degenerate bilinear function $f\colon V^2 \to GF(q)$. For a subspace $S < V$ let $S^\perp$ denote its orthogonal: $S^\perp = \{v \in V\colon f(u, v) = 0 \text{ for all } u \in S\}$. Also, define $\mathscr{F}^\perp = \{S^\perp\colon S \in \mathscr{F}\}$. Clearly $|\mathscr{F}^\perp| = |\mathscr{F}|$, $\mathscr{F}^\perp \subset \left[{V \atop n-k}\right]$.

Claim 3.1. *$\mathscr{F}^\perp$ is $(n - 2k + t)$-intersecting.*

Proof. Suppose $F, F' \in \mathscr{F}$ and let $S = \langle F, F' \rangle$ be the subspace generated by F and F'. Since $\dim F \cap F' \geqslant t$, $\dim S \leqslant 2k - t$ holds. Therefore $\dim S^\perp \geqslant n - 2k + t$ holds. Now the claim follows from $S^\perp = F^\perp \cap F'^\perp$. ∎

From $2k > n \geqslant 2k - t$ one infers

$$n > 2(n - k) \qquad \text{and} \qquad n - 2k + t > 0.$$

Thus applying the theorem gives $|\mathscr{F}| = |\mathscr{F}^\perp| \leqslant \left[{n - (n - 2k + t) \atop (n - k) - (n - 2k + t)}\right] = \left[{2k - t \atop k - t}\right] = \left[{2k - t \atop k}\right]$ as desired.

In the case of equality, equality holds for $|F^\perp|$ as well. Thus there exists a $(n - 2k + t)$-dimensional subspace T of V such that $T \leqslant F^\perp$ for all $F \in \mathscr{F}$, i.e., $F \leqslant T^\perp \in \left({V \atop 2k - t}\right]$ for all $F \in \mathscr{F}$. Since $|\mathscr{F}| = \left[{2k - t \atop k}\right]$, $\mathscr{F} = \left[{T^\perp \atop 2k - t}\right]$. ∎

4. The Spectrum of B_i

First let us note that given $S \in \left[{V \atop f}\right]$ the number of $(n - f)$-dimensional subspaces T with $S \cap T = \langle 0 \rangle$ is $q^{f(n-f)}$.

More generally, given $S \in \left[{V \atop i}\right]$, $S' \in \left[{V \atop f}\right]$, the number of $(n - e)$-dimensional spaces T with $S' \cap T = \langle 0 \rangle$, $S \leqslant T$ is $q^{f(e-i)}\left[{n - i - f \atop e - i}\right]$ or 0 according whether $\dim S \cap S' = 0$ or not.

This implies

$$W_{ie} \bar{W}_{ef} = q^{f(e-i)} \begin{bmatrix} n - i - f \\ e - i \end{bmatrix} \bar{W}_{if}, \qquad 0 \leqslant i \leqslant e. \tag{4.1}$$

By a simpler argument one has

$$W_{ie} W_{ef} = \begin{bmatrix} f - i \\ e - i \end{bmatrix} W_{if}, \qquad 0 \leqslant i \leqslant e \leqslant f. \tag{4.2}$$

Let us note that (4.2) shows $U_0 < U_1 < \cdots < U_t$, where U_i is the row space of W_{ik}. This justifies our definition of V_i as the orthogonal complement of U_{i-1} in U_i.

Let us recall three identities involving q-nomial coefficients:

$$\begin{bmatrix} -c \\ s \end{bmatrix} = (-1)^s q^{-cs-\binom{s}{2}} \begin{bmatrix} c+s-1 \\ s \end{bmatrix}, \tag{4.3}$$

$$\sum_{\substack{i+j=t \\ i,j\geqslant 0}} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} b \\ j \end{bmatrix} q^{(a-i)j} = \begin{bmatrix} a+b \\ t \end{bmatrix} = \sum_{i+j=t} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} b \\ j \end{bmatrix} q^{i(b-j)}, \tag{4.4}$$

$$\sum_{0\leqslant i\leqslant b} (-1)^i q^{\binom{i+1}{2}-bi} \begin{bmatrix} b \\ i \end{bmatrix} = 0$$

$$= \sum_{0\leqslant i\leqslant b} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} b \\ i \end{bmatrix}, b \geqslant 1. \tag{4.5}$$

Note that (4.5) can be derived from (4.4) substituting $a=-1$ and using (4.3).

Let us prove now

$$\bar{W}_{ef} = \sum_{0\leqslant i\leqslant \min\{e,f\}} (-1)^i q^{\binom{i}{2}} W_{ie}^T W_{if} \tag{4.6}$$

and

$$W_{ef} = \sum_{i=0}^{e} (-1)^i q^{\binom{i+1}{2}-ei} W_{ie}^T \bar{W}_{if}. \tag{4.7}$$

For $S\in [{V \atop e}]$, $T\in[{V \atop f}]$ let us compute the (S, T)-entry of the RHS of (4.6). Denoting $\dim S\cap T$ by b we obtain $\sum_{o\leqslant i\leqslant b}(-1)^i q^{\binom{i}{2}}[{b \atop i}]$, which (by (4.5)) is zero for $b>0$ and 1 for $b=0$. Similarly, the (S, T)-entry of the RHS of (4.7) is $\sum_{0\leqslant i\leqslant e-b}(-1)^i q^{\binom{i+1}{2}-ei} q^{bi}[{e-b \atop i}]$, which is zero—in view of (4.5) whenever $b<e$ and 1 for $b=e$.

Let us use (4.6) and (4.7) to compute:

$$(\bar{W}_{ek}^T W_{ek})(\bar{W}_{fk}^T W_{fk})$$

$$= q^{f(k-e)} \begin{bmatrix} n-e-f \\ k-e \end{bmatrix} \bar{W}_{ek}^T \bar{W}_{ef} W_{fk}$$

$$= q^{f(k-e)} \begin{bmatrix} n-e-f \\ k-e \end{bmatrix} \bar{W}_{ek}^T \left(\sum_{0\leqslant i\leqslant \min\{e,f\}} (-1)^i q^{\binom{i}{2}} W_{ie}^T W_{if} \right) W_{fk}$$

$$= q^{f(k-e)} \begin{bmatrix} n-e-f \\ k-e \end{bmatrix} \bar{W}_{ek}^T \sum (-1)^i q^{\binom{i}{2}} W_{ie}^T \begin{bmatrix} k-i \\ f-i \end{bmatrix} W_{ik}$$

$$= q^{f(k-e)} \begin{bmatrix} n-e-f \\ k-e \end{bmatrix}$$

$$\times \sum (-1)^i q^{\binom{i}{2}+k(e-i)} \begin{bmatrix} n-k-i \\ e-i \end{bmatrix} \begin{bmatrix} k-i \\ f-i \end{bmatrix} \bar{W}_{ik}^T W_{ik},$$

or equivalently,

$$B_e B_f = q^{f(k-e)} \begin{bmatrix} n-e-f \\ k-e \end{bmatrix}$$

$$\times \sum_{0 \leqslant i \leqslant \min\{e,f\}} (-1)^i q^{k(e-i)+\binom{i}{2}} \begin{bmatrix} n-k-i \\ e-i \end{bmatrix} \begin{bmatrix} k-i \\ f-i \end{bmatrix} B_i. \quad (4.8)$$

Note that it follows from (4.8) that $B_e B_f = B_f B_e$, which readily implies that the B_i can be diagonalized simultaneously.

Next we show that U_e is the row space of B_e and it has dimension $[{n \atop e}]$. Consider (4.7) for $e=f$. Then the LHS is the identity matrix of size $[{n \atop e}]$. Using (4.1) we infer ($\bar{W}_{ke} = \bar{W}_{ek}^T$):

$$I = \left(\sum_{i=0}^{e} (-1)^i q^{\binom{i+1}{2}-ei} q^{-e(k-i)} \begin{bmatrix} n-i-e \\ k-i \end{bmatrix}^{-1} W_{ie}^T W_{ik} \right) \bar{W}_{ek}^T$$

$$\text{or} \qquad I = C\bar{W}_{ek}^T,$$

and by the transpose of (4.1):

$$I = \left(\sum_{i=0}^{e} (-1)^i q^{\binom{i+1}{2}-ei} q^{-i(k-e)} \begin{bmatrix} n-i-e \\ k-e \end{bmatrix}^{-1} W_{ie}^T \bar{W}_{ik} \right) W_{ek}^T$$

or

$$I = DW_{ek}^T.$$

Consequently $CB_e D^T = C\bar{W}_{ek}^T W_{ek} D^T = I$. Since the rank of a product never exceeds the rank of the factors, rank $B_e =$ rank $W_{ek} = [{n \atop e}]$.

Now we are in a position to prove (2.6).

Let $\mathbf{x} \in V_e$. Since $\mathbf{x} \in U_e$, the row space of B_e, we have $\mathbf{x} = \mathbf{y}B_e$ for some vector $\mathbf{y}$. By definition for $i<e$ $\mathbf{x} \in U_i^\perp$ holds. As the rows of B_i (and the columns as B_i is symmetric) are in U_i, $\mathbf{x}B_i = \mathbf{0}$. Consequently $\mathbf{0} = (\mathbf{y}B_e) B_i = (\mathbf{y}B_i) B_e$, i.e., $\mathbf{y}B_i \in U_e^\perp$. But $\mathbf{y}B_i \in U_i$ holds as well, yielding $\mathbf{y}B_i = \mathbf{0}$.

Now for $f \geqslant e$ (4.8) implies

$$\begin{aligned}\mathbf{x}B_f &= \mathbf{y}B_eB_f\\ &= q^{f(k-e)}\begin{bmatrix} n-e-f \\ k-e \end{bmatrix} \sum_{0\leqslant i\leqslant e} (-1)^i q^{k(e-i)+\binom{i}{2}} \begin{bmatrix} n-k-i \\ e-i \end{bmatrix}\begin{bmatrix} k-i \\ f-i \end{bmatrix}\mathbf{y}B_i\\ &= (-1)^e q^{f(k-e)+\binom{e}{2}}\begin{bmatrix} n-e-f \\ k-e \end{bmatrix}\begin{bmatrix} k-e \\ f-e \end{bmatrix}\mathbf{y}B_e,\end{aligned}$$

proving (2.6).

5. The Spectrum of A

In this section we show that for $n \geqslant 2k$, A satisfies (2.2) with $c = \left[\begin{smallmatrix} n-t \\ k-t \end{smallmatrix}\right]$ which will prove $|\mathscr{F}| \leqslant \left[\begin{smallmatrix} n-t \\ k-t \end{smallmatrix}\right]$.

First note that V_i is an eigenspace of A for $i = 0, \ldots, k$. Let λ_i be the corresponding eigenvalue, (2.4) and (2.6) provide a complicated but closed form for λ_i. Let us recall that we must show $\lambda_0 \geqslant -1 + \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] / \left[\begin{smallmatrix} n-t \\ k-t \end{smallmatrix}\right]$ and $\lambda_i \geqslant -1$ for $i \geqslant 1$.

The next lemma shows that for $i = 0, 1, \ldots, t$ one has equality.

Lemma 5.1. $W_{tk}A = J_{tk} - W_{tk}$

Proof.

$$W_{tk}A = \sum_{i=0}^{t-1} (-1)^{t-1-i} q^{-k^2+k+\binom{t}{2}+(k-t)i+\binom{i}{2}} \begin{bmatrix} k-1-i \\ k-t \end{bmatrix} \times \left(\begin{bmatrix} n-k-t+i \\ k-t \end{bmatrix}\right)^{-1} (W_{tk}\bar{W}^T_{k-i,k})\, W_{k-i,k}.$$

Using (4.1) we may rewrite the RHS as

$$\sum_{i=0}^{t-1} (-1)^{t-1-i} q^{-k(t-1)+\binom{t}{2}+\binom{i}{2}} \begin{bmatrix} k-1-i \\ k-t \end{bmatrix} \bar{W}_{t,k-i} W_{k-i,k}.$$

The entry in row T and column K of $\bar{W}_{t,k-i}W_{k-i,k}$ is $q^{(k-i)l}\left[\begin{smallmatrix} k-l \\ k-i \end{smallmatrix}\right]$ if $\dim T \cap K = l$.

To complete the proof of the lemma one must show that

$$\sum_{0\leqslant i\leqslant t-1} (-1)^{t-i-1} q^{-k(t-1)+(k-i)l+\binom{t}{2}+\binom{i}{2}} \begin{bmatrix} k-1-i \\ k-t \end{bmatrix}\begin{bmatrix} k-l \\ k-i \end{bmatrix} = \begin{cases} 1 & \text{if } 0 \leqslant l < t, \\ 0 & \text{if } l = t. \end{cases}$$

For $l=t$ the expression is clearly 0. Suppose now $0 \leqslant l < t$. Using the first part of (4.4) with $a=t-k-1$, $b=k-l$, one may write

$$1=\begin{bmatrix} t-1-l \\ t-1-l \end{bmatrix}=\sum_{l\leqslant i\leqslant t-1} q^{(i-k)(i-l)}\begin{bmatrix} t-k-1 \\ t-i-1 \end{bmatrix}\begin{bmatrix} k-l \\ i-l \end{bmatrix},$$

or using (4.3) with $s=t-i-1$, $c=k-t+1$:

$$\begin{aligned}1&=\sum_{l\leqslant i\leqslant t-1} q^{(i-k)(i-l)}(-1)^{t-i-1}q^{-(t-i-1)(k-t+1)-\binom{t-i-1}{2}}\\ &\quad\times\begin{bmatrix} k-i-1 \\ t-i-1 \end{bmatrix}\begin{bmatrix} k-l \\ k-i \end{bmatrix}\\ &=\sum_{0\leqslant i\leqslant t-1}(-1)^{t-i-1}q^{-k(t-1)+(k-i)l+\binom{i}{2}+\binom{t}{2}}\\ &\quad\times\begin{bmatrix} k-1-i \\ k-t \end{bmatrix}\begin{bmatrix} k-l \\ k-i \end{bmatrix}. \quad\blacksquare\end{aligned}$$

At last we shall prove that $\lambda_e < -1$ holds for $e \geqslant t+1$. In view of (2.4), (2.6) we have

$$\begin{aligned}\lambda_e&=(-1)^e q^{-k^2+k+\binom{t}{2}}\sum_{i=0}^{t-1}(-1)^{t-1-i}\\ &\quad\times q^{(k-t)i+\binom{i}{2}}\begin{bmatrix} k-1-i \\ k-t \end{bmatrix}\begin{bmatrix} n-k-t+i \\ k-t \end{bmatrix}^{-1}\\ &\quad\times\begin{bmatrix} k-e \\ i \end{bmatrix}\begin{bmatrix} n-k+i-e \\ k-e \end{bmatrix}q^{(k-i)(k-e)+\binom{e}{2}}.\end{aligned} \tag{5.1}$$

First note that the expression for λ_e is an alternating sum. Our plan is to show that the terms decrease in absolute value, thus it is sufficient to show that the $i=0$ term has absolute value smaller than 1.

Then we show that the absolute value of this term strictly decreases as e increases. Therefore it is sufficient to check the case $e=t+1$ which we do by direct computation.

We use two simple inequalities:

$$\frac{a-1}{b-1}<\frac{a}{b} \qquad \text{for} \quad b>a\geqslant 1,$$

$$\frac{q^b-1}{q^a-1}<q^{b-a+1} \qquad \text{for} \quad a\geqslant 1, q\geqslant 2. \tag{5.2}$$

(a) Let us compute the absolute value of the ratio of consecutive terms (i.e., $i+1, i$) in the expression for λ_e. It is

$$q^{e-t+i}\frac{q^{t-1-i}-1}{q^{k-1-i}-1}\frac{q^{n-2k+i+1}-1}{q^{n-k-t+i+1}-1}\frac{q^{k-e-i}-1}{q^{i+1}-1}\frac{q^{n-k+i+1-e}-1}{q^{n-2k+i+1}-1}$$

$$<q^{e-t+i}q^{t-k}q^{t-e}\frac{q^{k-e-i}-1}{q^{i+1}-1}$$

$$<q^{t+i-k}q^{k-e-2i}=q^{t-e-i}\leqslant q^{-1-i}<1.$$

(b) Let us consider now the absolute value of the $i=0$ term in (5.1). It is

$$q^{-k^2+k+\binom{t}{2}}q^{k(k-e)+\binom{e}{2}}\begin{bmatrix}k-1\\k-t\end{bmatrix}\begin{bmatrix}n-k-t\\k-t\end{bmatrix}^{-1}\begin{bmatrix}n-k-e\\k-e\end{bmatrix}.$$

As $\left[\begin{smallmatrix}n-k-e\\k-e\end{smallmatrix}\right]\leqslant\left[\begin{smallmatrix}n-k-e-1\\k-e-1\end{smallmatrix}\right]$ and also $q^{k(k-e)+\binom{e}{2}}$ decreases as e increases (the derivative of the exponent is $-k+e-\frac{1}{2}<0$), the whole expression is a decreasing function of e.

(c) Finally consider the absolute value of the $i=0$ term in (5.1) when $e=t+1$. It is

$$q^{t^2-tk}\begin{bmatrix}k-1\\k-t\end{bmatrix}\begin{bmatrix}n-k-t\\k-t\end{bmatrix}^{-1}\begin{bmatrix}n-k-t-1\\k-t-1\end{bmatrix}=q^{-t(k-t)}\begin{bmatrix}k-1\\k-t\end{bmatrix}\frac{q^{k-t}-1}{q^{n-k-t}-1}.$$

Since $n\geqslant 2k$, it is sufficient to show that

$$\begin{bmatrix}k-1\\k-t\end{bmatrix}=\prod_{j=0}^{k-t-1}\frac{q^{k-1-j}-1}{q^{k-t-j}-1}<q^{t(k-t)}$$

but this is clear from (5.2). ▮

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