Weighted cross-intersecting families

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Abstract

In this paper we investigate weighted cross-intersecting families: if \( \alpha, \beta > 0 \) are given constants, we want to find the maximum of \( \alpha |A| + \beta |B| \) for \( A, B \) uniform cross-intersecting families. We determine the maximum sum, even if we have restrictions of the size of \( A \).

As corollaries, we will obtain some new bounds on the shadows and the shades of uniform families. We give direct proofs for these bounds, as well, and show that the theorems for cross-intersecting families also follow from these results.

Finally, we will generalize the LYM inequality not only for cross-intersecting families, but also for arbitrary Sperner families.

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1. Introduction

In this paper we will investigate subsets of an \( n \)-element underlying set \( X \). In place of \( X \) we will also use the short form \( [n] = \{1, 2, \ldots, n\} \). \( \binom{X}{k} \) will denote the collection of all \( k \)-element subsets of \( X \). A family \( \mathcal{A} \) is said to be \( k \)-uniform if \( \mathcal{A} \subseteq \binom{X}{k} \). The family of all subsets of \( X \) is denoted by \( 2^X \).

**Definition 1.** Let \( \mathcal{A} \subseteq \binom{X}{k} \) and \( \mathcal{B} \subseteq \binom{X}{l} \) be collections of \( k \)-subsets and \( l \)-subsets of \( X \), respectively. These two families, \( \mathcal{A}, \mathcal{B} \), are called cross-intersecting if \( A \cap B \neq \emptyset \) for all \( A \in \mathcal{A}, B \in \mathcal{B} \).

This definition is a natural generalization of intersecting families, since \( \mathcal{A} = \mathcal{F}, \mathcal{B} = \mathcal{F} \) are cross-intersecting families for any intersecting family \( \mathcal{F} \).

It is a natural goal to find the maximal cross-intersecting families, i.e. to maximize \( |\mathcal{A}| + |\mathcal{B}| \). Wu [18] gave a simple proof that this sum is maximum for \( |X| > k + l \) if either \( \mathcal{A} \) is empty and \( \mathcal{B} = \binom{X}{l} \), or \( \mathcal{A} = \binom{X}{k} \) and \( \mathcal{B} \) is empty.

Assuming \( \mathcal{A}, \mathcal{B} \) are \( k \)-uniform non-empty cross-intersecting families, Hilton and Milner [7] showed \( |\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k} + 1 \).

Frankl and Tokushige [6,16] generalized the above results in several ways, and determined the maximum sizes of cross-intersecting families if certain bounds on the size of \( \mathcal{A} \) are given.

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In [10] we have computed the maximum of $|\mathcal{A}| + |\mathcal{B}|$ if $u \leq |\mathcal{A}| \leq v$ holds for arbitrary natural numbers $u, v$. We have also determined the extremal pairs $(|\mathcal{A}|, |\mathcal{B}|)$ if $u \leq |\mathcal{A}| \leq v, u' \leq |\mathcal{B}| \leq v'$ hold.

In general, a more natural goal is to characterize the region $G^n_{k,l} = \{(x, y) \in \mathbb{R}^2_+ : x + y \leq n\}$: there is an intersecting pair $(\mathcal{A}, \mathcal{B})$ with parameters $n, k, l$ such that $|\mathcal{A}| \geq x, |\mathcal{B}| \geq y$.

To achieve this, we investigate weighted cross-intersecting families in Sections 2 and 3 of the present paper: if $x, \beta$ are given positive constants, we want to find the maximum of $x|\mathcal{A}| + \beta|\mathcal{B}|$ for $\mathcal{A}, \mathcal{B}$ uniform cross-intersecting families. Following the idea of [10], we provide results for this problem if certain constraints $u \leq |\mathcal{A}| \leq v$ are given. The considered problem clearly homogenous in the coefficients $x$ and $\beta$: it depends only on their ratio. To save notation, we will often use $\lambda = x/\beta$.

**Definition 2.** A family $\mathcal{A} \subseteq 2^X$ is said to be Sperner or an antichain if for all distinct members $B, C \in \mathcal{A}$ we have $B \nsubseteq C$.

Let us introduce the notation $\mathcal{A}^c = \{X - A : A \in \mathcal{A}\}$. Clearly, $|\mathcal{A}| = |\mathcal{A}^c|$ and $\mathcal{A} \subseteq \left(\begin{smallmatrix} X \\ n-k \end{smallmatrix}\right)$ if $\mathcal{A} \subseteq \left(\begin{smallmatrix} X \\ k \end{smallmatrix}\right)$. It is easy to see that two families $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting if and only if $B \nsubseteq A^c$ for every $B \in \mathcal{B}, A \in \mathcal{A}^c$. Thus, we obtain the following simple

**Proposition 1.** Let $\mathcal{A} \subseteq \left(\begin{smallmatrix} X \\ k \end{smallmatrix}\right)$ and $\mathcal{B} \subseteq \left(\begin{smallmatrix} X \\ l \end{smallmatrix}\right)$ be given. These two families are cross-intersecting exactly if $\mathcal{A}^c \cup \mathcal{B}$ is an antichain on levels $n - k$ and $l$.

Let $\mathcal{F} \subseteq \left(\begin{smallmatrix} [n] \\ k \end{smallmatrix}\right)$ be a family of $k$-element sets; for $l < k$, the $l$-shadow of $\mathcal{F}$ is defined as

$$\Delta_l \mathcal{F} = \{G : |G| = l, \exists F \in \mathcal{F} \text{ such that } G \subseteq F\}.$$  

Similarly, for $t > k$, the $t$-shade of $\mathcal{F}$ is

$$\nabla_t \mathcal{F} = \{H : |H| = t, \exists F \in \mathcal{F} \text{ such that } H \supseteq F\}.$$  

The next proposition establishes the connection between the shadow and the shade.

**Proposition 2.** For every $k$-uniform $\mathcal{F}, l < k$, it holds $(\Delta_l \mathcal{F})^c = \nabla_{n-l}(\mathcal{F}^c)$.

By a double counting argument, Sperner [15] has obtained lower estimations on the shadow and the shade. A repeated application of his result yields

$$\frac{|\Delta_l \mathcal{F}|}{\binom{n}{l}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}, \quad \frac{|\nabla_t \mathcal{F}|}{\binom{n}{t}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}$$

for any $\mathcal{F} \subseteq \left(\begin{smallmatrix} [n] \\ k \end{smallmatrix}\right), l < k < t$.

Given two sets $A, B \subseteq [n]$, we say that $A$ is smaller than $B$ in the squashed or colex order if the largest element of the symmetric difference of $A$ and $B$ is in $B$. Let $\mathcal{F}(k, m)$ and $\mathcal{L}(k, m)$ be the first and the last $m$ consecutive $k$-subsets of $X$ in squashed order, respectively. The following simple facts arise from the definition of the colex order.

**Proposition 3.** For $1 \leq k \leq a \leq n, 0 \leq m \leq \binom{n}{k}$ we have

$$\mathcal{F}(k, \binom{a}{k}) = \left\lfloor \frac{|a|}{k} \right\rfloor,$$

$$\mathcal{F}(k, m) = \mathcal{L}(n - k, m),$$

$$\mathcal{L}(k, \binom{n}{k} - \binom{a}{k}) = \left\lfloor \frac{|n|}{k} \right\rfloor - \left\lfloor \frac{|a|}{k} \right\rfloor.$$  

Note that (1) provides only a lower bound on the shadow and the shade. The exact value of the minimum shadow is given by Kruskal [11] and Katona [9].
Theorem 1 (Kruskal–Katona). If $\mathcal{F}$ runs over all $k$-uniform systems for which $|\mathcal{F}| = m$, then

$$\min |\Delta_{l}(\mathcal{F})| = |\Delta_{l}(\mathcal{F}(k, m))|.$$  

By Propositions 2 and 3, the shade version of the Kruskal–Katona theorem states that for $\mathcal{F} \subseteq \binom{X}{k}$, $|\mathcal{F}| = m$,

$$|\nabla_{l}\mathcal{F}| \geq |\nabla_{l}\mathcal{L}(k, m)|.$$  

It is known that for a fixed positive integer $k$, every natural number $m$ can be written in the unique form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_{k-q}}{k-q},$$  

(2)

where $a_k > a_{k-1} > \cdots > a_{k-q} \geq k - q \geq 1$ are integers. This is called the $k$-binomial representation of $m$. It is easy to see that the order of the natural numbers and the lexicographical order of the corresponding vectors $(a_k, a_{k-1}, \ldots, a_{k-q}, 0, \ldots, 0)$ are the same.

The Kruskal–Katona function, $F_{k,l}^{L}$, is the following: $F_{k,l}^{L}(0) = 0$, and if $m$ is in the form (2) then

$$F_{k,l}^{L}(m) = \binom{a_k}{l} + \binom{a_{k-1}}{l-1} + \cdots + \binom{a_{k-q}}{l-q}.$$  

We use the convention $\binom{i}{j} = 0$ if $i < j$ or $j < 0$. The numerical version of the Kruskal–Katona theorem is then the following.

Theorem 2 (Numerical Kruskal–Katona). If $\mathcal{F}$ runs over all $k$-uniform systems for which $|\mathcal{F}| = m$, then

$$\min |\Delta_{l}(\mathcal{F})| = F_{k,l}^{L}(m).$$  

Although the Kruskal–Katona theorem provides the best possible lower bound for shadows, it is sometimes inconvenient in an analytical sense, cf. [5]. In some cases, (1) or the Lovász bound [12] that uses the real $k$-binomial representation are more useful for computations.

It is worth mentioning that Katona [8] has verified that if $\mathcal{F}$ is a $k$-uniform $r$-intersecting family and $1 \leq k - l \leq r$, then

$$\frac{|\Delta_{l}\mathcal{F}|}{\binom{2k-r}{l}} \geq \frac{|\mathcal{F}|}{\binom{2k-r}{k}}.$$  

($\mathcal{F}$ is $r$-intersecting if for all $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \geq r$ holds.)

In Section 4, we give further lower bounds for the shadows and the shades of $k$-uniform families if we have restrictions on their sizes.

For $0 \leq i \leq n$ we denote by $\mathcal{A}_{i}$ the collection of $i$-element sets in $\mathcal{A}$, i.e. $\mathcal{A}_{i} = \{A \in \mathcal{A} : |A| = i\}$. The following LYM inequality, due to Bollobás [3], Lubell [13], Meshalkin [14] and Yamamoto [17], is maybe one of the most known results for Sperner families. If $\mathcal{A}$ is an antichain, then

$$\sum_{i=0}^{n} \frac{|\mathcal{A}_{i}|}{\binom{n}{i}} \leq 1.$$  

(3)

By Proposition 1, the LYM inequality turns to the following form for cross-intersecting families:

$$\frac{|\mathcal{A}|}{\binom{n}{k}} + \frac{|\mathcal{B}|}{\binom{n}{l}} \leq 1.$$  

(4)

The LYM inequality is generalized in many ways. For example, P.L. Erdős et al. [4] sharpened it by raising the coefficients $1/\binom{n}{i}$ depending on the profile vector of $\mathcal{A}$. In their paper, they proved a sequence of inequalities each of which strengthens the LYM inequality.
Ahlsweede and Zhang [1] brought up the LYM inequality to an identity for arbitrary set system by involving certain intersections of its elements.

Recently, Bey [2] gave a polynomial LYM inequality for antichains by adding to the original LYM inequality all possible products of the fractions \(|\mathcal{A}|/\binom{n}{i}\) with suitable coefficients.

In the last section of this paper, we present a generalization of the LYM inequality in another direction.

2. The local minimum method

In [10] we introduced this method in order to solve the problem of maximum-sized cross-intersecting families in the non-weighted case. In this section, we generalize that method to a certain extent.

Let us take cross-intersecting families \(\mathcal{A} \subseteq \binom{X}{k}, \mathcal{B} \subseteq \binom{X}{l}\). By Proposition 1, we know that \(\mathcal{A}^c \cup \mathcal{B}\) is an antichain. Suppose that \(|X| = n > k + l\), this is equivalent to \(A(I,\mathcal{A}^c) \cap \mathcal{B} = \emptyset\).

Since here \(A(I,\mathcal{A}^c)\) and \(\mathcal{B}\) are disjoint families of \(l\)-element subsets of \(X\), we have

\[
|A(I,\mathcal{A}^c)| + |\mathcal{B}| \leq \binom{n}{l}.
\]

Consequently, the numerical version of the Kruskal–Katona theorem says that if \(a, \beta > 0\) are given constants, then

\[
\max_{|\mathcal{A}|=m} (\alpha|\mathcal{A}| + \beta|\mathcal{B}|) = \beta \binom{n}{l} - \beta F_{l-k}^n(m) + \alpha m.
\] (5)

So our aim is to minimize the function \(\beta F_{l-k}^n(m) - \alpha m\), or, equivalently, \(F_{l-k}^n(m) - \lambda m\) with \(\lambda = \alpha/\beta\), in a certain interval \([u, v]\).

Definition 3. Suppose that a function \(f : \mathbb{N} \to \mathbb{N}\) is given, and take a set \(W = \{w_1, w_2, \ldots\} \subseteq \mathbb{N}\), where \(w_1 < w_2 < \cdots\). We call a \(w_j\) \((j \geq 2)\) a local minimum place of \(f\) along \(W\) if \(f(w_j) \geq f(w_{j-1})\) and \(f(w_j) \leq f(w_{j+1})\). We denote the set of local minimum places by \(L_f(W)\).

The local minimum method is based on the following argument. We are looking for the minimum of the function \(f\) in the interval \([u, v]\), so let \(W_0 = \mathbb{N}\), and let us take a sequence of sets \(W_i \subseteq \mathbb{N}\) \((i = 1, 2, \ldots)\), such that \(L_f(W_i) \subseteq W_{i+1}\) for every \(i \geq 0\). If \(f\) has a minimum in \(w \in [u, v]\), then \(w\) is either an endpoint of the interval, i.e., \(w = u\) or \(w = v\), or \(w\) is a local minimum place, thus \(w \in W_1\). But the restriction of \(f\) on \(W_1\) is still minimal in \(w\), hence \(w\) is either an endpoint of \(W_1 \cap [u, v]\), or it is a local minimum place of \(f\) on this set, so \(w \in W_2\). Continuing this way, we obtain that

\[
w \in W^* = \{u_0, u_1, \ldots, u_z, v_z, \ldots, v_1, v_0\},
\] (6)

where

\[
u_i = \min W_i \cap [u, v], \quad \nu_i = \max W_i \cap [u, v]
\] (7)

and \(W_{z+1} \cap [u, v]\) is the first set which is empty (we will see that such a \(z\) exists in our case). In other words, it is necessary only to minimize \(f\) on \(W^*\) instead of the whole interval \([u, v]\). Of course, we will apply this method to the function \(F_{l-k}^n(m) - \lambda m\).

For fixed \(h\), let us introduce the following notation:

\[
W_{p,r} = \left\{ a_h^\frac{h}{h} + a_{h-1}^\frac{h-1}{h-1} + \cdots + a_{h-q}^\frac{h-q}{h-q} : a_h > \cdots > a_{h-q} \geq h - q + r, \ h - q \geq p \right\}.
\]

The next two lemmas are devoted to the local minimums of \(F_{l-k}^n(m) - \lambda m\); we will apply them for \(h = n - k\) in later sections.

Lemma 1. Let \(h, s, p, r\) be fixed positive integers with \(h, p > s\) and \(f(m) = F_{h-s}^h(m) - \lambda m \ (m \in \mathbb{N})\), where \(\lambda > 0\). Then for the local minimums we have

\[
L_f(W_{p,r}) \subseteq W_{p+1,r+1}.
\]
Proof. Let $m = \left( \frac{a_0}{h} \right) + \cdots + \left( \frac{a_{h-q}}{h-q} \right) \in W_{p,r}$, and denote by $m_1$ the biggest element of the set \( \{ m' \in W_{p,r} : m' < m \} \) and by $m_2$ the smallest element of \( \{ m'' \in W_{p,r} : m'' > m \} \). Suppose that $m \notin W_{p+1,r+1}$. To save notation, let $t = h - q$.

**Case (i):** Assume first that $t = p$, $a_t = t + r$. Then $m_1 = \left( \frac{a_0}{h} \right) + \cdots + \left( \frac{a_{t+1}}{t+1} \right)$ and $m_2 = \left( \frac{a_0}{h} \right) + \cdots + \left( \frac{a_{t+y}}{y} \right)$, where $y$ is the largest integer for which $a_y = a_t + y - t$ holds. It is easy to see that $F^{h}_{h-s}(m_1) = F^{h}_{h-s}(m) - \left( \frac{a_t}{t-s} \right)$, so $f(m_1) = f(m) + \lambda \left( \frac{a_t}{t-s} \right)$, thus, $f(m) \leq f(m_1)$ exactly if
\[
\frac{a_t}{t-s} \leq \lambda. \tag{8}
\]
Since $m_2 = m + \left( \frac{a_t}{t-1} \right)$, it is easy to check that $F^{h}_{h-s}(m_2) = F^{h}_{h-s}(m) + \left( \frac{a_t}{t-1-s} \right)$, so $f(m_2) = f(m) - \lambda \left( \frac{a_t}{t-1} \right) + \left( \frac{a_t}{t-1-s} \right)$. Thus, $f(m) \leq f(m_2)$ exactly if
\[
\lambda \leq \frac{a_t}{t-1-s} \tag{9}
\]
If $m$ would be a local minimum place, then
\[
\frac{a_t}{t-s} \leq \frac{a_t}{t-1-s} \leq \frac{a_t}{t-1} \leq \frac{a_t}{t-1-s}
\]
would follow from (8) and (9); equivalently,
\[
\frac{a_t - t + 1 + s}{t-s} \leq \frac{a_t - t + 1}{t}
\]
would hold, which is not true since $s > 0$. Thus, $m$ is not a local minimum place.

**Case (ii):** Let now $t = p$, $a_t > t + r + 1$. Here $m_1 = \left( \frac{a_0}{h} \right) + \cdots + \left( \frac{a_{t-1}}{t} \right)$ and $m_2$ is the same as defined in (i). We have $m_1 = m - \left( \frac{a_t-1}{t-1} \right)$ and $F^{h}_{h-s}(m_1) = F^{h}_{h-s}(m) - \left( \frac{a_t-1}{t-1-s} \right)$, so $f(m_1) = f(m) + \lambda \left( \frac{a_t-1}{t-1} \right) - \left( \frac{a_t-1}{t-1-s} \right)$. Consequently, $f(m) \leq f(m_1)$ if and only if
\[
\frac{a_t-1}{t-1-s} \leq \frac{a_t-1}{t-1} \leq \frac{a_t-1}{t-1-s}
\]
As in case (i), we know that $f(m) \leq f(m_2)$ exactly if (9) holds. So, if $m$ is a local minimum place then
\[
\frac{a_t-1}{t-1-s} \leq \frac{a_t-1}{t-1} \leq \frac{a_t-1}{t-1-s},
\]
which is equivalent to
\[
\frac{a_t - t + 1 + s}{a_t} \leq \frac{a_t - t + 1}{a_t},
\]
a contradiction to $s > 0$.

**Case (iii):** Finally, let $t \geq p + 1$, $a_t = t + r$. Now $m_1 = \left( \frac{a_0}{h} \right) + \cdots + \left( \frac{a_{t+1}}{t+1} \right)$ and $m_2 = \left( \frac{a_0}{h} \right) + \cdots + \left( \frac{a_{t-1}}{t-1} \right)$. As in case (i), $f(m) \leq f(m_1)$ iff (8) holds. Since $m_2 = m + \left( \frac{a_t-1}{t-1} \right)$ and $F^{h}_{h-s}(m) = F^{h}_{h-s}(m) + \left( \frac{a_t-1}{t-1-s} \right)$,
it holds \( f(m_2) = f(m) - \lambda \left( \frac{a_{r-1}}{t-1-s} \right) \). Thus, \( f(m) \leq f(m_2) \) holds exactly if

\[
\lambda \leq \frac{a_{r-1}}{t-1-s}.
\]

We get that if \( m \) would be a local minimum place then

\[
\frac{a_r}{t-s} \leq \frac{a_t}{t},
\]

or, equivalently,

\[
\frac{a_r}{t-s} \leq \frac{a_t}{t},
\]

which is not true as \( s > 0 \). So we obtain that if \( t = p \) or \( a_t = t + r \) then \( m \) is not a local minimum place, and we are done.

\[\square\]

**Lemma 2.** Let \( h > s \) be fixed positive integers, \( f(m) = F_{h-s}^h(m) - \lambda m \) (\( m \in \mathbb{N} \)), where \( \lambda > 0 \) fixed. Let \( W_0 = \mathbb{N} \) and

\[
W_i = \left\{ \left( \frac{a_h}{h} \right) + \left( \frac{a_{h-1}}{h-1} \right) + \cdots + \left( \frac{a_{h-q}}{h-q} \right) : a_h > \cdots > a_{h-q} \geq h - q + i, \ h - q \geq s + i \right\}.
\]

Then for the local minimum places of \( f \) along \( W_i \) we have

\[
L_f(W_i) \subseteq W_{i+1}.
\]

**Proof.** For \( i \geq 1 \), Lemma 1 shows the statement with \( r = i, \ p = s + i \), so we only have to prove it when \( i = 0 \). Put \( t = h - q \), again, and let \( m = \left( \frac{a_h}{h} \right) + \left( \frac{a_{h-1}}{h-1} \right) + \cdots + \left( \frac{a_y}{y} \right) \notin W_i \). We distinguish three cases.

Case (i): If \( t = 1 \) then \( m+1 = \left( \frac{a_h}{h} \right) + \cdots + \left( \frac{a_y+1}{y} \right) \), where \( y \geq t \) is the largest integer for which \( a_y = a_t + y - t \) holds. So \( F_{h-s}^h(m) = F_{h-s}^h(m+1) \) and \( f(m) = f(m+1) + \lambda \), thus \( m \notin L_f(W_0) \).

Case (ii): If \( 1 < t \leq s \) then \( m+1 = \left( \frac{a_h}{h} \right) + \cdots + \left( \frac{t-1}{t-1} \right) \), so again \( F_{h-s}^h(m) = F_{h-s}^h(m+1) \) and \( f(m) = f(m+1) + \lambda \), thus \( m \notin L_f(W_0) \).

Case (iii): Let \( t \geq s + 1, \ a_t = t \). Here \( m-1 = \left( \frac{a_h}{h} \right) + \cdots + \left( \frac{a_t+1}{t+1} \right) \) and, as we have seen earlier, \( f(m-1) = f(m) + \lambda - \left( \frac{t-1}{t-s} \right) \). Thus, \( f(m) \leq f(m-1) \) if

\[
\frac{t}{t-s} \leq \lambda.
\]

Since \( m+1 = \left( \frac{a_h}{h} \right) + \cdots + \left( \frac{a_t}{t} \right) + \left( \frac{t-1}{t-1} \right) \), we have, as previously, \( f(m+1) = f(m) - \lambda + \left( \frac{t-1}{t-1-s} \right) \). Consequently, \( f(m) \leq f(m+1) \) holds exactly if

\[
\lambda \leq \frac{t-1}{t-1-s}.
\]

But \( \left( \frac{t-1}{t-1-s} \right) < \left( \frac{t}{t-s} \right) \) since \( \left( \frac{t-1}{t-s} \right) > 0 \), thus \( m \notin L_f(W_0) \). \[\square\]
3. Weighted results

Throughout the section we will assume that \( n > k + l \), since the case \( n \leq k + l \) is trivial.

We first state a theorem that can be regarded as a generalization of (4), the LYM inequality for cross-intersecting families.

**Theorem 3.** Suppose that \( \mathcal{A} \subseteq \binom{X}{k} \), \( \mathcal{B} \subseteq \binom{X}{l} \) are cross-intersecting families. Let \( a = 0 \) and \( n - k \leq b \leq n \), or \( n - k \leq a < b \leq n \) be integers. Then \( x|\mathcal{A}| + y|\mathcal{B}| \) is maximal for

\[
\binom{a}{n-k} \leq |\mathcal{A}| \leq \binom{b}{n-k}
\]

if \( |\mathcal{A}| = \binom{a}{n-k} \) or \( |\mathcal{A}| = \binom{b}{n-k} \).

**Proof.** We will give a proof similar to those in [10].

Remember that we gave a numerical version of the problem by (5). Applying Lemma 2 to the function \( f(m) = F^{n-k}_{l}(m) - \lambda m \) (i.e. \( h = n - k \), \( s = n - k - l \)), we obtain that the minimum places of \( f \) are in the set \( W^* \) defined by (6) and (7), where \( u = \binom{a}{n-k} \), \( v = \binom{b}{n-k} \) and

\[
W_i = \left\{ \binom{a_{n-k}}{n-k} + \cdots + \binom{a_{n-k-q}}{n-k-q} : a_{n-k} > \cdots > a_{n-k-q} \geq n - k - q + i, \ l \geq q + i \right\}
\]

for \( i \geq 1 \). Note that \( W_i = \emptyset \) for \( i > l \), so there exists an integer \( z \leq l \), such that \( W_{z+1} \cap \left[ \binom{a}{n-k}, \binom{b}{n-k} \right] \) is empty.

In case \( a = 0 \): \( u_0 = 0 \) by definition, and \( u_i = \binom{n-k+i}{n-k} \) for \( 1 \leq i \leq z \), because \( \binom{n-k+i}{n-k} \in W_i \) is obvious, and if \( a' = \binom{a_{n-k}}{n-k} + \cdots + \binom{a_{n-k-p}}{n-k-p} \in W_i \) then \( a'_{n-k} - (n - k) \geq a'_{n-k-p} - (n - k - p) \geq i \), hence \( a'_{n-k} \geq n - k + i \) and \( a' \geq \binom{n-k+i}{n-k} \).

In case \( a \geq n - k \): if \( n - k + i > a \) then \( u_i = \binom{n-k+i}{n-k} \) for the same reason, while if \( a \geq n - k + i \) then clearly \( \binom{a}{n-k} \in W_i \), thus \( u_i = \binom{a}{n-k} \).

It is also easy to check that \( v_0 = \cdots = v_z = \binom{b}{n-k} \); let \( v_i = \binom{b_{n-k}}{n-k} + \cdots + \binom{b_{n-k-q}}{n-k-q} \), then \( b - (n - k) \geq b_{n-k} - (n - k) \geq b_{n-k-q} - (n - k - q) \geq i \), so \( \binom{b}{n-k} \in W_i \).

In conclusion, we only have to find the minimum of the sequence

\[
g(j) = f \left( \binom{j}{n-k} \right) = \binom{j}{l-1} - \lambda \binom{j}{n-k},
\]

where \( j \) is integer, \( a \leq j \leq b \) for \( a \geq n - k \), and \( j = 0 \), \( n - k \leq j \leq b \) for \( a = 0 \).

We will verify that this sequence is unimodal, i.e. there is an \( x_j \), such that \( g(j) < g(j+1) \) for \( j < x_j \), \( g(j) = g(j+1) \) for \( j = x_j \) and \( g(j) > g(j+1) \) for \( j > x_j \); consequently, it has a minimum exactly if \( j = a \) or \( j = b \). First, let \( a \geq n - k \). It is easy to see that \( g(j) \leq g(j+1) \) is equivalent to

\[
\lambda \binom{j}{n-k-1} \leq \binom{j}{l-1},
\]

that is,

\[
\frac{(j-l+1)(j-l) \cdots (j-n+k+2)}{l(l+1) \cdots (n-k-1)} \leq 1/\lambda.
\]
Clearly, the polynomial
\[ p(x) = \frac{(x - l + 1)(x - l) \cdots (x - n + k + 2)}{l(l + 1) \cdots (n - k - 1)} \]
is positive and strictly increasing for \( x > n - k - 2 \), so there exists a unique \( x_2 \) for which \( p(x_2) = 1/\lambda \), and we are done.

Similar argument shows the case \( a = 0 \). \( \square \)

It is easy to see that Theorem 3 is equivalent to the following generalization of the LYM inequality for cross-intersecting families. Clearly, \( a = 0 \) and \( b = n \) give (4).

**Theorem 4.** Suppose that \( \mathcal{A} \subseteq \binom{X}{k} \), \( \mathcal{B} \subseteq \binom{X}{l} \) are cross-intersecting families. Let \( \binom{a}{n-k} \leq |\mathcal{A}| \leq \binom{b}{n-k} \), where \( a, b, n \) are integers. Then
\[
\frac{|\mathcal{A}| - \binom{a}{n-k}}{\binom{b}{n-k} - \binom{a}{n-k}} + \frac{|\mathcal{B}| - \binom{b}{n-k}}{\binom{b}{n-k} - \binom{a}{n-k}} \leq 1. \tag{12}
\]

Theorem 4 yields that \((|\mathcal{A}|, |\mathcal{B}|) \in \mathbb{Z}^2\) lies under the line connecting \( \left( \binom{a}{n-k}, \binom{b}{n-k} \right) \) and \( \left( \binom{b+1}{n-k}, \binom{b+1}{n-k} \right) \) for every cross-intersecting pair, \( \mathcal{A}, \mathcal{B} \), satisfying the conditions of Theorem 3. Thus, Theorem 3 follows.

**Remark 1.** By the property of \( g(j) \) described in the proof of Theorem 3, the inequality is strict in (12) except \( |\mathcal{A}| = \binom{a}{n-k} \) or \( \binom{b}{n-k} \).

The next theorem is a slight generalization of Theorem 3.

**Theorem 5.** Suppose that \( \mathcal{A} \subseteq \binom{X}{k} \), \( \mathcal{B} \subseteq \binom{X}{l} \) are cross-intersecting families, and let
\[
\binom{a}{n-k} \leq |\mathcal{A}| \leq \binom{b}{n-k} = \binom{b}{n-k} + \binom{b-1}{n-k-1} + \cdots + \binom{b-q}{n-k-q},
\]
where \( a, b, n \) are integers. Then the maximum of \( |\mathcal{A}| + \beta |\mathcal{B}| \) is achieved in one of the cases \( |\mathcal{A}| = \binom{a}{n-k} \) or \( |\mathcal{A}| = \binom{b}{n-k} \).

**Proof.** By Theorem 3, we know that if \( \binom{a}{n-k} \leq |\mathcal{A}| \leq \binom{b}{n-k} \) then the maximum is obtained for either \( |\mathcal{A}| = \binom{a}{n-k} \) or \( |\mathcal{A}| = \binom{b}{n-k} \).

Introduce now the notation
\[ m_i = \binom{b}{n-k} + \cdots + \binom{b-i+1}{n-k-i+1} \]
for \( i = 1, \ldots, q + 1 \). By a similar argument to those in Theorem 3, one can see that in order to maximize \( |\mathcal{A}| + \beta |\mathcal{B}| \) for \( \binom{a}{n-k} \leq |\mathcal{A}| \leq \binom{b}{n-k} \), we have to find the minimum of \( f(m) = f^n_k(m) - \lambda m \) on the set \( \{m_i : 1 \leq i \leq q+1\} \) (\( \lambda = \alpha/\beta \)).

We show that the sequence \( f(m_i) \) is unimodal. Let
\[
p_i(x) = \frac{(x - l)(x - l - 1) \cdots (x - n + k + 1)}{(l - i + 1)(l - i + 2) \cdots (n - k - i)}
\]
for \( i = 1, \ldots, l \). It is easy to see that \( f(m_i) \geq f(m_{i+1}) \) iff
\[
\frac{b-i}{l-i} \leq \frac{b-i}{n-k-i}.
\]
This is true if \( i > l \), while for \( i \leq l \) it is equivalent to \( p_i(b) \geq 1/\lambda \). But \( p_{i-1}(b) < p_i(b) \) holds for \( i > 1 \), so \( f(m_{i-1}) \geq f(m_1) \) implies \( f(m_i) \geq f(m_{i+1}) \). Thus, we have maximum if either \(|\mathcal{A}| = \binom{b-k}{n-k} \) or \(|\mathcal{A}| = b^* \).

It remains to show that \( g(a) > g(b) \) and \( f(m_1) < f(b^*) \) never hold simultaneously (\( g(a) \) and \( g(b) \) are defined by (11)). We have seen that the sequence \( g(j) \) is unimodal; thus, \( g(a) > g(b) \) implies \( g(b - 1) > g(b) \). This means

\[
\frac{b - 1}{l - 1} < \lambda \frac{b - 1}{n - k - 1},
\]

that is \( f(m_1) > f(m_2) \), and so \( f(m_1) > f(m_{q+1}) = f(b^*) \). \( \square \)

A simple special case of Theorem 5 is the following: if \( \mathcal{A} \) and \( \mathcal{B} \) are non-empty families, then we have

\[
1 \leq |\mathcal{A}| \leq \binom{n}{k} - \binom{n - l}{k},
\]

which is the assumption of Theorem 5 with \( a = n - k, b = n - 1 \) and \( q = l - 1 \). We obtain:

**Corollary 1.** Assume that \( \mathcal{A} \subseteq \binom{X}{k}, \mathcal{B} \subseteq \binom{X}{l} \) are non-empty cross-intersecting families. Then

\[
x|\mathcal{A}| + \beta|\mathcal{B}| \leq \max \left\{ x + \beta \binom{n}{l} - \beta \binom{n - k}{l}, x \binom{n}{k} - x \binom{n - l}{k} + \beta \right\}.
\]

Our last theorem of this type is based on Theorem 5.

**Theorem 6.** Suppose that \( \mathcal{A} \subseteq \binom{X}{k}, \mathcal{B} \subseteq \binom{X}{l} \) are cross-intersecting families. Let \( n - k \leq a \leq b < n \) be integers. Then the maximum of \( x|\mathcal{A}| + \beta|\mathcal{B}| \) for

\[
\left( \frac{a}{n-k} \right) + \cdots + \left( \frac{a - q}{n-k - q} \right) = a^* \leq |\mathcal{A}| \leq b^* = \left( \frac{b}{n-k} \right) + \cdots + \left( \frac{b - q'}{n-k - q'} \right)
\]

is reached in one of the cases \( |\mathcal{A}| = a^* \) or \( |\mathcal{A}| = \binom{a + 1}{n-k} \) or \( |\mathcal{A}| = b^* \).

**Proof.** For \( a = b \), this is Theorem 5 with \( a = 0 \) there. If \( a < b \), let us observe first the interval \( a^* \leq |\mathcal{A}| \leq \binom{a+1}{n-k} \). Using the local minimum method, one can check that for the set \( W^* \), defined by (6), (7) and Lemma 2, we have \( W^* = \left\{ a^*, \binom{a+1}{n-k} \right\} \). Hence, \( x|\mathcal{A}| + \beta|\mathcal{B}| \) is maximal here if \( |\mathcal{A}| = a^* \) or \( |\mathcal{A}| = \binom{a+1}{n-k} \).

Application of Theorem 5 to \( \binom{a+1}{n-k} \leq |\mathcal{A}| \leq b^* \) completes the proof. \( \square \)

Finally, we mention that the above described method may yield further results in other particular cases, but we do not discuss them here.

4. New bounds for shadows and shades

In this section lower bounds for the sizes of the shadow and the shade of set families are given. We give two proofs for the next theorem: the first one uses the results of Section 3, while the second one is independent from them. We will see that Theorem 4 and so Theorem 3 follow from this theorem; thus, we will get a second proof for those results, too.

**Theorem 7.** Let \( \mathcal{F} \) be a \( k \)-uniform system of sets. Assume that \( \binom{a}{k} \leq |\mathcal{F}| \leq \binom{b}{k} \), where \( a = 0, k \leq b \leq n \) or \( k \leq a < b \leq n \) hold. Then

\[
\frac{|\Delta_i\mathcal{F}| - \binom{a}{l}}{|\mathcal{F}| - \binom{a}{k}} \geq \frac{|\mathcal{F}| - \binom{a}{k}}{|\mathcal{F}| - \binom{a}{k}}.
\]  

(13)
that is, an initial segment of the 2nd proof.

We know that \( \beta = \binom{b}{k} - \binom{a}{l} \) and \( \binom{a}{l} \leq |\mathcal{F}| \leq \binom{b}{k} \). By (5), this is nothing else but Theorem 3. This yields that the minimum shadow is reached if \( |\mathcal{F}| = \binom{a}{l} \) or \( |\mathcal{F}| = \binom{b}{k} \). Both cases give

\[
\beta |\Delta_1 \mathcal{F}| - \beta |\mathcal{F}| \geq \binom{a}{l} \binom{b}{k} - \binom{a}{l} \binom{b}{k},
\]

which is equivalent to (13). \( \square \)

2nd proof. We know that \( |\Delta_1 \mathcal{F}| \geq |\Delta_1 \mathcal{F}(k, |\mathcal{F}|)| \) by the Kruskal–Katona theorem, hence we will prove (13) if \( \mathcal{F} \) is an initial segment of \( \binom{x}{k} \) in the colex order. Let \( a \leq c \leq b \) be an integer such that \( \binom{c}{k} \leq |\mathcal{F}| \leq \binom{c+1}{k} \). By Proposition 3, we have \( \binom{|c|}{k} \subseteq \mathcal{F} \subseteq \binom{|c|+1}{k} \). Let us count the pairs \((F, G)\) where \( c+1 \in F \in \mathcal{F}, c+1 \in G \in \Delta_1 \mathcal{F} \) and \( G \subseteq F \) in two ways. We obtain:

\[
\left( |\Delta_1 \mathcal{F}| - \binom{c}{l} \right) \binom{c-l+1}{k-l} \geq \left( |\mathcal{F}| - \binom{c}{k} \right) \binom{k-l-1}{l-1};
\]

that is,

\[
\frac{|\Delta_1 \mathcal{F}| - \binom{c}{l}}{\binom{c}{l-1}} \geq \frac{|\mathcal{F}| - \binom{c}{k}}{\binom{c}{k-1}},
\]

which is (13) with \( c, c+1 \) stand in place of \( a, b \) respectively. We prove that if (13) holds for \( c, d \) (in place of \( a, b \) resp.), then it also holds for \( c, d+1 \). This immediately follows from the next technical lemma.

Lemma 3. If \( k \leq c < d \) then

\[
\frac{\binom{d+1}{l}}{\binom{d+1}{k}} - \frac{\binom{c}{l}}{\binom{c}{k}} \leq \frac{\binom{d}{l}}{\binom{d}{k}} - \frac{\binom{c}{l}}{\binom{c}{k}}.
\] (14)

Proof. It is easy to see that (14) is equivalent to

\[
\frac{\binom{d}{l-1}}{\binom{d}{k-1}} \leq \frac{\binom{d-1}{l-1}}{\binom{d-1}{k-1}} = \frac{\binom{d-1}{l-1} + \cdots + \binom{c}{l-1}}{\binom{d-1}{k-1} + \cdots + \binom{c}{k-1}}.
\] (15)

Since \( l \leq k \leq c \), we have

\[
\frac{\binom{d}{l-1}}{\binom{d}{k-1}} \leq \frac{\binom{d-1}{l-1}}{\binom{d-1}{k-1}} \leq \cdots \leq \frac{\binom{c}{l-1}}{\binom{c}{k-1}},
\]

thus, (15) and so (14) hold. \( \square \)

Consequently, (13) is true for \( c, b \). We now verify that if (13) holds for \( d, b \), then it also holds for \( d-1, b \). This is done by the next lemma.

Lemma 4. If \( k \leq d < b \) and \( \binom{d}{k} \leq |\mathcal{F}| \leq \binom{b}{k} \), then

\[
\frac{\binom{b}{l}}{\binom{b}{k}} - \frac{\binom{d}{l}}{\binom{d}{k}} \left( |\mathcal{F}| - \binom{d}{k} \right) + \frac{\binom{d}{l}}{\binom{d}{k}} \leq \frac{\binom{b}{l}}{\binom{b}{k}} - \frac{\binom{d}{l}}{\binom{d}{k}} \left( |\mathcal{F}| - \binom{d-1}{k} \right) + \left( \binom{d-1}{l} \right).
\]
Proof. This holds with equality for \(|\mathcal{F}| = \binom{b}{k}\), so it is necessary only to show that
\[
\frac{\binom{b}{l} - \binom{d}{l}}{\binom{b}{k} - \binom{d}{k}} \leq \frac{\binom{b}{l} - \binom{d-1}{l}}{\binom{b}{k} - \binom{d-1}{k}}.
\]
As in the proof of the previous lemma, this is equivalent to
\[
\frac{\binom{d-1}{l-1}}{\binom{d-1}{k-1}} \geq \frac{\binom{b}{l} - \binom{d}{l}}{\binom{b}{k} - \binom{d}{k}} \geq \cdots \geq \frac{\binom{b}{l} - \binom{d}{l}}{\binom{b}{k} - \binom{d}{k}}.
\]
Again, since \(l \leq k \leq d\), we have
\[
\frac{\binom{d-1}{l-1}}{\binom{d-1}{k-1}} \geq \frac{\binom{d}{l}}{\binom{d}{k}} \geq \cdots \geq \frac{\binom{d-1}{l-1}}{\binom{d-1}{k-1}},
\]
and we have proved the lemma. □

The last lemma deals with the case \(a = 0\). We prove that if (13) holds for \(d, b\), then it also holds for \(0, b\).

Lemma 5. If \(k \leq d < b\) and \(\binom{d}{k} \leq |\mathcal{F}| \leq \binom{b}{k}\), then
\[
\frac{\binom{b}{l} - \binom{d}{l}}{\binom{b}{k} - \binom{d}{k}} \left( |\mathcal{F}| - \binom{d}{k} \right) + \binom{d}{l} \geq \binom{b}{l} |\mathcal{F}|.
\]
Proof. Again, this holds with equality for \(|\mathcal{F}| = \binom{b}{k}\), and it is easy to see that
\[
\frac{\binom{b}{l} - \binom{d}{l}}{\binom{b}{k} - \binom{d}{k}},
\]
which shows the statement. □

The above three technical lemmas assure that (13) holds for the given \(a, b\), so we are done with the proof of Theorem 7. □

If \(\mathcal{A} \subseteq \binom{X}{k}\), \(\mathcal{B} \subseteq \binom{X}{l}\) are cross-intersecting families, then we know that \(|\mathcal{A} \cup \mathcal{B}| \leq \binom{n}{k}\), and an easy calculation shows that (12) follows from (13) with \(\mathcal{F} = \mathcal{A}^c\) and \(n - k\) instead of \(k\). In this way, we have obtained a second proof also for Theorem 3.

By Proposition 2, we can reformulate Theorem 7 for shades:

Corollary 2. If \(\mathcal{F} \subseteq \binom{X}{k}\) is a collection of sets, such that \(\binom{a}{n-k} \leq |\mathcal{F}| \leq \binom{b}{n-k}\), where \(a = 0, n-k \leq b \leq n\) or \(n-k \leq a < b \leq n\), then
\[
\frac{|\nabla, \mathcal{F}| - \binom{a}{n-t}}{\binom{b}{n-t} - \binom{a}{n-t}} \geq \frac{|\mathcal{F}| - \binom{a}{n-k}}{\binom{b}{n-k} - \binom{a}{n-k}}.
\]

The next theorem is a shade-type version of Theorem 6. We give only the sketch of the two possible proofs.
Theorem 8. Let $\mathcal{F}$ be a collection of $k$-element sets, such that \( \binom{n}{k} - \binom{b}{k} \leq |\mathcal{F}| \leq \binom{n}{k} - \binom{a}{k} \) with $k \leq a < b$, $t \leq b \leq n$. Then for the $t$-shade of $\mathcal{F}$ we have

\[
\left| \nabla_t \mathcal{F} \right| - \binom{n}{b} + \binom{b}{t} \geq \frac{|\mathcal{F}| - \binom{n}{k} + \binom{b}{k}}{(b) - \binom{n}{k}}.
\]

1st proof. As earlier, the statement can be rewritten as follows: find the minimum of $\beta|\nabla_t \mathcal{F}| - \alpha|\mathcal{F}|$, or, by Proposition 2, the minimum of $\beta|A_{n-t} \mathcal{F}^c| - \alpha|\mathcal{F}^c|$ where $\alpha = \binom{b}{t} - \binom{n}{k}$, $\beta = \binom{b}{k} - \binom{a}{k}$ and

\[
\binom{n-1}{n-k} + \ldots + \binom{b}{n-k+1} \leq |\mathcal{F}| \leq \binom{n-1}{n-k} + \ldots + \binom{a}{a-k+1}
\]

if $b < n$. Basically, this is Theorem 6. It states that we have minimum if $|\mathcal{F}| = \binom{n}{k} - \binom{a}{k}$ or $|\mathcal{F}| = \binom{n}{k} - \binom{b}{k}$. Both cases give (16) by the shade version of the numerical Kruskal–Katona theorem. For $b = n$, Theorem 5 applies. \( \square \)

2nd proof. The shade version of the Kruskal–Katona theorem implies that $|\nabla_t \mathcal{F}| \geq |\nabla_t \mathcal{L}(k, |\mathcal{F}|)|$, hence we assume that $\mathcal{F}$ is a terminal segment of $\binom{X}{k}$ in the colex order. Let $a \leq c \leq b$ be an integer such that $\binom{n}{c} - \binom{c+1}{k} \leq |\mathcal{F}| \leq \binom{n}{k} - \binom{c}{k}$. By Proposition 3, we have $\binom{n}{k} \setminus \binom{c+1}{k} \subseteq \mathcal{F} \subseteq \binom{n}{k} \setminus \binom{c}{k}$. We count the pairs $(F, G)$ where $c+1 \in F \in \mathcal{F}$, $c + 2, \ldots, n \notin F$, $c + 1 \in G \in \nabla_t \mathcal{F}$, $c + 2, \ldots, n \notin G$ and $G \supseteq F$ in two ways. We obtain

\[
\left( |\mathcal{F}| - \left[ \binom{n}{k} - \binom{c+1}{k} \right] \right) \binom{c-k+1}{t-k} \leq \left( |\nabla_t \mathcal{F}| - \left[ \binom{n}{t} - \binom{c+1}{t} \right] \right) \binom{t-1}{k-1},
\]

that is,

\[
\frac{|\nabla_t \mathcal{F}| - \binom{n}{c+1} + \binom{c+1}{t}}{(c+1)_{t-1}} \geq \frac{|\mathcal{F}| - \binom{n}{k} + \binom{c+1}{k}}{(c)_{k-1}},
\]

which is (16) if $c, c+1$ stand instead of $a, b$ respectively.

Similar computations to those in Lemmas 3–5 finish the proof. \( \square \)

Finally, we have a corollary of Theorem 8 for shadows.

Corollary 3. Let $\mathcal{F}$ be a $k$-uniform family of sets for which $\binom{n}{n-k} - \binom{b}{n-k} \leq |\mathcal{F}| \leq \binom{n}{n-k} - \binom{a}{n-k}$ with $n-k \leq a < b$ and $n-l \leq b \leq n$. Then

\[
\frac{|\mathcal{A}_l \mathcal{F}| - \binom{n}{n-l} + \binom{b}{n-l}}{(b)_{n-l} - (a)_{n-l}} \geq \frac{|\mathcal{F}| - \binom{n}{n-k} + \binom{b}{n-k}}{(b)_{n-k} - (a)_{n-k}}
\]

holds.

5. Generalization of the LYM inequality

In the previous sections we have obtained some results for antichains on two levels and for shadows and shades of uniform families. We are now ready to generalize the LYM inequality. Remember that for $\mathcal{A} \subseteq 2^n$, $\mathcal{A}_i$ denotes the collection of $i$-element sets of $\mathcal{A}$.
Theorem 9. Let $\mathcal{A}$ be an antichain on $[n]$ with largest and smallest set size $k$ and $l$ respectively, $k > l$. Assume that $a$ and $b$ are integers such that $\binom{a}{k} \leq |\mathcal{A}_k|$, $\binom{n}{l} \leq |\mathcal{A}_l|$ with $a = 0, k \leq b \leq n$ or $k \leq a < b \leq n$. Then

$$\frac{|\mathcal{A}_k| - \binom{a}{k}}{\binom{b}{k}} + \sum_{i=l+1}^{k-1} \frac{|\mathcal{A}_i| - \binom{b}{i}}{\binom{b}{i} - \binom{a}{i}} + \frac{|\mathcal{A}_l| - \binom{n}{l} - \binom{b}{l}}{\binom{b}{l} - \binom{a}{l}} \leq 1 \tag{17}$$

holds.

Proof. We proceed by induction on the number of non-empty levels $\mathcal{A}_i$.

If this is 2, i.e. $\mathcal{A}_i \neq \emptyset$ only for $i = l, k$, then $|\mathcal{A}_k| \leq \binom{b}{k}$, since otherwise $|\mathcal{A}_l, \mathcal{A}_k| > \binom{b}{l}$ would hold, which is impossible, because $|\mathcal{A}_l, \mathcal{A}_k| + |\mathcal{A}_l| \leq \binom{b}{l}$. In this case Theorem 4 implies (17).

Assume now that there exists a largest $h, l < h < k$ for which $\mathcal{A}_h \neq \emptyset$. Let us change $\mathcal{A}_k$ by $\mathcal{A}_h, \mathcal{A}_k$, that is $\mathcal{A'} = \mathcal{A}\setminus \mathcal{A}_k \cup \mathcal{A}_h, \mathcal{A}_k$. Clearly, $\mathcal{A'}$ is an antichain, and the number of non-empty levels $\mathcal{A}_i'$ decreased. It is clear that $|\mathcal{A'}_h| = |\mathcal{A}_h, \mathcal{A}_k| + |\mathcal{A}_h| \geq \binom{a}{h} + |\mathcal{A}_l|$, so by the induction hypothesis, we have

$$\frac{|\mathcal{A'}_h| - \binom{a}{h}}{\binom{b}{h} - \binom{a}{h}} + \sum_{i=l+1}^{h-1} \frac{|\mathcal{A'}_i| - \binom{b}{i}}{\binom{b}{i} - \binom{a}{i}} + \frac{|\mathcal{A'}_l| - \binom{n}{l} - \binom{b}{l}}{\binom{b}{l} - \binom{a}{l}} \leq 1 \tag{18}$$

Obviously, also $|\mathcal{A'}_h| \leq \binom{b}{h}$ holds, hence, by Theorem 7, we obtain

$$\frac{|\mathcal{A'}_h| - \binom{a}{h}}{\binom{b}{h} - \binom{a}{h}} = \frac{|\mathcal{A}_h, \mathcal{A}_k| + |\mathcal{A}_h| - \binom{a}{h}}{\binom{b}{h} - \binom{a}{h}}$$

$$\geq \frac{|\mathcal{A}_k| - \binom{a}{k}}{\binom{b}{k} - \binom{a}{k}} + \frac{|\mathcal{A}_l| - \binom{n}{l} - \binom{b}{l}}{\binom{b}{l} - \binom{a}{l}}.$$ 

Thus, (18) implies (17). □

Of course, this theorem could be proved by applying Theorems 4 and 8, too. Note that (17) gives the original LYM inequality with $a = 0, b = n$.

References