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Complexes of discrete Morse functions

Manoj K. Chari[a], Michael Joswig[b]

[a]Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA
[b]Institut für Mathematik, MA 6-2, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany

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Abstract

We investigate properties of the set of discrete Morse functions on a fixed simplicial complex as defined by Forman [5]. It is not difficult to see that the pairings of discrete Morse functions of Δ again form a simplicial complex, the complex of discrete Morse functions of Δ. It turns out that several known results from combinatorial topology and enumerative combinatorics, which previously seemed to be unrelated, can be re-interpreted in the setting of these complexes of discrete Morse functions.

Keywords: Discrete Morse Theory; Matrix-Tree Theorem; Collapsibility

1. Introduction

In the paper [6], Forman introduced the notion of a discrete Morse function on an abstract simplicial complex and developed a combinatorial analog of classical Morse theory. Discrete Morse theory has proved to be an extremely useful tool in the study of certain combinatorially defined spaces. In particular, it has been applied quite successful to the study of the topology of monotone graph properties (see the papers of Babson et al. [1], Shareshian [15], and Jonsson [10]). In this paper, we initiate the investigation of the set of all possible Morse functions on a given simplicial complex. We give this set the structure of a simplicial complex, which we call the *complex of discrete Morse functions* associated with the given simplicial complex. This simplicial structure is rather natural in the context of a graph-theoretical interpretation of discrete Morse functions which is discussed in the

E-mail address: joswig@math.tu-berlin.de (M. Joswig).

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aforementioned papers of Chari [3], Shareshian [15], and Jonsson [10]. For the special case of one-dimensional simplicial complexes, that is graphs, discrete Morse complexes reduce to complexes of rooted forests on graphs whose study was undertaken independently by Kozlov [12].

The set of discrete Morse functions of a given finite simplicial complex can be considered as a discrete analog to the space of vector fields on a given manifold as studied in global analysis, e.g., see Palis and de Melo [14]. More importantly, our point of view yields a unified perspective on a variety of combinatorial concepts including the set of forests of a given graph (certain subsets of) the set of perfect matchings of a bipartite graph, and the set of collapsing strategies of a given collapsible complex.

On a different account, we hope that our investigation stimulates research on geometrically inspired heuristics for the computation of simplicial homology: although efficient modular algorithms for the Smith-Normal-Form computation of integer matrices are known, it seems that for the case of boundary matrices of finite simplicial complexes, elimination techniques are superior in many cases, see Dumas et al. [5]. The connection to the topic studied in this paper comes from the fact that a sequence of pivots in a Gauss-type elimination algorithm, taken over all the boundary matrices, essentially is the same as a discrete Morse function.

In what follows, we present our results in the topological and enumerative study of complexes of discrete Morse functions.

2. Preliminaries

Throughout the following let Δ be a finite abstract simplicial complex.

A (*discrete*) *Morse function* on Δ is a function $m : \Delta \to \mathbb{N}$ with the following properties: For each k-face $f \in \Delta$ there is at most one $(k+1)$-face g containing f with $m(g) < m(f)$, and there is at most one $(k-1)$-face e contained in f with $m(e) > m(f)$. The k-face f is *critical* with respect to m if m attains a higher value at all $(k+1)$-faces containing f and a lower value at all $(k-1)$-faces contained in f. We can phrase it as follows: f is critical with respect to m if and only if, locally at f, the function m is strictly increasing with the dimension.

A key result of Forman is that if m is a discrete Morse function on Δ with critical faces $f_1^{k_1}, \ldots, f_n^{k_n}$, where $\dim f_i^{k_i} = k_i$, then Δ is homotopic to a CW-complex with n cells of respective dimensions $k_1, \ldots, k_n$. This is the direct combinatorial equivalent to what is known from classical Morse theory and for an introduction see Milnor's book [13]. Observe that the function $f \mapsto \dim f$ is a discrete Morse function where all faces are critical. In particular, a Morse function always exists. In order to understand the topological structure of Δ one needs a good Morse function, that is, a Morse function with few critical faces.

For any discrete Morse function m, it can be shown that for any non-critical face f, *exactly one* of the following is true:

(i) there exists a (unique) $(k+1)$-face g containing f with $m(g) < m(f)$,
(ii) there exists a (unique) $(k-1)$-face g contained in f with $m(g) > m(f)$.

Therefore, the set of non-critical faces with respect to m can be uniquely partitioned in to pairs (f, g) where f is a maximal face of g and $m(f) > m(g)$. Now, consider the Hasse diagram of Δ as a directed graph; we direct all edges downward, that is, from the larger faces to the smaller ones. The previous observation implies that the non-critical pairs form a matching in the Hasse diagram. If we reverse the orientation of the arrows in this matching, it can be shown that the resulting directed graph obtained is acyclic. We will call such a matching on the Hasse diagram of the given simplicial complex *acyclic*. Conversely, given an acyclic matching in the Hasse diagram, one can construct a discrete Morse function with the matching edges corresponding precisely to the non-critical pairs. The critical faces are precisely those with no matching edges incident to them. We call two discrete Morse functions on Δ *equivalent* if they induce the same acyclic matching. In the following we usually do not distinguish between equivalent discrete Morse functions, that is, we identify a discrete Morse function with its associated acyclic matching. For further details of this interpretation of discrete Morse functions and its applications, we refer to the papers of Chari [3], Shareshian [15] and Jonsson [10].

The purpose of this paper is to study the set of all possible Morse functions on a given simplicial complex by using this above identification. We define the *complex of discrete Morse functions* $\mathfrak{M}(\Delta)$ of Δ, on the set of edges of the Hasse diagram of Δ, as the set of subsets of edges of the Hasse diagram of Δ which form acyclic matchings. This is clearly an abstract simplicial complex on the given vertex set. Note that even if Δ is pure, the discrete Morse complex $\mathfrak{M}(\Delta)$ is not necessarily pure itself. Often it will be useful instead to consider $\mathfrak{M}_{\text{pure}}(\Delta)$, the *pure complex of discrete Morse functions* of Δ, the subcomplex of $\mathfrak{M}(\Delta)$ generated by the facets of maximal dimension. The facets of maximal dimension correspond to Morse functions which are optimal in the sense that they lead to cell decompositions with as few cells as possible. Note that, for a collapsible simplicial complex such an optimal Morse function corresponds to a collapsing strategy (up to a reordering of the elementary collapses) and vice versa.

It would be interesting to know to what extent topological properties of the complex of discrete Morse functions of a simplicial complex Δ encode topological properties of Δ. For instance, does the homotopy (homeomorphism) type of $\mathfrak{M}(\Delta)$ determine the homotopy (homeomorphism) type of Δ? We suspect that the answer is "no" in both cases, but we do not have any examples.

The topology of the complex of discrete Morse functions as defined above defines two acyclic matchings to be "near" if one can be obtained from the other by "few" extension and deletion steps. It would be interesting to know if this topology is related to the concept of cancellation of critical points, see Forman [6, Theorem 11.1]. In this context, it seems to be an open question whether any two acyclic matchings of a finite simplicial complex are related by a sequence of cancellation steps.

3. Complexes of discrete Morse functions of graphs

As can be expected, complexes of discrete Morse functions are typically very large and very complicated spaces. To obtain some sort of intuition about these spaces, it is helpful to consider the one-dimensional case, that is, graphs. We first observe that the (undirected) Hasse diagram of a graph Γ is obtained by subdividing each edge of the graph exactly once. Now a Morse matching on such a complex gives us pairs (of non-critical faces) which are all of the type (v, e) where v is a node in Γ and e is an edge of Γ with v as one of its end points. Consider the subgraph $S(M)$ of Γ of all such edges e (with both endpoints included) which appear in M and orient each edge e away from v in $S(M)$. The matching property applied to e ensures that this construction is well defined while the matching property at each node v ensures that the out-degree at each node is at most one. From the acyclic property of M, we can deduce that the subgraph $S(M)$ as an undirected graph contains no cycles and hence is a forest. Since the outdegree at each node of $S(M)$ is at most one, each component of the forest has a unique "sink" (often called a "root") with respect to the given orientation. Given any graph Γ, we call an oriented subset F of edges of G, a *rooted forest* of Γ if F is forest as an undirected graph and further, every component of F has a unique root with respect to the given orientation. We have argued above that every acyclic matching for the Hasse diagram of a graph corresponds in a natural way to a rooted forest in a graph and this can be easily reversed to yield the following.

Proposition 3.1. *The set of Morse functions on a graph Γ is in one-to-one correspondence with the set of rooted forests of Γ.*

Complexes of rooted trees and forests of graphs have independently been investigated by Kozlov [12] and the above proposition shows that the complexes he considers are, in fact, complexes of discrete Morse functions of graphs.

In particular, the facets of the complex $\mathfrak{M}(\Gamma)$ correspond precisely to the rooted spanning trees of Γ. Each facet gives rise to a "different" proof of the elementary facet that a (connected) graph with m edges and n nodes is homotopy equivalent to a wedge of $m-n+1$ circles. The rooted spanning tree (in isolation) can be collapsed according to the orientation to a point represented by the root node. The rest of the $m-n+1$ edges form the $m-n+1$ critical 1-cells giving the homotopy type. This simple result yields some interesting enumerative consequences for the *f*-vector of the complex $\mathfrak{M}(\Gamma)$ which we now discuss. Recall that the *f*-vector of a complex just lists for all i, the number f_i of i-dimensional faces. From well-known formulae for the number of rooted forests on n nodes with k-trees, we get an explicit formula for the *f*-vector of $\mathfrak{M}(K_n)$.

Corollary 3.2. *The f-vector of the complex of discrete Morse functions of the complete graph K_n on n nodes is given by*

$$f_{i-1} = \binom{n}{i}(n-i)n^{i-1}.$$

For general graphs, we can relate the *f*-vector of $\mathfrak{M}(\Gamma)$ to the characteristic polynomial of the Laplacian matrix of the graph using the proposition above. The spectrum of the Laplacian is a fundamental algebraic object associated with a graph and has been studied extensively (see Biggs [2]). In what follows, we assume familiarity with the basic notions of algebraic graph theory and we will follow the notation and terminology of Biggs [2]. Given a (connected) graph Γ (which we assume for convenience to be connected), let $\mathbf{Q}(\Gamma)$ be the Laplacian of the graph and let $\sigma(\Gamma;\mu)$ be the characteristic polynomial of $\mathbf{Q}(\Gamma)$ given by $\sigma(\Gamma;\mu) = \det(\mu\mathbf{I} - \mathbf{Q}(\Gamma))$.

Corollary 3.3. *If* $(f_0, f_1, \dots)$ *is the f-vector of* $\mathfrak{M}(\Gamma)$ *then*

$$\sigma(\Gamma; \mu) = \sum f_{i-1}(-1)^i \mu^{n-i}.$$

The proof of this is immediate from the above proposition and Theorem 7.5 of [2].

Now we move on to the topological properties of complexes of discrete Morse functions of graphs. We will frequently use a certain special case of a general result about homotopy colimits of diagrams of spaces, cf. Welker et al. [17].

Proposition 3.4. *Let A, B be subcomplexes of* Δ *such that both inclusion maps* $A \cap B \hookrightarrow A$ *and* $A \cap B \hookrightarrow B$ *are homotopic to the constant map. Then* $A \cup B \simeq A \vee \mathrm{susp}(A \cap B) \vee B$.

Proof. Consider the diagram $\mathscr{D}$ associated to the poset $(\{A \cap B, A, B\}, \geqslant)$. Note that inclusion is reversed such that $A \cap B$ becomes $\widehat{1}$. The inclusion maps $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$ clearly are closed cofibrations. By the Projection Lemma [17, Lemma 4.5] we have that $A \cup B \simeq \mathrm{hocolim}\, \mathscr{D}$.

The inclusion maps being homotopic to the constant map, it follows from the Wedge Lemma [17, Lemma 4.9] that $\mathrm{hocolim}\, \mathscr{D} \simeq A * \emptyset \vee (A \cap B) * \mathbb{S}^0 \vee B * \emptyset \simeq A \vee \mathrm{susp}(A \cap B) \vee B$. □

Two instances of the preceding proposition are particularly relevant for our discussion.

Corollary 3.5. *Let A, B be contractible. Then* $A \cup B \simeq \mathrm{susp}(A \cap B)$.

Corollary 3.6. *Assume that* $A \simeq \mathbb{S}^n \simeq B$ *and* $A \cap B \simeq \mathbb{S}^r$ *with* $r < n$. *Then* $A \cup B \simeq \mathbb{S}^{r+1} \vee \mathbb{S}^n \vee \mathbb{S}^n$.

4. The pure complex of discrete Morse functions of a circle

Let C_n be the cyclic graph on n nodes. Obviously, C_n is homeomorphic to the circle $\mathbb{S}^1$. We choose the following notation. The nodes of C_n are denoted by $x_0, x_1, \dots, x_{n-1}$ with edges (x_i, x_{i+1}); all indices are taken modulo n. The $2n$ vertices of $\mathfrak{M}(C_n)$ are identified with numbers $0, 1, 2, \dots, 2n-1$ such that the vertex $2i$ corresponds to the pair $(x_i, (x_i, x_{i+1}))$ and $2i+1$ corresponds to the pair $(x_{i+1}, (x_i, x_{i+1}))$, see Fig. 1.

The complex of discrete Morse functions $\mathfrak{M}(C_n)$ is pure if and only if $n \leqslant 5$. Its *f*-vector is known to be

$$f_i = \frac{2n}{2n-i-1}\binom{2n-i-1}{i-1},$$

see Ref. [16, Section 2.3.4]. We note that for any n, the pure complex of discrete Morse functions $\mathfrak{M}_{\mathrm{pure}}(C_n)$ is of dimension $n-2$.

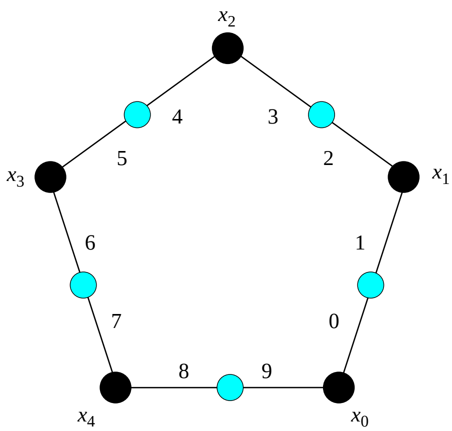

Fig. 1. Graph C_5 and numbering of the vertices of $\mathfrak{M}(C_5)$.

Theorem 4.1. *Let* $n \geqslant 4$. *Then the pure complex of discrete Morse functions* $\mathfrak{M}_{\text{pure}}(C_n)$ *is homotopic to* $\mathbb{S}^2 \vee \mathbb{S}^{n-2} \vee \mathbb{S}^{n-2}$.

Kozlov computed the homotopy type of the complex of discrete Morse functions $\mathfrak{M}(C_n)$. Our result (Theorem 4.1) was obtained independently.

Theorem 4.2 (*Kozlov [12, Proposition 5.2]*).

$$\mathfrak{M}(C_n) \simeq \begin{cases} \mathbb{S}^{2k-1} \vee \mathbb{S}^{2k-1} \vee \mathbb{S}^{3k-2} \vee \mathbb{S}^{3k-2} & \text{if } n = 3k, \\ \mathbb{S}^{2k} \vee \mathbb{S}^{3k-1} \vee \mathbb{S}^{3k-1} & \text{if } n = 3k+1, \\ \mathbb{S}^{2k} \vee \mathbb{S}^{3k} \vee \mathbb{S}^{3k} & \text{if } n = 3k+2. \end{cases}$$

Before we prove Theorem 4.1, we will establish the following useful lemma.

Lemma 4.3. *The pure complex of discrete Morse functions of any path* (*with more than two nodes*) *is collapsible*.

Proof. The poset of faces of a path is obtained by a subdividing each edge of the path, when constructing Morse matchings, the acyclic property is trivially satisfied. Therefore, the pure complex of discrete Morse functions of the path with n edges is simply the complex of partial matchings that are extendible to perfect matchings for the path with $2n$ edges. We will call this the *pure matching complex* for the path with $2n$ edges. Assuming that the $2n$ edges are labeled $1, 2, \ldots, 2n$, it is clear that there is exactly one perfect matching which contains 2, namely $F = \{2, 4, \ldots, 2n\}$ and every other perfect matching contains the edge 1. It is easy to see that F can be collapsed onto the face $\{4, \ldots, 2n\}$ which is contained in the facet $\{1, 4, \ldots, 2n\}$. As a result, we are left with a cone with apex 1, which is obviously collapsible.

Note that a path with exactly two nodes, that is, an interval has a complex of discrete Morse functions isomorphic to $\mathbb{S}^0$. □

Proof of Theorem 4.1. Observe that $\mathfrak{M}_{\text{pure}}(C_n) = \text{St0} \cup \text{St1} \cup \text{St2} \cup \text{St3}$, that is, it is the union of the stars of four vertices in the complex of discrete Morse functions, that is, edges in the Hasse diagram of C_n.

We claim that $\text{St0} \cap \text{St2} = \text{St02} \cup \{4, 6, 8, \dots, 2\text{n} - 2\}$. To show this assume that F is a maximal face of $\text{St0} \cap \text{St2} \backslash \text{St02}$. Then both $F \cup 0$ and $F \cup 2$ are facets of St0 and St2, respectively. It follows that F consists of $n - 2$ elements and it cannot possibly contain 1, 3 or $2n - 1$ and therefore these elements must come from $\{4, 5, 6, \dots, 2n - 2\}$. From the matching property that is required it follows that there is exactly one possibility, that is, $F = \{4, 6, 8, \dots, 2n - 2\}$. Also, to any proper subset of F, say G, one can always add the (Hasse diagram) edges 0 and 2 to obtain a Morse matching, which is obviously an element of St02. This completes the proof of the claim.

Now the star St02 is clearly contractible, and as shown above the boundary of the face $\{4, 6, 8, \dots, 2n - 2\}$ is entirely contained in St02. We infer that $\text{St0} \cap \text{St2}$ is homotopic to $\mathbb{S}^{n-3}$. Similarly, $\text{St1} \cap \text{St3} \simeq \mathbb{S}^{n-3}$.

As the stars St0 and St2 are contractible, we can apply Corollary 3.5 to derive that $\text{St0} \cup \text{St2}$ is homotopic to the suspension of the intersection $\text{St0} \cap \text{St2} \simeq \mathbb{S}^{n-3}$. Thus $\text{St0} \cup \text{St2} \simeq \mathbb{S}^{n-2}$. Similarly, $\text{St1} \cup \text{St3} \simeq \mathbb{S}^{n-2}$.

Next we consider $(\text{St0} \cup \text{St2}) \cap (\text{St1} \cup \text{St3}) = (\text{St0} \cap \text{St1}) \cup (\text{St0} \cap \text{St3}) \cup (\text{St2} \cap \text{St1}) \cup (\text{St2} \cap \text{St3})$, which we claim is equal to $\text{St03} \cup \text{B}$, where B is the pure matching complex of the path with edges $\{4, 5, \dots, 2n - 1\}$. To show this, assume F is a facet of the intersection. It is easy to see from the matching requirement that only way that F can have $n - 1$ elements is if $0 \in F$ and $3 \in F$, that is F is a facet of St03. It is also evident that in this instance, the set $G = F \backslash \{0, 3\}$ is a facet of the pure matching complex of the path on the edges $\{5, 6, \dots, 2n - 2\}$.

Now the facets of $(\text{St0} \cup \text{St2}) \cap (\text{St1} \cup \text{St3}) \backslash \text{St03}$ are subsets of $\{4, 5, \dots, 2n - 1\}$). It is clear that any such facet F is also a facet of the pure matching complex of the path on the edges $\{4, 5, \dots, 2n - 1\}$. Conversely, to any facet F of the pure matching complex of this path we can add either 0 or 2 to F to get a facet of $(\text{St0} \cup \text{St2})$ *and* we can add either 1 or 3 to get a facet of $(\text{St1} \cup \text{St3})$ so that F will be facet of $(\text{St0} \cup \text{St2}) \cap (\text{St1} \cup \text{St3})$.

Thus we have $(\text{St0} \cup \text{St2}) \cap (\text{St1} \cup \text{St3})$ is the union of the two contractible complexes St03 and B. Note that if we have a facet of the pure matching complex of the path on $2n$ edges, consecutively labeled starting with an odd number, then every facet consists of a string of odd edges (possibly empty) followed by an even string of vertices (possibly empty). On the other hand, if the labeling starts with an even number, then the facets consist of an even string followed by an odd string. This observation is useful in determining $\text{St03} \cap \text{B}$, which, following the above arguments, is the intersection of the pure matching complex on the path $\{4, 5, \dots, 2n - 1\}$ with that of the pure matching complex on the path $\{5, \dots, 2n - 2\}$. It follows that any face in this intersection consists entirely of odd vertices or entirely of even vertices. Now the two faces $\{6, 8, \dots, 2n - 2\}$ and $\{5, 7, \dots, 2n - 1\}$ are in the intersection and obviously, they are the unique maximal even and odd sets, respectively, in the intersection. Therefore, we have shown that the intersection $\text{St03} \cap \text{B}$ is a disjoint union of two non-empty simplices. That is, it is homotopic to $\mathbb{S}^0$. Due to Corollary 3.5 we have that $\mathrm{St03} \cup \mathrm{B} = (\mathrm{St0} \cup \mathrm{St2}) \cap (\mathrm{St1} \cup \mathrm{St3}) \simeq \mathbb{S}^1$.

Finally, the claim follows from Corollary 3.6. □

We remark here that the small cases of n can also be treated by shelling techniques.

5. The complex of discrete Morse functions of the simplex

Let Δ_d be the d-dimensional simplex, that is Δ_d is the Boolean lattice on $d+1$ points. The Hasse diagram of Δ_d is isomorphic to the graph of the $(d+1)$-dimensional cube: as the vertices of the cube take all 0/1-vectors of length $d+1$, the linear function $x_0 + \cdots + x_d$ induces an acyclic orientation on the graph of the 0/1-cube. Mapping a subset of $\{0, 1, \ldots, d\}$ to its characteristic function yields the desired isomorphism. By Γ_{d+1} we will denote the directed graph of the $(n+1)$-cube whose arcs point towards lower values of the named linear function.

Except for very small d it seems to be extremely difficult to determine the topological types of $\mathfrak{M}(\Delta_d)$ and $\mathfrak{M}_{\mathrm{pure}}(\Delta_d)$. It even seems to be hard to compute the generating function of the f-vector.

Proposition 5.1. *The complex of discrete Morse functions of Δ_d is homotopic to*

(a) *the 0-sphere $\mathbb{S}^0$ if $d = 1$,*
(b) *the wedge $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$ if $d = 2$.*

Proof. A line segment can be oriented in two different ways. Hence the result for $d = 1$.

Now consider the 2-simplex with vertices a, b, c. Label the edges of the Hasse diagram from 0 to 8 as in Fig. 2. For any subset $X \subseteq \Delta_2$ let $\mathfrak{M}[X]$ be the complex of discrete Morse functions of the subcomplex generated by X. Moreover, $\mathfrak{M} = \mathfrak{M}[\Delta_2]$.

Clearly, each maximal Morse matching contains either 6 or 7 or 8, that is, $\mathfrak{M} = \mathrm{St6} \cup \mathrm{St7} \cup \mathrm{St8}$. Now $\mathrm{St6} = 6 * \mathfrak{M}[\mathrm{ac}, \mathrm{bc}]$, that is, the star of 6 is the cone over $\mathfrak{M}[ac, bc]$ with apex 6. Similarly, $\mathrm{St7} = 7 * \mathfrak{M}[\mathrm{ab}, \mathrm{bc}]$, and $\mathrm{St8} = 8 * \mathfrak{M}[\mathrm{ab}, \mathrm{ac}]$.

Moreover, the intersection $\mathrm{St6} \cap \mathrm{St7}$ equals $\mathfrak{M}[bc]$ which consists of two isolated points. Thus $\mathrm{St6} \cup \mathrm{St7} \simeq \mathbb{S}^1$.

Observe that $(\mathrm{St6} \cup \mathrm{St7}) \cap \mathrm{St8} = \mathfrak{M}[\mathrm{ab}] \cup \mathfrak{M}[\mathrm{ac}]$ has four isolated points, that is, it is equal to $\mathbb{S}^0 \dot{\cup} \mathbb{S}^0$. We infer that $\mathfrak{M} \simeq \mathbb{S}^1 \vee \mathrm{susp}(\mathbb{S}^0 \dot{\cup} \mathbb{S}^0) \simeq \bigvee_4 \mathbb{S}^1$. □

Note that for $n \geqslant 3$ the complex of discrete Morse functions of Δ_n is no longer pure. For an example of a maximal Morse matching of the 3-simplex which is not perfect see Fig. 3.

We computed the homology of the (pure) complex of discrete Morse functions of the 3-simplex using two independent software implementations by Heckenbach [9] and by Gawrilow and Joswig [7]. See Table 1 for the results.

We now turn our attention to the number of faces of the complex of discrete Morse functions of the simplex. For collapsible complexes, such as the simplex, the *perfect Morse matchings*, that is, the facets of the pure part of the complex of discrete Morse functions,

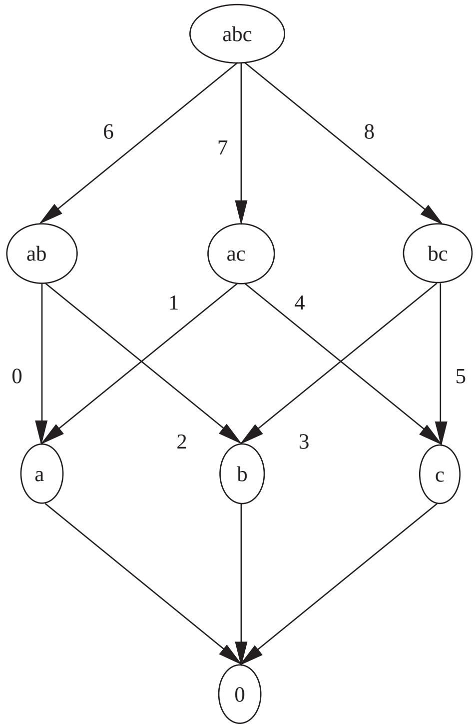

Fig. 2. Hasse diagram of the 2-simplex.

Table 1
Complex of discrete Morse functions of the 3-simplex. All vectors are written from left to right with increasing dimension

	f-vector	Reduced integer homology
$\mathfrak{M}(\Delta_3)$	(28, 300, 1544, 3932, 4632, 2128, 256)	$(0, 0, 0, 0, \mathbb{Z}^{99}, 0, 0)$
$\mathfrak{M}_{\text{pure}}(\Delta_3)$	(28, 300, 1544, 3680, 3672, 1600, 256)	$(0, 0, 0, \mathbb{Z}^{81}, 0, 0, 0)$

correspond to collapsing strategies (modulo ordering of the elementary collapses). Given a perfect Morse matching μ of any collapsible complex each k-face is paired to a unique l-face where $l \in \{k-1, k+1\}$. Now let μ be a perfect Morse matching of the $(n-1)$-simplex. Define $T_k(\mu)$ to be the set of k-faces of Δ_{n-1} which are paired to $(k-1)$-faces (via μ). We now relate these sets to certain subcomplexes to simplices which were studied by Kalai [11].

A (k, n)-*tree* T is a subset of Δ_{n-1}^k of cardinality $\binom{n-1}{k}$ which has the property that for $\Delta(T) = \Delta_{n-1}^{\leqslant k-1} \cup T$ we have $H_k(\Delta(T)) = 0$. Here $\Delta_{n-1}^{\leqslant k-1}$ denotes the complete $(k-1)$-skeleton of the $(n-1)$-dimensional simplex, whose faces correspond to the subsets of

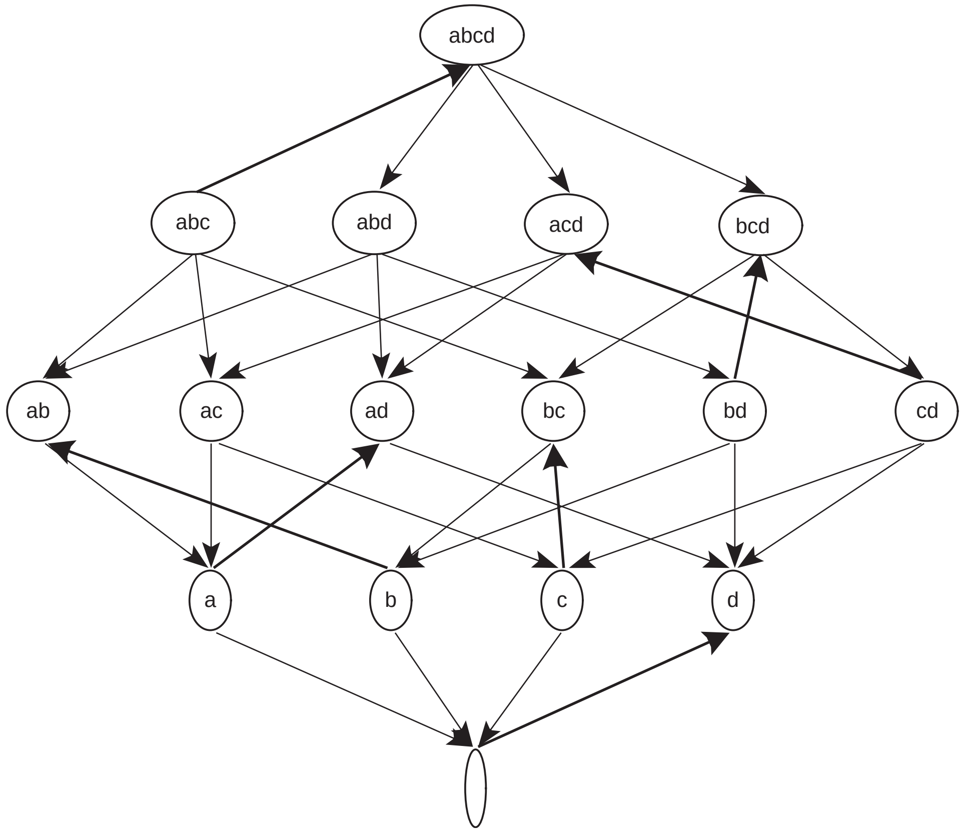

Fig. 3. Maximal acyclic matching in the Hasse diagram of the 3-simplex.

size at most k of a set of n-elements. From an Euler characteristic argument we infer that $H_{k-1}(\Delta(T))$ is finite. The following lemma is immediate from the definition of $T_k(\mu)$.

Lemma 5.2. *For each μ and each k the set $T_k(\mu)$ is a (k, n)-tree.*

Lemma 5.3. *Let μ, μ' be perfect Morse matchings of Δ_{n-1} with $(T_{n-1}(\mu), \ldots, T_0(\mu)) = (T_{n-1}(\mu'), \ldots, T_0(\mu'))$. Then $\mu = \mu'$.*

Proof. On the contrary assume that $\mu \neq \mu'$. Hence, for some k the restrictions $\mu|_{T_k(\mu)}$ and $\mu'|_{T_k(\mu')}$differ. Abbreviate T_k for $T_k(\mu) = T_k(\mu')$ and T_{k-1} for $T_{k-1}(\mu) = T_{k-1}(\mu')$, respectively. Let Γ be the subgraph of the Hasse diagram of Δ_{n-1} which is induced on the vertex set $T_k \cup (\Delta^{k-1} \backslash T_{k-1})$. Both, μ and μ' induce perfect matchings of the bipartite graph Γ. So their symmetric difference is a union of cycles. We arrive at a contradiction because of the acyclicity condition on Morse matchings. □

Let $\mathscr{C}(n, k)$ be the set of (k, n)-trees.

Theorem 5.4 (*Kalai [11]*). *For arbitrary n and k we have*

$$\sum_{C\in\mathscr{C}(n,k)} |H_{k-1}(\Delta(C))|^2 = n^{\binom{n-2}{k}}$$

and

$$|\mathscr{C}(n,k)| \leqslant \left(\frac{en}{k+1}\right)^{\binom{n-1}{k}}$$

where e is Euler's constant.

The preceding results immediately yield an upper bound on the number $f(n)$ of perfect Morse matchings of the n-simplex.

Corollary 5.5. *The number of perfect Morse matchings of the n-simplex is bounded from above by*

$$f(n) \leqslant (n+1)^{2^{n-1}}.$$

Note that, $f(1) = 2$, $f(2) = 9$, and $f(3) = 256$; that is, in principal, the upper bound is tight. However, for larger values of n the estimate becomes increasingly inaccurate for two obvious reasons. Firstly, the formula in Theorem 5.4 also counts (k, n)-trees T for which $\Delta(T)$ is not collapsible. Secondly, each summand is weighted whereas here we are only interested in the number of summands.

So it seems reasonable to look for a better upper bound. A possible way is straightforward from the definition of a perfect Morse matching as a special type of perfect matching. This leads to the problem of counting perfect matchings in the graph of the $(n+1)$-dimensional cube. There is an asymptotic solution to this problem, which is due to Clark et al. [4]. Here we are interested only in the upper bound.

Theorem 5.6 (*Clark et al. [4]*). *The number of perfect matchings of the graph of the* $(n+1)$-*dimensional cube is bounded from above by*

$$f(n) \leqslant (n+1)!^{2^n/(n+1)}.$$

Unfortunately, a direct computation shows that the bound from Corollary 5.5 is always better than the one derived from Theorem 5.6.

What about lower bounds? The interpretation of the Hasse diagram of the n-simplex as the directed graph Γ_{n+1} of the $(n+1)$-cube suggests a way of constructing perfect Morse matchings recursively. Recall that the vertices of Γ_{n+1} are the vertices of the $(n+1)$-dimensional 0/1-cube. For arbitrary $i \in \{0, \dots, n\}$ the vertices satisfying the equation $x_i = 0$ are the vertices of an n-cube, whose graph we denote by $\Gamma^-_{n+1,i}$; similarly, we obtain the graph $\Gamma^+_{n+1,i}$ of another n-cube for $x_i = 1$. We call $\Gamma^+_{n+1,i}$ and $\Gamma^-_{n+1,i}$ *bottom* and *top*, respectively. Observe that all arcs in between point from bottom to top. Thus any perfect acyclic matching of $\Gamma^+_{n+1,i}$, combined with any perfect acyclic matching of $\Gamma^-_{n+1,i}$ yields a perfect acyclic matching of Γ_{n+1}. In principal, we can do this for every $i \in \{0, \ldots, n\}$, but we may obtain the same matching for different i.

Proposition 5.7. *Let* $r(1) = 1$, $r(2) = 2$, $r(3) = 9$, *and, for* $n \geqslant 3$, *recursively,*

$$r(n+1) = \frac{(n+1)(n-1)}{n}\, r(n)^2.$$

The number of perfect Morse matchings of the n-simplex is bounded from below by

$$f(n) \geqslant r(n+1).$$

Proof. The numbers of perfect matchings of the graph of the $(n+1)$-cubes for $n+1 \leqslant 3$ are easy to determine. All these matchings are acyclic and thus are Morse matchings of the respective n-simplex.

We say that an edge of Γ_{n+1} is *in direction i* if its vertices differ in the ith coordinate. In the following we construct perfect acyclic matchings of cubes which contain edges of all but one direction. Observe that all perfect matchings of Γ_1, Γ_2 and Γ_3 are of this kind.

Choose $k \in \{0, \ldots, n\}$. Suppose we have two such matchings μ^+, μ^- in $\Gamma^+_{n+1,k}$ and $\Gamma^-_{n+1,k}$, respectively. Then $\mu = \mu^+ \cup \mu^-$ is a perfect acyclic matching of Γ_{n+1}. Now μ contains edges of either $n-1$ or n directions. Fix μ^+, and let $i \in \{0, \ldots, k-1, k+1, \ldots, n\}$ be the unique direction which μ^+ does not contain an edge of. Now μ contains edges from n directions if and only if μ^- contains edges of direction i.

If there are r perfect acyclic matchings of the n-cube containing edges of all but one direction, then $r(n-1)/n$ of them contain edges of a given direction. This gives $r^2(n-1)/n$ different perfect acyclic matchings of the Γ_{n+1}, which contain edges from all but direction n.

Now there are $n+1$ choices for k, and all of them yield different matchings. □

The number $p(n)$ of perfect matchings of the graph of the n-cube is known for small values of n: In addition to the obvious values Graham and Harary [8] computed $p(4) = 272$ and $p(5) = 589, 185$. Moreover, in [4] the authors mention that Weidemann showed that $p(6) = 16, 332, 454, 526, 976$.

Note that the graph of the 4-cube has a perfect acyclic matching using all directions; it can be constructed from a Hamiltonian cycle.

In order to give a vague idea about the growth of the function r from Proposition 5.7 one can unroll the recursion.

Corollary 5.8. *The number of perfect Morse matchings of the n-simplex is bounded from below by*

$$f(n) \geqslant r(n+1) > \prod_{k=1}^{n-1} k^{2^{n-k-1}}.$$

Proof. We will prove the result by induction on n. The initial case $1^{2^0} = 1 < 2 = r(2)$ is clear. Further,

$$\prod_{k=1}^{n-1} k^{2^{n-k-1}} = (n-1)\left[\prod_{k=1}^{n-2} k^{2^{n-k-2}}\right]^2$$
$$< (n-1)\ r(n)^2$$
$$< \frac{(n+1)(n-1)}{n}\ r(n)^2 = r(n+1). \qquad \square$$

This way we obtain a growth rate for the number of perfect Morse matchings of the n-simplex which is approximately $(1.289)^{2^n}$. We conjecture that the precise value of $f(n)$ is a function of n which goes to infinity with n as the base of this double exponent.

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