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Spectrum of the $\overline{\partial}$-Neumann Laplacian on the Fock space ☆

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ABSTRACT

The spectrum of the $\overline{\partial}$-Neumann Laplacian on the Fock space $L^2(\mathbb{C}^n, e^{-|z|^2})$ is explicitly computed. It turns out that it consists of positive integer eigenvalues, each of which is of infinite multiplicity. Spectral analysis of the $\overline{\partial}$-Neumann Laplacian on the Fock space is closely related to Schrödinger operators with magnetic fields and to the complex Witten Laplacian.

1. Introduction

The spectrum of the $\overline{\partial}$-Neumann Laplacian for the ball and annulus was explicitly computed by Folland [3]. Fu [5] determined the spectrum for the polydisc, showing that it need not be purely discrete like for the usual Dirichlet Laplacian. Here we will exhibit the weighted case, where the weight function is $\varphi(z) = |z|^2$, showing that the essential spectrum is non-empty, which is equivalent to the fact that the $\overline{\partial}$-Neumann operator (the inverse to the $\overline{\partial}$-Neumann Laplacian) fails to be compact [2].

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function, and define the space

$$L^2(\mathbb{C}^n, e^{-\varphi}) = \left\{ f : \mathbb{C}^n \longrightarrow \mathbb{C} : \int_{\mathbb{C}^n} |f|^2 \, e^{-\varphi} \, d\lambda < \infty \right\},$$

where λ denotes the Lebesgue measure, the space $L^2_{(0,q)}(\mathbb{C}^n, e^{\varphi})$ of $(0, q)$-forms with coefficients in $L^2(\mathbb{C}^n, e^{-\varphi})$, for $1 \le q \le n$. Let

$$(f, g)_\varphi = \int_{\mathbb{C}^n} f \, \overline{g} e^{-\varphi} \, d\lambda$$

denote the inner product and

$$\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda$$

the norm in $L^2(\mathbb{C}^n, e^{-\varphi})$.

We consider the weighted $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n, e^{-\varphi}),$$

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where for $(0, q)$-forms $u = \sum'_{|J|=q} u_J \, d\overline{z}_J$ with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ we have

$$\overline{\partial} u = \sum_{|J|=q}{}' \sum_{j=1}^n \frac{\partial u_J}{\partial \overline{z}_j} \, d\overline{z}_j \wedge d\overline{z}_J,$$

and

$$\overline{\partial}_\varphi^* u = - \sum_{|K|=q-1}{}' \sum_{k=1}^n \delta_k u_{kK} \, d\overline{z}_K,$$

where $\delta_k = \frac{\partial}{\partial z_k} - \frac{\partial \varphi}{\partial z_k}$.

There is an interesting connection between $\overline{\partial}$ and the theory of Schrödinger operators with magnetic fields; see, for example, [6] for recent contributions exploiting this point of view.

The complex Laplacian on $(0, q)$-forms is defined as

$$\Box_{\varphi,q} := \overline{\partial} \, \overline{\partial}_\varphi^* + \overline{\partial}_\varphi^* \overline{\partial},$$

where the symbol $\Box_{\varphi,q}$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $\mathcal{C}_0^\infty$, i.e., the space of smooth functions with compact support.

$\Box_{\varphi,q}$ is a self-adjoint and positive operator, which means that

$$(\Box_{\varphi,q} f, f)_\varphi \geq 0, \quad \text{for } f \in \text{dom}(\Box_\varphi).$$

The associated Dirichlet form is denoted by

$$Q_\varphi(f, g) = (\overline{\partial} f, \overline{\partial} g)_\varphi + (\overline{\partial}_\varphi^* f, \overline{\partial}_\varphi^* g)_\varphi, \tag{1.1}$$

for $f, g \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi^*)$. The weighted $\overline{\partial}$-Neumann operator $N_{\varphi,q}$ is – if it exists – the bounded inverse of $\Box_{\varphi,q}$.

We indicate that a square integrable $(0, 1)$-form $f = \sum_{j=1}^n f_j \, d\overline{z}_j$ belongs to $\text{dom}(\overline{\partial}_\varphi^*)$ if and only if

$$e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right) \in L^2(\mathbb{C}^n, e^{-\varphi}),$$

where the derivative is to be taken in the sense of distributions, and that forms with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ are dense in $\text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}_\varphi^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|_\varphi^2 + \|\overline{\partial}_\varphi^* f\|_\varphi^2)^{\frac{1}{2}}$ (see [8]).

We consider the Levi matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}$$

of φ, and suppose that the sum s_q of any q (equivalently: the smallest q) eigenvalues of M_φ satisfies

$$\liminf_{|z| \to \infty} s_q(z) > 0. \tag{1.2}$$

We show that (1.2) implies that there exists a continuous linear operator

$$N_{\varphi,q} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}),$$

such that $\Box_{\varphi,q} \circ N_{\varphi,q} u = u$, for any $u \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$.

If we suppose that the sum s_q of any q (equivalently: the smallest q) eigenvalues of M_φ satisfies

$$\lim_{|z| \to \infty} s_q(z) = \infty, \tag{1.3}$$

then the $\overline{\partial}$-Neumann operator $N_{\varphi,q} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$ is compact (see [9,10] for further details).

To find the canonical solution to $\overline{\partial} f = u$, where $u \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ is a given $(0, 1)$-form such that $\overline{\partial} u = 0$, one can take $f = \overline{\partial}_\varphi^* N_{\varphi,1} u$, and f will also satisfy $f \perp \text{Ker} \overline{\partial}$. For further results on the canonical solution operator to $\overline{\partial}$, see [11,14].

If the weight function is $\varphi(z) = |z|^2$, the corresponding Levi matrix M_φ is the identity. The space $A^2(\mathbb{C}^n, e^{-|z|^2})$ of entire functions belonging to $L^2(\mathbb{C}^n, e^{-|z|^2})$ is the Fock space, which plays an important role in quantum mechanics. In this case,

$$\Box_{\varphi,0} \, u = \overline{\partial}_\varphi^* \, \overline{\partial} u = -\frac{1}{4} \, \Delta u + \sum_{j=1}^n \overline{z}_j \, u_{\overline{z}_j}, \tag{1.4}$$

where $u \in \operatorname{dom} \Box_{\varphi,0} \subseteq L^2(\mathbb{C}^n, e^{-|z|^2})$, and

$$\Box_{\varphi,n} u = \overline{\partial}\, \overline{\partial}^*_{\varphi} u = -\frac{1}{4} \triangle u + \sum_{j=1}^{n} \overline{z}_j\, u_{\overline{z}_j} + n\, u, \tag{1.5}$$

where $u \in \operatorname{dom} \Box_{\varphi,n} \subseteq L^2_{(0,n)}(\mathbb{C}^n, e^{-|z|^2})$.

For $n = 1$, there is a connection to Schrödinger operators with magnetic fields (see [1] for properties of its spectrum), and to Dirac and Pauli operators [13]: the operators

$$P_+ = e^{-|z|^2/2}\, \overline{\partial}\, \overline{\partial}^*_{\varphi}\, e^{|z|^2/2}, \qquad P_- = e^{-|z|^2/2}\, \overline{\partial}^*_{\varphi}\, \overline{\partial}\, e^{|z|^2/2}$$

defined on $L^2(\mathbb{C})$ are the Pauli operators; P_+ is also a Schrödinger operator with magnetic field, and the square of the corresponding Dirac operator satisfies

$$\mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}.$$

For $n > 1$ and $1 \le q \le n-1$, the $\overline{\partial}$-Neumann Laplacian $\Box_{\varphi,q}$ acts diagonally (see [12]): for

$$u = {\sum_{|J|=q}}' u_J\, d\overline{z}_J \in \operatorname{dom} \Box_{\varphi,q} \subseteq L^2_{(0,q)}(\mathbb{C}^n, e^{-|z|^2}),$$

we have

$$\Box_{\varphi,q} u = (\overline{\partial}\, \overline{\partial}^*_{\varphi} + \overline{\partial}^*_{\varphi}\, \overline{\partial})\, u = {\sum_{|J|=q}}' \left(-\frac{1}{4} \triangle u_J + \sum_{j=1}^{n} \overline{z}_j\, u_{J\overline{z}_j} + q\, u_J \right) d\overline{z}_J. \tag{1.6}$$

2. Determination of the spectrum

In order to determine the spectrum of $\Box_{\varphi,q}$ for $\varphi(z) = |z|^2$, we use the following lemma (see [2, Lemma 1.2.2]).

Lemma 2.1. *Let H be a symmetric operator on a Hilbert space $\mathcal{H}$ with domain L, and let $(f_k)_{k=1}^{\infty}$ be a complete orthonormal set in $\mathcal{H}$. If each f_k lies in L and there exist $\mu_k \in \mathbb{R}$ such that $Hf_k = \mu_k f_k$ for every k, then H is essentially self-adjoint. Moreover, the spectrum of $\overline{H}$ is the closure in $\mathbb{R}$ of the set of all μ_k.*

For the sake of simplicity, and in order to explain the general method, we start with the complex one-dimensional case. Looking for the eigenvalues μ of $\Box_{\varphi,0}$, we have, by (1.4),

$$\Box_{\varphi,0} u = -u_{z\overline{z}} + \overline{z} u_{\overline{z}} = \mu u. \tag{2.1}$$

It is clear that the space $A^2(\mathbb{C}^n, e^{-|z|^2})$ is contained in the eigenspace of the eigenvalue $\mu = 0$.

For any positive integer k, the antiholomorphic monomial $\overline{z}^k$ is an eigenfunction for the eigenvalue $\mu = k$.

In the following, we denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2.2. *Let $n = 1$. For $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, the functions*

$$u_{k,m}(z, \overline{z}) = \overline{z}^{k+m} z^m + \sum_{j=1}^{m} \frac{(-1)^j (k+m)!\, m!}{j!\, (k+m-j)!\, (m-j)!}\, \overline{z}^{k+m-j} z^{m-j} \tag{2.2}$$

are eigenfunctions for the eigenvalue $k+m$ of the operator $\Box_{\varphi,0} u = -u_{z\overline{z}} + \overline{z} u_{\overline{z}}$.

For $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, the functions

$$v_{k,m}(z, \overline{z}) = \overline{z}^{k} z^{k+m} + \sum_{j=1}^{k} \frac{(-1)^j (k+m)!\, k!}{j!\, (k+m-j)!\, (k-j)!}\, \overline{z}^{k-j} z^{k+m-j} \tag{2.3}$$

are eigenfunctions for the eigenvalue k of the operator $\Box_{\varphi,0} u = -u_{z\overline{z}} + \overline{z} u_{\overline{z}}$.

Proof. To prove (2.2), we set

$$u_{k,m}(z, \overline{z}) = \overline{z}^{k+m} z^m + a_1 \overline{z}^{k+m-1} z^{m-1} + a_2 \overline{z}^{k+m-2} z^{m-2} + \cdots + a_{m-1} \overline{z}^{k+1} z + a_m \overline{z}^k,$$

and compute

$$\frac{\partial^2}{\partial z \partial \overline{z}} u_{k,m}(z, \overline{z}) = (k+m) m \overline{z}^{k+m-1} z^{m-1} + a_1 (k+m-1)(m-1) \overline{z}^{k+m-2} z^{m-2} + \cdots + a_{m-1}(k+1) \overline{z}^k$$

as well as

$$\overline{z}\,\frac{\partial}{\partial \overline{z}}\,u_{k,m}(z,\overline{z}) = (k+m)\overline{z}^{k+m}\,z^m + a_1(k+m-1)\overline{z}^{k+m-1}\,z^{m-1} + \cdots + a_{m-1}(k+1)\overline{z}^{k+1}\,z + a_m k\overline{z}^k,$$

which implies that the function $u_{k,m}$ is an eigenfunction for the eigenvalue $\mu = k+m$, and from (2.1) we obtain, comparing the highest exponents of $\overline{z}$ and z,

$$(k+m)m - a_1(k+m-1) = -(k+m)a_1;$$

hence $a_1 = -(k+m)m$. Comparing the next lower exponents, we get

$$a_1(k+m-1)(m-1) - a_2(k+m-2) = -a_2(k+m)$$

and $a_2 = \frac{1}{2}\,(k+m)(k+m-1)m(m-1)$. In general, we find that, for $j = 1, 2, \ldots, m$,

$$a_j = \frac{(-1)^j (k+m)!\, m!}{j!\,(k+m-j)!\,(m-j)!},$$

which proves (2.2).

In order to show (2.3), we set

$$v_{k,m}(z,\overline{z}) = \overline{z}^k\,z^{k+m} + b_1\overline{z}^{k-1}\,z^{k+m-1} + b_2\overline{z}^{k-2}\,z^{k+m-2} + \cdots + b_{k-1}\overline{z}\,z^{m+1} + b_k z^m,$$

and compute

$$\frac{\partial^2}{\partial z \partial \overline{z}}\,v_{k,m}(z,\overline{z}) = k(k+m)\overline{z}^{k-1}\,z^{k+m-1} + b_1(k-1)(k+m-1)\overline{z}^{k-2}\,z^{k+m-2} + \cdots + b_{k-1}\overline{z}\,z^{m+1}$$

as well as

$$\overline{z}\,\frac{\partial}{\partial \overline{z}}\,v_{k,m}(z,\overline{z}) = k\overline{z}^k\,z^{k+m} + b_1(k-1)\overline{z}^{k-1}\,z^{k+m-1} + \cdots + b_{k-1}\overline{z}\,z^{m+1}$$

which implies that the function $v_{k,m}$ is an eigenfunction for the eigenvalue $\mu = k$, for each $m \in \mathbb{N}$, and from (2.1) we obtain, comparing the highest exponents of $\overline{z}$ and z,

$$k(k+m) - b_1(k-1) = -kb_1;$$

hence $b_1 = -(k+m)k$. Comparing the next lower exponents, we get

$$b_1(k-1)(k+m-1) - b_2(k-2) = -b_2 k$$

and $b_2 = \frac{1}{2}\,(k+m)(k+m-1)k(k-1)$. In general, we find that, for $j = 1, 2, \ldots, k$,

$$b_j = \frac{(-1)^j (k+m)!\, k!}{j!\,(k+m-j)!\,(k-j)!},$$

which proves (2.3). $\square$

Now we are able to prove the following theorem.

Theorem 2.3. *Let $n = 1$ and $\varphi(z) = |z|^2$. The spectrum of $\square_{\varphi,0}$ consists of all non-negative integers $\{0, 1, 2, \ldots\}$, each of which is of infinite multiplicity, so 0 is the bottom of the essential spectrum. The spectrum of $\square_{\varphi,1}$ consists of all positive integers $\{1, 2, 3, \ldots\}$, each of which is of infinite multiplicity.*

Proof. We already know that the whole Bergman space $A^2(\mathbb{C}, e^{-|z|^2})$ is contained in the eigenspace of the eigenvalue 0 of the operator $\square_{\varphi,0}$ and, for any positive integer k, the antiholomorphic monomial $\overline{z}^k$ is an eigenfunction for the eigenvalue $\mu = k$. In addition, all functions of the form $\overline{z}^\nu z^\kappa$ with $\nu, \kappa \in \mathbb{N}_0$ can be expressed as a linear combination of functions of the form (2.2) and (2.3). For a fixed $k \in \mathbb{N}$, the functions of (2.3) have infinite multiplicity, as the parameter $m \in \mathbb{N}_0$ is free. So all eigenvalues are of infinite multiplicity. All the eigenfunctions considered so far yield a complete orthogonal basis of $L^2(\mathbb{C}, e^{-|z|^2})$, since the Hermite polynomials $\{H_0(x)H_k(y), H_1(x)H_{k-1}(y), \ldots, H_k(x)H_0(y)\}$ for $k = 0, 1, 2, \ldots$ form a complete orthogonal system in $L^2(\mathbb{R}^2, e^{-x^2-y^2})$ (see for instance [4]), and, since $x = 1/2(z+\overline{z})$, $y = i/2(\overline{z}-z)$, we can apply Lemma 2.1 and obtain that the spectrum of $\square_{\varphi,0}$ is $\mathbb{N}_0$. The statement for the spectrum of $\square_{\varphi,1}$ follows from (1.5). $\square$

For several variables we can adopt the method from above to obtain the following result.

Theorem 2.4. *Let $n > 1$, $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$, and $0 \leq q \leq n$. The spectrum of $\square_{\varphi,q}$ consists of all integers $\{q, q+1, q+2, \ldots\}$, each of which is of infinite multiplicity.*

Proof. Recall that the $\overline{\partial}$-Neumann Laplacian $\Box_{\varphi,q}$ acts diagonally, and that

$$\Box_{\varphi,q}\, u = {\sum_{|J|=q}}' \mathrm{dom}\left(-\frac{1}{4}\,\triangle u_J + \sum_{j=1}^{n} \overline{z}_j\, u_{J\overline{z}_j} + q\, u_J\right)\, d\overline{z}_J.$$

The factor q in the last formula is responsible for the fact that the eigenvalues start with q, which can be seen, in each component separately, by

$$-\frac{1}{4}\,\triangle u_J + \sum_{j=1}^{n} \overline{z}_j\, u_{J\overline{z}_j} = (\mu - q)\, u_J.$$

Now let $k_1, k_2, \ldots, k_n \in \mathbb{N}_0$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}$. Then the function

$$u_{k_1,m_1}(z_1, \overline{z}_1)\, u_{k_2,m_2}(z_2, \overline{z}_2) \cdots u_{k_n,m_n}(z_n, \overline{z}_n)$$

is a component of an eigenfunction for the eigenvalue $\sum_{j=1}^{n}(k_j + m_j)$ of the operator $\Box_{\varphi,q}$, which follows from (1.6) and (2.2).

Similarly, it follows from (1.6) and (2.3) that, for $k_1, k_2, \ldots, k_n \in \mathbb{N}$ and $m_1, m_2, \ldots, m_n \in \mathbb{N}_0$, the function

$$v_{k_1,m_1}(z_1, \overline{z}_1)\, v_{k_2,m_2}(z_2, \overline{z}_2) \cdots v_{k_n,m_n}(z_n, \overline{z}_n)$$

is an eigenfunction for the eigenvalue $\sum_{j=1}^{n} k_j$.

All other possible n-fold products with factors u_{k_j,m_j} or v_{k_j,m_j} (also mixed) appear as eigenfunctions of $\Box_{\varphi,q}$.

From this we obtain that all expressions of the form $z_1^{\alpha_1}\,\overline{z}_1^{\beta_1}\,\cdots\, z_n^{\alpha_n}\,\overline{z}_n^{\beta_n}$ for arbitrary $\alpha_j, \beta_j \in \mathbb{N}_0,\ j = 1, \ldots, n$, can be written as a linear combination of eigenfunctions of $\Box_{\varphi,q}$, which proves that all these eigenfunctions constitute a complete basis in $L^2_{(0,q)}(\mathbb{C}^n, e^{-|z|^2})$; see the proof of Theorem 2.3. So we can again apply Lemma 2.1. $\Box$

Remark 2.5. (i) Since in all cases the essential spectrum is non-empty, the corresponding $\overline{\partial}$-Neumann operator fails to be with compact resolvent (see for instance [2]).

(ii) If one considers the weight function

$$\varphi(z) = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{\alpha} \quad \text{for } \alpha > 1,$$

the situation is completely different: the operators $\Box_{\varphi,q}$ are with compact resolvent (see [13]), so the essential spectrum must be empty.

We can use the results from above to settle the corresponding questions for the so-called Witten Laplacian which is defined on $L^2(\mathbb{C}^n)$.

For this purpose, we set $Z_k = \frac{\partial}{\partial \overline{z}_k} + \frac{1}{2}\,\frac{\partial \varphi}{\partial \overline{z}_k}$ and $Z_k^* = -\frac{\partial}{\partial z_k} + \frac{1}{2}\,\frac{\partial \varphi}{\partial z_k}$, and we consider $(0, q)$-forms $h = {\sum_{|J|=q}}'\, h_J\, d\overline{z}_J$, where $\sum'$ means that we sum up only increasing multiindices $J = (j_1, \ldots, j_q)$ and where $d\overline{z}_J = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$. We define

$$\overline{D}_{q+1} h = \sum_{k=1}^{n} {\sum_{|J|=q}}' Z_k(h_J)\, d\overline{z}_k \wedge d\overline{z}_J \tag{2.4}$$

and

$$\overline{D}_q^* h = \sum_{k=1}^{n} {\sum_{|J|=q}}' Z_k^*(h_J)\, d\overline{z}_k \rfloor d\overline{z}_J, \tag{2.5}$$

where $d\overline{z}_k \rfloor d\overline{z}_J$ denotes the contraction, or interior multiplication by $d\overline{z}_k$, i.e. we have

$$\langle \alpha, d\overline{z}_k \rfloor d\overline{z}_J \rangle = \langle d\overline{z}_k \wedge \alpha, d\overline{z}_J \rangle$$

for each $(0, q-1)$-form α.

The complex Witten Laplacian on $(0, q)$-forms is then given by

$$\Delta_\varphi^{(0,q)} = \overline{D}_q\, \overline{D}_q^* + \overline{D}_{q+1}^*\, \overline{D}_{q+1}, \tag{2.6}$$

for $q = 1, \ldots, n-1$.

The general $\overline{D}$-complex has the form

$$L^2_{(0,q-1)}(\mathbb{C}^n) \underset{\overline{D}_q^*}{\overset{\overline{D}_q}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n) \underset{\overline{D}_{q+1}^*}{\overset{\overline{D}_{q+1}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n). \tag{2.7}$$

It follows that

$$\overline{D}_{q+1}\, \Delta_\varphi^{(0,q)} = \Delta_\varphi^{(0,q+1)}\, \overline{D}_{q+1} \quad \text{and} \quad \overline{D}_{q+1}^*\, \Delta_\varphi^{(0,q+1)} = \Delta_\varphi^{(0,q)}\, \overline{D}_{q+1}^*. \tag{2.8}$$

We remark that

$$\overline{D}_q^* h = \sum_{k=1}^n {\sum_{|J|=q}}' Z_k^*(h_J)\, d\overline{z}_k \rfloor d\overline{z}_J = {\sum_{|K|=q-1}}' \sum_{k=1}^n Z_k^*(h_{kK})\, d\overline{z}_K. \tag{2.9}$$

In particular, we get, for a function $v \in L^2(\mathbb{C}^n)$,

$$\Delta_\varphi^{(0,0)} v = \overline{D}_1^* \overline{D}_1 v = \sum_{j=1}^n Z_j^* Z_j(v), \tag{2.10}$$

and, for a (0, 1)-form $g = \sum_{\ell=1}^n g_\ell \, d\overline{z}_\ell \in L^2_{(0,1)}(\mathbb{C}^n)$, we obtain

$$\Delta_\varphi^{(0,1)} g = (\overline{D}_1 \overline{D}_1^* + \overline{D}_2^* \overline{D}_2) g = (\Delta_\varphi^{(0,0)} \otimes I) g + M_\varphi g, \tag{2.11}$$

where we set

$$M_\varphi g = \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j}\, g_k \right) d\overline{z}_j$$

and

$$(\Delta_\varphi^{(0,0)} \otimes I)\, g = \sum_{k=1}^n \Delta_\varphi^{(0,0)} g_k \, d\overline{z}_k.$$

In general, we have that

$$\Delta_\varphi^{(0,q)} = e^{-\varphi/2}\, \Box_{\varphi,q}\, e^{\varphi/2}, \tag{2.12}$$

for $q = 0, 1 \ldots, n$.

For more details, see [13,7].

In our case $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$, we get

$$\Delta_\varphi^{(0,q)} h = {\sum_{|J|=q}}' \left(-\frac{1}{4} \triangle h_J + \frac{1}{2} \sum_{j=1}^n (\overline{z}_j\, h_{J\overline{z}_j} - z_j\, h_{Jz_j}) + \frac{1}{4}\, |z|^2\, h_J + \left(q - \frac{n}{2}\right) h_J \right) d\overline{z}_J, \tag{2.13}$$

for

$$h = {\sum_{|J|=q}}' h_J \, d\overline{z}_J \in \operatorname{dom} \Delta_\varphi^{(0,q)} \subseteq L^2_{(0,q)}(\mathbb{C}^n).$$

The spectrum of $\Delta_\varphi^{(0,0)}$, in an even more general form, was calculated by Ma and Marinescu; see [15] and [16].

Using (2.12) and Lemma 2.1, we get that $\Delta_\varphi^{(0,q)}$ and $\Box_{\varphi,q}$ have the same spectrum. Hence, by Theorem 2.4, we obtain the following theorem.

Theorem 2.6. *Let $\varphi(z) = |z_1|^2 + \cdots + |z_n|^2$ and $0 \le q \le n$. The spectrum of the Witten Laplacian $\Delta_\varphi^{(0,q)}$ consists of all integers $\{q, q+1, q+2, \ldots\}$, each of which is of infinite multiplicity.*

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