JOURNAL OF COMBINATORIAL THEORY, Series A **74**, 345–350 (1996)
ARTICLE NO. 0057

Note

An Infinite Family of Non-embeddable Quasi-Residual Designs with $k < v/2$

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Communicated by the Managing Editors

Received March 17, 1995

parameters $2-(2(3^{d+1})-2, 2(3^d), 3^d)$, where $d \geqslant 1$.

1. INTRODUCTION

Prior to 1978, most of the known examples of non-embeddable quasi-residual designs were either those designs having parameters $2-(16, 6, 3)$ i.e. the parameters of the Bhattacharya design (see, for example [3, 4, 5 and 10]) or were trivial, in the sense that the associated symmetric design does not exist. Since 1978, much work has been done in this area, both on non-embeddable quasi-residual designs with large block size ($k \geqslant v/2$) (see for example [8, 9 and 11]) and in the case where $k < v/2$, (see for example [12, 13, 14 and 15]). The aim of this paper is to demonstrate the existence of a previously unknown infinite family of non-embeddable quasi-residual designs with parameters $2-(2(3^{d+1})-2, 2(3^d), 3^d)$, where $d \geqslant 1$.

2. DEFINITIONS AND NOTATION

The definitions and notation used in this paper are standard, see, for example, [1]. A $t-(v, k, \lambda)$ design D is an $(\mathscr{P}, \mathscr{B}, I)$ incidence structure with the following properties:

(i) The point set $\mathscr{P}$ of D has cardinality v;

(ii) every block $B \in \mathscr{B}$ is incident with exactly k points;

(iii) every t distinct points are together incident with exactly λ blocks.

0097-3165/96 $18.00

Further the cardinality of $\mathscr{B}$ is b, and every point is incident with r blocks, where r and b are dependent on v, k, and λ. Since all designs in this paper are $2-(v, k, \lambda)$ designs we will omit the 2, and simply use the notation (v, k, λ) design. In particular, a symmetric design is one in which $|\mathscr{P}| = |\mathscr{B}| = v$, and $r = k$. The derived design of a symmetric design D is the design obtained by deleting a block and retaining those points incident with the block. The derived design is a $(k, \lambda, \lambda - 1)$ design. The residual of a symmetric (v, k, λ) design D, is the design obtained by deleting a block of D and retaining those points not incident with the block. The residual design is a $(v-k, k-\lambda, \lambda)$ design. A design is said to be quasi-residual if it has the property that $r = k + \lambda$.

A design is said to be affine resolvable if its block set can be partitioned into sets of equal size (parallel classes) such that: the blocks in any given parallel class are pairwise disjoint; each point of the design appears on exactly one block of each parallel class; and blocks from distinct parallel classes meet in a constant number of points. A design which has the first two properties, but not the third (i.e. blocks from distinct parallel classes need not intersect in a constant number of points) is called resolvable.

3. Construction

In [6, available from the author] it is shown that the existence of a symmetric $2-(v, k, \lambda)$ design, D, possessing a resolvable derived design, implies the existence of a $2-(v+ek, 2k, k)$ design, where $e = v/k$. If, in particular, we take for D a member of the infinite family of Mitchell designs ([7]), then D is a $2-(q^{d+1}-q+1, q^d, q^{d-1})$ design, where $q > 2$ is a prime power, and $d \geqslant 2$, and, further if we note that the derived design of D is a $(q-1)$-multiple of $AG_{d-1}(d, q)$, we can modify Theorem 2 of [6] as follows:

Theorem 1. *Given a symmetric design D, where D is a member of the Mitchell family, together with q copies of $AG_{d-1}(d, q)$, $A_1, A_2, ..., A_q$, there exists a $(2q^{d+1}-q+1, 2q^d, q^d)$ design, R.*

Proof. The proof is constructive and is similar to that given in [6], but requires modifying to allow for use of $AG_{d-1}(d, q)$ in place of the derived design of D. Let the blocks of D be $\{b_1, b_2, ..., b_s\}$, where $s = q^{d+1}-q+1$. For the points of R we take the points of D together with the points of each of $A_1, A_2, ..., A_q$, giving us $(q^{d+1}-q+1)+q(q^d) = 2q^{d+1}-q+1$ points, as is required.

All but $q(q+1)/2$ blocks of R will consist of one block from each of $A_1, A_2, ..., A_q$ together with a block from D, giving a block size of

$q(q^{d-1}) + q^d = 2q^d$, as required. Note that, each of the A_i, $i = 1, 2, ..., q$ has $(q^d - 1)/(q-1)$ parallel classes, each of which contain q blocks. Select one such parallel class from each of these designs, giving us q^2 blocks. We want to form blocks of R by pasting together a block from each parallel class of distinct copies of A_i together with a block of D subject to the following conditions:

1. Each pair of blocks from distinct copies of A_i appear together exactly once on a block of R;

2. Blocks from the same copy of A_i do not appear together on a block of R; and

3. We use the minimum number of blocks of D, such that each block of each of the A_i appears exactly once with the chosen blocks from D in a block of R.

We accomplish this by viewing are chosen q^2 blocks as the points of an affine plane of order q. We then construct the affine plane A on these points. It is necessary to reject a parallel class of A, since one class will have lines whose points correspond to blocks from the same A_i, and would thus violate condition 2. above. By viewing the construction in this way, we see that condition 1 is met, and that we need q blocks from D, (one per parallel class of A, other than the rejected class), that each of the blocks of D is used q times (once for each line in a parallel class of A) and further we see that we have been able to construct q^2 blocks of R.

If we now repeat this procedure for the remaining parallel classes of the A_i, ensuring that each parallel class is only chosen once, and that blocks of D are only chosen once, we have constructed $((q^d - 1)/(q-1))(q^2)$ blocks of R, and further, we have used $((q^d - 1)/(q-1))(q)$ blocks of D, say $b_1, b_2, ..., b_t$ where $t = ((q^d - 1)/(q-1))(q)$. Further, we call this structure M_1. Now, repeat this structure $q-2$ times and form $M_2, M_3, ..., M_{q-1}$, by replacing each occurrence of b_j, $j = 1, 2, ..., t$ in M_k, $k = 2, 3, ..., q-1$ with $b_{j+(k-1)t}$. This gives a total of $q^2(q-1)((q^d - 1)/(q-1)) = q^2(q^d - 1)$ blocks of R. We also note that we have used $(q-1)\, q((q^d - 1)/(q-1)) = q^{d+1} - q$ blocks of D to form $M_1, M_2, ..., M_{q-1}$, specifically we have an as yet unused block of D.

We form q more blocks of R by taking the points of each copy of A_i together with the unused block of D. Since each of the A_i has q^d points and each block of D also has q^d points, these blocks have size $2q^d$.

The final $q(q-1)/2$ blocks of R are formed by taking all possible pairs of the A_i's, and then forming the block of R by adjoining all the points of each of our chosen pair of A_i's. Each of these blocks will, clearly, have $2q^d$ points.

Certainly, the number of points, the number of blocks, and the block size of R are as required.

To verify that the replication number is q^{d+1}, consider two cases:

1. A point from D–each point of D appears on q^d blocks of D, each of these blocks is part of q blocks of R, thus each such point appears q^{d+1} times.

2. A point from one of the A_i's—each point of A_i appears on $(q^d-1)/(q-1)$ blocks of A_i, each of these blocks appears as part of $q(q-1)$ blocks of R in the first part of the construction. Further each such point appears on q of the final $q(q+1)/2$ blocks of R. Also giving a replication number of q^{d+1}.

We now verify that every pair of points appear q^d times. We consider four cases:

1. A pair of points from one copy of A_i. Each pair of such points appears on $(q^{d-1}-1)/(q-1)$ blocks of A_i. Each of these blocks appears as part of $q(q-1)((q^{d-1}-1)/(q-1))$ blocks of R in the first part of the construction. Further, each pair of points from A_i appears q times in the final $q(q+1)/2$ blocks of R. Thus, we have each such pair appearing q^d times.

2. A pair of points from D. Each pair of such points appears on q^{d-1} blocks of D. Each of these blocks appear as part of q blocks of R. Therefore, these pairs appears q^d times.

3. A pair of points from distinct copies of A_i. Each point of one copy of A_i will be paired with every point of a distinct copy of A_i once per choice of parallel class from the A_i's. Further these blocks are repeated $q-1$ times. Also, each pair will appear once more when we form blocks by taking all possible pairs of copies of the A_i's. Such pairs then appear q^d times.

4. A pair of points, one from one of the A_i's, the other from D. Each time a point of A_i appears on a block of R given in the first and second parts of the construction, that block contains a distinct block from D. Further, since each point of A_i appears $q^{d+1}-(q-1)=q^{d+1}-q+1$ times in such blocks, we know that these points appear with every block of D exactly once. Therefore, the number of times these pairs occur is equal to the replication number of D, i.e., q^d.

We note that the Mitchell family of designs require that $q>2$ and $d\geqslant 2$. The construction given above works for $q=2$ and $d\geqslant 2$, and for $q>2$ with $d=1$. In these two cases the symmetric design is the complement of a projective geometry design or a projective geometry design.

4. The Infinite Family of Quasi-Residual Designs

If we take $q=3$ in the above construction, the design R is a $(2(3^{d+1})-2, 2(3^d), 3^d)$ design with replication number 3^{d+1}, i.e. R is quasi-residual. Further, consider a block of M_1, say, l_i, $i=1, 2, ..., 3t$ and a block of M_2, say l_j, $j=3t+1, 3t+2, ..., 6t$ where $j-i=3t$ then any such pair of blocks of R are identical in their first 3^d positions, and intersect in 3^{d-1} points in their last 3^d positions. Thus, these pairs of blocks have intersection size 3^d+3^{d-1}, and so these designs are non-embeddable quasi-residual designs of Bhattacharya type.

We can also obtain a second family of non-embeddable quasi-residual designs using Theorem 2 of [6]. If we take as our symmetric design D a $(9(2^{2d+1})-2, 3(2^{2d+1}), 2^{2d+1})$ having a resolvable derived design. Then R would be a $((9(2^{2d+2})-2, 3(2^{2d+2}), 3(2^{2d+1}))$ design with replication number $9(2^{2d+1})$, i.e. R is a quasi-residual design. Such designs are trivially non-embeddable since the associated symmetric design does not exist by the Bruck–Ryser–Chowla Theorem. The first two examples in this family would take for D a $(16, 6, 2)$ design $(d=0)$ and a $(70, 24, 8)$ design $(d=1)$. We, further note that the design D does not itself have to have a resolvable derived design, but there must exist a design having the parameters of D which has a resolvable derived design.

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