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An alternative approach to comprehensive Gröbner bases

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Abstract

We give an alternative definition of comprehensive Gröbner bases in terms of Gröbner bases in polynomial rings over commutative Von Neumann regular rings. Our comprehensive Gröbner bases are defined as Gröbner bases in polynomial rings over certain commutative Von Neumann regular rings, hence they have two important properties which do not hold in standard comprehensive Gröbner bases. One is that they have canonical forms in a natural way. Another one is that we can define monomial reductions which are compatible with any instantiation. Our comprehensive Gröbner bases are wider than Weispfenning's original comprehensive Gröbner bases. That is there exists a polynomial ideal generated by our comprehensive Gröbner basis which cannot be generated by any of Weispfenning's original comprehensive Gröbner bases.

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1. Introduction

Let R be a commutative ring and S be any non-empty set. Then the set of all functions from S to R denoted by R^S becomes a commutative ring by naturally defining an addition and a multiplication of functions. Furthermore, this ring becomes a commutative Von Neumann regular ring if R is a commutative Von Neumann regular ring. Therefore, in case it is computable, we can construct Gröbner bases in polynomial rings over R^S. For such Gröbner bases, we have the following theorem.

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Theorem. *Let $G = \{g_1, \dots, g_k\}$ be a reduced Gröbner basis of an ideal $\langle f_1, \dots, f_l \rangle$ in a polynomial ring $R^S[\bar{X}]$, then for each element a of S, $\{g_1(a), \dots, g_k(a)\}$ becomes a reduced Gröbner basis of the ideal $\langle f_1(a), \dots, f_l(a) \rangle$ in the polynomial ring $R[\bar{X}]$. Here $h(a)$ denotes a polynomial in $R[\bar{X}]$ given from a polynomial h of $R^S[\bar{X}]$ with replacing each coefficient c in h by $c(a)$ (see Theorem* 2.3 *of Weispfenning (1989)).*

This observation leads us to an alternative definition of comprehensive Gröbner bases. Let K be an infinite field and $f_1(A_1, \dots, A_m, \bar{X}), \dots, f_k(A_1, \dots, A_m, \bar{X})$ be polynomials in $K[A, \dots, A_m, \bar{X}]$ with parameters $A_1, \dots, A_m$. Considering each polynomial $f(A_1, \dots, A_m)$ in $K[A_1, \dots, A_m]$ as a function from K^m to K, $f_1(A_1, \dots, A_m, \bar{X}), \dots, f_k(A_1, \dots, A_m, \bar{X})$ become polynomials in $K^{(K^m)}[\bar{X}]$. If we can construct a reduced Gröbner basis G of the ideal $\langle f_1(A_1, \dots, A_m, \bar{X}), \dots, f_k(A_1, \dots, A_m, \bar{X}) \rangle$ in the polynomial ring $K^{(K^m)}[\bar{X}]$ over the commutative Von Neumann regular ring $K^{(K^m)}$ somehow, then we can consider G as a kind of comprehensive Gröbner basis of $\langle f_1(A_1, \dots, A_m, \bar{X}), \dots, f_k(A_1, \dots, A_m, \bar{X}) \rangle$ with parameters $A_1, \dots, A_m$, since an instantiation of $A_1, \dots, A_m$ with any elements $a_1, \dots, a_m$ of K becomes a reduced Gröbner basis of the ideal $\langle f_1(a_1, \dots, a_m, \bar{X}), \dots, f_k(a_1, \dots, a_m, \bar{X}) \rangle$ in $K[\bar{X}]$ by the theorem above.

In order to enable the above computation, it suffices to establish a way to handle the smallest commutative Von Neumann regular ring extending the canonical image of $K[A_1, \dots, A_m]$. If the quotient field $K(A_1, \dots, A_m)$ would correspond to it, the situation would be very nice. Unfortunately, however, it does not work. Consider the inverse A_1^{-1} of A_1 in the commutative Von Neumann regular ring $K^{(K^m)}$. Since $A_1(a_1, \dots, a_m) = a_1$ for any $a_1, \dots, a_m$ in K, A_1^{-1} should be the function φ from K^m to K such that $\varphi(0, a_2, \dots, a_m) = 0$ and $\varphi(a_1, \dots, a_m) = 1/a_1$ if $a_1 \neq 0$. Certainly φ is not a member of $K(A_1, \dots, A_m)$.

In order to overcome this situation, we define a new algebraic structure called a *terrace*, which enables us to handle the smallest commutative Von Neumann regular ring extending the canonical image of $K[A_1, \dots, A_m]$. Using terraces we can compute a Gröbner basis in a polynomial ring over $K^{(K^m)}$. We call it an *ACGB* (alternative comprehensive Gröbner basis). ACGB have the following two nice properties, which do not hold in standard comprehensive Gröbner bases (Weispfenning, 1992).

(1) There is a canonical form of an ACGB in a natural way.
    Since an ACGB is already in a form of a Gröbner basis in a polynomial ring over a commutative Von Neumann regular ring, we can use a stratified Gröbner basis as a canonical form of an ACGB.
(2) We can use monomial reductions of an ACGB.
    Because of the same reason as above, we can use monomial reductions of an ACGB. Moreover, it will be shown that monomial reductions are compatible with any instantiation of parameters.

In this paper we introduce our work on ACGB. We concentrate on the case that K is algebraically closed. We give some algorithms to handle terraces using the classical Gröbner bases technique.

Our plan is as follows. In Section 2, we give a quick review for Gröbner bases for polynomial rings over Von Neumann regular rings. The reader is referred to

Weispfenning (1989), Sato (1998), or Sato and Suzuki (2001) for more detailed descriptions. In Section 3, we give a definition of terraces with several algorithms to handle them. In Section 4, we give a definition of ACGB. We prove several nice properties they have. In Section 5, we show that the class of ACGB is wider than the class of Weispfenning's original comprehensive Gröbner bases. In Section 6, we give some computation examples we got through our implementation. In Section 7, we show several methods to find the properties of systems of polynomial equations over functions using ACGB.

2. Von Neumann regular ring and Gröbner basis

A commutative ring R with identity 1 is called a *Von Neumann regular ring* if it has the following property:

$$\forall a \in R\ \exists b \in R \qquad a^2 b = a.$$

For such a b, $a^* = ab$ and $a^{-1} = ab^2$ are uniquely determined and satisfy $aa^* = a$, $aa^{-1} = a^*$, and $(a^*)^2 = a^*$.

Note that every direct product of fields is a Von Neumann regular ring. Conversely, any Von Neumann regular ring is shown to be isomorphic to a subring of direct product of fields as follows.

Definition 2.1. Let R be a Von Neumann regular ring. If we define $\neg a = 1 - a$, $a \wedge b = ab$ and $a \vee b = \neg(\neg a \wedge \neg b)$ for each $a, b \in R$ such that $a^2 = a, b^2 = b$, $(\{x \in R : x^2 = x\}, \neg, \wedge, \vee)$ becomes a Boolean algebra, which is denoted by $B(R)$.

Considering $B(R)$ as a Boolean ring, the Stone representation theorem gives the following isomorphism Φ from $B(R)$ to a subring of $\prod_{I \in St(B(R))} B(R)/I$ by $\Phi(x) = \prod_{I \in St(B(R))} [x]_I$, where $St(B(R))$ is the set of all maximal ideals of $B(R)$. This representation of $B(R)$ is extended to a representation of R as follows.

Theorem 2.2 (Saracino–Weispfenning). *For a maximal ideal I of $B(R)$, if we put $I_R = \{xy : x \in R, y \in I\}$, then I_R is a maximal ideal of R. If we define a map Φ from R into $\prod_{I \in St(B(R))} R/I_R$ by $\Phi(x) = \prod_{I \in St(B(R))} [x]_{I_R}$, then Φ is a ring embedding.*

In the following unless mentioned, Greek letters α, β, γ are used for terms, Roman letters a, b, c for elements of R, and f, g, h for polynomials over R. Throughout this section, we work in a polynomial ring over R and assume that some total admissible order on the set of terms is given. The leading term of f is denoted by $lt(f)$ and its coefficient by $lc(f)$. The leading monomial of f, i.e., $lc(f)lt(f)$ is denoted by $lm(f)$.

We redescribe some definitions and results which we need for our comprehensive Gröbner bases. The detailed argument is given in Weispfenning (1989) and Sato and Suzuki (2001).

Definition 2.3. For a polynomial $f = a\alpha + g$ with $lm(f) = a\alpha$, a monomial reduction $\rightarrow_f$ is defined as follows:

$$b\alpha\beta + h \rightarrow_f b\alpha\beta + h - ba^{-1}\beta(a\alpha + g)$$

where $ab \neq 0$ and $b\alpha\beta$ need not be the leading monomial of $b\alpha\beta + h$.

A monomial reduction $\rightarrow_F$ by a set F of polynomials is also naturally defined. Using this monomial reduction, we can construct a Gröbner basis of the ideal generated by a given finite set of polynomials. Using the following properties, we can see that the algorithm is almost the same as Buchberger's.

Definition 2.4. A polynomial f is called *Boolean closed* if $(lc(f))^* f = f$. $(lc(f))^* f$ is called a *Boolean closure* of f and denoted by $bc(f)$. Note that the Boolean closure of any polynomial is Boolean closed.

We can construct a set of Boolean closed polynomials H from any given set of polynomials F such that $\langle F \rangle = \langle H \rangle$. Though H is not determined uniquely, we abuse the notation $bc(F)$ to denote one of such H.

Theorem 2.5. *Let F be a set of Boolean closed polynomials. Then the equivalence relation $\overset{*}{\leftrightarrow}$ coincides with the equivalence relation induced by the ideal $\langle F \rangle$.*

Using our monomial reductions, Gröbner bases are defined as follows.

Definition 2.6. A finite set G of polynomials is called a *Gröbner basis*, if it satisfies the following two properties:

- $f \overset{*}{\leftrightarrow}_G g$ iff $f - g \in \langle G \rangle$ for each polynomial f and g,
- $\rightarrow_G$ has a Church Rosser property,
  i.e., for each polynomial f and g, $f \overset{*}{\leftrightarrow}_G g$ iff there exists a polynomial h such that $f \overset{*}{\rightarrow}_G h$ and $g \overset{*}{\rightarrow}_G h$.

Definition 2.7. For each pair of polynomials $f = a\alpha\gamma + f'$ and $g = b\beta\gamma + g'$, where $lm(f) = a\alpha\gamma$, $lm(g) = b\beta\gamma$, and $\mathrm{GCD}(\alpha, \beta) = 1$, the polynomial $b\beta f - a\alpha g = b\beta f' - a\alpha g'$ is called the *S-polynomial* of f and g and denoted by $SP(f, g)$.

We can also characterize Gröbner bases in terms of S-polynomials as in polynomial rings over fields.

Theorem 2.8. *Let G be a finite set of Boolean closed polynomials. Then G is a Gröbner basis iff $SP(f, g) \overset{*}{\rightarrow}_G 0$ for any pair f and g of polynomials in G.*

This theorem enables us to construct a Gröbner basis G for a given finite set F of polynomials such that $\langle G \rangle = \langle F \rangle$. We can repeat computations of Boolean closures and S-polynomials until we get a desired Gröbner basis G, each element of which is Boolean closed.

We describe some important properties of Gröbner bases.

Theorem 2.9. *Let G be a reduced Gröbner basis, then any element of G is Boolean closed.*

Definition 2.10. A reduced Gröbner basis G in a polynomial ring over a commutative Von Neumann regular ring is called a *stratified* Gröbner basis, when it satisfies the following two properties:

- $lc(g) = lc(g)^*$ for each $g \in G$,
- $lt(f) \neq lt(g)$ for any distinct elements f and g in G.

Theorem 2.11. *A stratified Gröbner basis is determined uniquely. That is two stratified Gröbner bases G and G' such that $\langle G\rangle = \langle G'\rangle$ must be identical.*

3. Terrace

In this section, we define a computable ring T and operations on T which witness that T forms a Von Neumann regular ring. For an arbitrary polynomial $f \in K[A_1, \dots, A_n]$, we can consider it as a mapping $f : K^n \to K$, i.e., $f \in K^{(K^n)}$. So we can define the canonical embedding

$$\varphi : K[A_1, \dots, A_n] \to K^{(K^n)}.$$

Let T be the closure of the image $\varphi[K[A_1, \dots, A_n]]$ under addition, multiplication, and inverse in the Von Neumann regular ring $K^{(K^n)}$, hence T becomes a Von Neumann regular ring. We show a way to describe each element of T and define computable operations on T.

In the rest of this section, we fix an algebraically closed field K and a natural number n. We use the symbols $A_1, \dots, A_n$ as variables. For each finite set of polynomials $\{f_1, \dots, f_l\}$ in $K[A_1, \dots, A_n]$, we denote the affine variety by $V(\{f_1, \dots, f_l\})$, i.e.,

$$\begin{aligned} V(\{f_1, \dots, f_l\}) = \{(a_1, \dots, a_n) \in K^n :\\ f_1(a_1, \dots, a_n) = \cdots = f_l(a_1, \dots, a_n) = 0\}. \end{aligned}$$

We set $V(\emptyset) = K^n$ and $V(\{1\}) = \emptyset$ for convenience.

Example 3.1. Let t be a function from $\mathbb{C}^2$ to $\mathbb{C}$ defined by

$$t(a, b) = \begin{cases} a - b, & \text{if } (a, b) \in \mathbb{C}^2 \backslash V(\{a - b\}), \text{ i.e., } a \neq b, \\ 0, & \text{otherwise.} \end{cases}$$

Then the inverse is

$$t^{-1}(a, b) = \begin{cases} \frac{1}{a-b}, & \text{if } (a, b) \in \mathbb{C}^2 \backslash V(\{a - b\}), \\ 0, & \text{otherwise.} \end{cases}$$

The addition of t and t^{-1} is

$$(t + t^{-1})(a, b) = \begin{cases} \frac{a^2-2ab+b^2+1}{a-b}, & \text{if } (a, b) \in \mathbb{C}^2 \backslash V(\{a - b\}), \\ 0, & \text{otherwise.} \end{cases}$$

And the multiplication of t and t^{-1} is

$$(t \cdot t^{-1})(a, b) = \begin{cases} 1, & \text{if } (a, b) \in \mathbb{C}^2 \backslash V(\{a - b\}), \\ 0, & \text{otherwise.} \end{cases}$$

In order to handle elements of T such as $t \cdot t^{-1}$, we define an algebraic structure called a terrace.

3.1. Definition of preterraces

Definition 3.1. A triple $\langle s, t, r\rangle$ is called a *preterrace* on $K[A_1, \ldots, A_n]$ if s and t are finite sets of polynomials in $K[A_1, \ldots, A_n]$ and $r = g/h$ for some $g, h \in K[A_1, \ldots, A_n]$ which satisfy

(1) $V(s) \subseteq V(t)$,

(2) $(V(\{g\}) \cup V(\{h\})) \cap (V(t)\backslash V(s)) = \emptyset$, i.e., $g(a_1, \ldots, a_n) \neq 0$ and $h(a_1, \ldots, a_n) \neq 0$ for any $(a_1, \ldots, a_n) \in V(t)\backslash V(s)$.

For a given preterrace $p = \langle s, t, r\rangle$, the *support* of p (supp(p)) is the set $V(t)\backslash V(s) \subseteq K^n$. For a preterrace $p = \langle s, t, g/h\rangle$ on $K[A_1, \ldots, A_n]$ and $(a_1, \ldots, a_n) \in K^n$, we define $p(a_1, \ldots, a_n) \in K$ by

$$p(a_1, \ldots, a_n) = \begin{cases} \frac{g(a_1,\ldots,a_n)}{h(a_1,\ldots,a_n)}, & \text{if } (a_1, \ldots, a_n) \in \operatorname{supp}(p), \\ 0, & \text{otherwise.} \end{cases}$$

So p can be considered as a member of T. For given preterraces p_1 and p_2, we define a relation $p_1 \equiv p_2$ by $\operatorname{supp}(p_1) = \operatorname{supp}(p_2)$ and $p_1(a_1, \ldots, a_n) = p_2(a_1, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in \operatorname{supp}(p_1)$. We can easily check that $\equiv$ is an equivalence relation on the set of preterraces.

For an arbitrary polynomial $f \in K[A_1, \ldots, A_n]$, we define the corresponding preterrace pre(f) as follows:

$$\operatorname{pre}(f) = \langle\{f\}, \emptyset, f/1\rangle.$$

Note that

$$\begin{aligned} \operatorname{supp}(\operatorname{pre}(f)) &= V(\emptyset)\backslash V(\{f\}) \\ &= \{(a_1, \ldots, a_n) \in K^n \ : \ f(a_1, \ldots, a_n) \neq 0\}. \end{aligned}$$

Then we can easily see that $f(a_1, \ldots, a_n) = \operatorname{pre}(f)(a_1, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in K^n$.

Next we define the inverse and multiplicative operations on preterraces. The inverse p^{-1} of a preterrace $p = \langle s, t, g/h\rangle$ is defined by $p^{-1} = \langle s, t, h/g\rangle$ without changing the support. Note that we have

$$\begin{cases} p(a_1, \ldots, a_n)^{-1} = p^{-1}(a_1, \ldots, a_n), & \text{if } (a_1, \ldots, a_n) \in \operatorname{supp}(p) = \operatorname{supp}(p^{-1}) \\ p(a_1, \ldots, a_n) = p^{-1}(a_1, \ldots, a_n) = 0, & \text{otherwise.} \end{cases}$$

Hence p^{-1} represents the inverse of p in T.

In order to define the multiplication $p_1 \cdot p_2$ of preterraces $p_1 = \langle s_1, t_1, r_1\rangle$ and $p_2 = \langle s_2, t_2, r_2\rangle$ to represent the multiplication as elements of T, we need that

$$(p_1 \cdot p_2)(a_1, \ldots, a_n) = \begin{cases} p_1(a_1, \ldots, a_n) \cdot p_2(a_1, \ldots, a_n), \\ \qquad \text{if } (a_1, \ldots, a_n) \in \operatorname{supp}(p_1) \cap \operatorname{supp}(p_2), \\ 0, \quad \text{otherwise.} \end{cases}$$

Note that we have

$$\begin{aligned}\mathrm{supp}(p_1) \cap \mathrm{supp}(p_2) &= (V(t_1)\backslash V(s_1)) \cap (V(t_2)\backslash V(s_2))\\ &= (V(t_1) \cap V(t_2))\backslash(V(s_1) \cup V(s_2))\\ &= (V(t_1) \cap V(t_2))\backslash((V(s_1) \cap V(t_2)) \cup (V(s_2) \cap V(t_1)))\\ &= V(t_1 \cup t_2)\backslash V(\mathbf{Prod}(s_1 \cup t_2, s_2 \cup t_1)),\end{aligned}$$

where, for a finite set s, t of polynomials,

$$\mathbf{Prod}(s,t) = \{f \cdot g \ : \ f \in s, g \in t\}.$$

So we define the multiplication by

$$p_1 \cdot p_2 = \langle \mathbf{Prod}(s_1 \cup t_2, s_2 \cup t_1), t_1 \cup t_2, r_1 \cdot r_2\rangle.$$

We can easily check that $p_1 \cdot p_2 \equiv p_2 \cdot p_1$ (actually $p_1 \cdot p_2 = p_2 \cdot p_1$), $(p_1 \cdot p_2) \cdot p_3 \equiv p_1 \cdot (p_2 \cdot p_3)$, and $p_1 \cdot \langle\{1\}, \emptyset, 1\rangle \equiv p_1$ for any preterraces p_1, p_2, and p_3. Note that, for a preterrace $p = \langle s, t, r\rangle$, we have $p \cdot p^{-1} \equiv \langle s, t, 1\rangle$, which might not be equal to $\langle\{1\}, \emptyset, 1\rangle$ in T in general.

3.2. Definition of terraces

A sum of two preterraces as an element of T is not generally represented by a preterrace. We need another definition.

Definition 3.2. A finite set $\{p_1, \dots, p_l\}$ is called a *terrace* on $K[A_1, \dots, A_n]$ if each p_i $(i = 1, \dots, l)$ is a preterrace on $K[A_1, \dots, A_n]$ such that $\mathrm{supp}(p_i) \neq \emptyset$ and $\mathrm{supp}(p_i) \cap \mathrm{supp}(p_j) = \emptyset$ for any distinct $i, j \in \{1, \dots, l\}$. The *support* of a terrace t is defined by

$$\mathrm{supp}(t) = \bigcup_{p\in t} \mathrm{supp}(p) \subseteq K^n.$$

For a given terrace t and a sequence $(a_1, \dots, a_n) \in K^n$, we define

$$t(a_1, \dots, a_n) = \begin{cases} p(a_1, \dots, a_n), & \text{if } \exists p \in t)(a_1, \dots, a_n) \in \mathrm{supp}(p),\\ 0, & \text{otherwise.}\end{cases}$$

(The well-definedness is derived from the disjunctiveness of the supports of the preterraces.) Hence, we consider t as an element of $K^{(K^m)}$, actually it is an element of T since t represents $p_1 + \cdots + p_l$ in T. Intuitively a terrace is a representation of an element of T as a finite set of pairs of a rational function and a partition of K^m such that the rational function is not equal to 0 everywhere on its partition.

For a given finite set of preterraces, we can decide whether it forms a terrace or not by using the following algorithm **PreterraceIsZERO**. Indeed, for two given preterraces p and q, $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset$ iff **PreterraceIsZERO** $(p \cdot q)$ returns True.

Algorithm (PreterraceIsZERO).

Specification: PreterraceIsZERO(P)

check whether a preterrace P satisfies $\mathrm{supp}(P) = \emptyset$ or not

Input: P a preterrace on $K[A_1, \ldots, A_n]$
Output: return True if $\mathrm{supp}(P) = \emptyset$, return False otherwise

```
⟨S, T, R⟩ := P
IF V(S) = V(T) THEN
  RETURN True
ELSE
  RETURN False
```

For a given preterrace p, we see that $p(a_1, \ldots, a_n) \neq 0$ for some $(a_1, \ldots, a_n) \in K^n$ if and only if $\mathrm{supp}(p) \neq \emptyset$ by the definition of preterraces. So the previous algorithm works as we desire.

The addition $t_1 + t_2$, the multiplication $t_1 \cdot t_2$, and the inverse t_1^{-1} of terraces t_1 and t_2 as elements of T are given as follows:

(1) $(t_1 + t_2)(a_1, \ldots, a_n) = t_1(a_1, \ldots, a_n) + t_2(a_1, \ldots, a_n)$,
(2) $(t_1 \cdot t_2)(a_1, \ldots, a_n) = t_1(a_1, \ldots, a_n) \cdot t_2(a_1, \ldots, a_n)$,
(3) $t_1^{-1}(a_1, \ldots, a_n) = \begin{cases} 1/t_1(a_1, \ldots, a_n), & \text{if } t_1(a_1, \ldots, a_n) \neq 0, \\ 0, & \text{if } t_1(a_1, \ldots, a_n) = 0. \end{cases}$

We will define $t_1 + t_2$, $t_1 \cdot t_2$, and t_1^{-1} as terraces satisfying these properties. For the addition of two terraces t_1 and t_2, we require that,

$$(t_1 + t_2)(a_1, \ldots, a_n) = \begin{cases} t_1(a_1, \ldots, a_n) + t_2(a_1, \ldots, a_n), & \text{if } (a_1, \ldots, a_n) \in \mathrm{supp}(t_1) \cap \mathrm{supp}(t_2), \\ t_1(a_1, \ldots, a_n), & \text{if } (a_1, \ldots, a_n) \in \mathrm{supp}(t_1) \backslash \mathrm{supp}(t_2), \\ t_2(a_1, \ldots, a_n), & \text{if } (a_1, \ldots, a_n) \in \mathrm{supp}(t_2) \backslash \mathrm{supp}(t_1), \\ 0, & \text{otherwise,} \end{cases}$$

for any $(a_1, \ldots, a_n) \in K^n$.

We first concentrate on the case that t_1 and t_2 are singletons of preterraces, say $t_1 = \{p_1\}$ and $t_2 = \{p_2\}$ where $p_1 = \langle s_1, t_1, r_1 \rangle$ and $p_2 = \langle s_2, t_2, r_2 \rangle$. Note that $\mathrm{supp}(t_1) = \mathrm{supp}(p_1)$ and $\mathrm{supp}(t_2) = \mathrm{supp}(p_2)$.

Consider the case $(a_1, \ldots, a_n) \in \mathrm{supp}(t_1) \cap \mathrm{supp}(t_2)$. Although, as we saw, $\mathrm{supp}(t_1) \cap \mathrm{supp}(t_2) = \mathrm{supp}(p_1) \cap \mathrm{supp}(p_2) = V(t_1 \cup t_2) \backslash V(\mathbf{Prod}(s_1 \cup t_2, s_2 \cup t_1))$, the triple $\langle \mathbf{Prod}(s_1 \cup t_2, s_2 \cup t_1), t_1 \cup t_2, r_1 + r_2 \rangle$ might not form a preterrace, since $r_1(a_1, \ldots, a_n) + r_2(a_1, \ldots, a_n) = 0$ may occur for some $(a_1, \ldots, a_n) \in \mathrm{supp}(t_1) \cap \mathrm{supp}(t_2)$. So, we shrink the support in order to ensure the definition of preterraces. Present $r_1 + r_2$ as an irreducible form g/h as an element of $K(A_1, \ldots, A_n)$. Note that we already have that $V(\{h\}) \cap (V(t_1 \cup t_2) \backslash V(\mathbf{Prod}(s_1 \cup t_2, s_2 \cup t_1))) = \emptyset$ by the definition of preterraces. Let

$$p^{\cap}_{p_1, p_2} = \langle \mathbf{Prod}(\mathbf{Prod}(s_1 \cup t_2, s_2 \cup t_1), g), t_1 \cup t_2, g/h \rangle,$$

then $p^{\cap}_{p_1, p_2}$ forms a preterrace, and we have

$$\begin{aligned} p^{\cap}_{p_1, p_2}(a_1, \ldots, a_n) &= r_1(a_1, \ldots, a_n) + r_2(a_1, \ldots, a_n) \\ &= t_1(a_1, \ldots, a_n) + t_2(a_1, \ldots, a_n) \end{aligned}$$

for any $(a_1, \ldots, a_n) \in \mathrm{supp}(t_1) \cap \mathrm{supp}(t_2)$.

For the case $(a_1, \ldots, a_n) \in \mathrm{supp}(t_1)\backslash\mathrm{supp}(t_2)$, using the property $V(s_1) \subseteq V(t_1)$ and $V(s_2) \subseteq V(t_2)$, we can check the following equation by easy calculation of elementary set theory:

$$\begin{aligned}
\mathrm{supp}(t_1)\backslash\mathrm{supp}(t_2) &= \mathrm{supp}(p_1)\backslash\mathrm{supp}(p_2)\\
&= (V(t_1)\backslash V(s_1))\backslash(V(t_2)\backslash V(s_2))\\
&= (V(t_1)\backslash(V(s_1) \cup V(t_2))) \sqcup ((V(t_1) \cap V(s_2))\backslash V(s_1))\\
&= (V(t_1)\backslash(V(s_1) \cup (V(t_1) \cap V(t_2)))\\
&\quad \sqcup ((V(t_1) \cap V(s_2))\backslash(V(s_1) \cap V(s_2))),
\end{aligned}$$

where $a \sqcup b$ denotes $a \cup b$ with the property $a \cap b = \emptyset$. Then we have

$$\begin{aligned}
&V(t_1) \subseteq V(s_1) \cup (V(t_1) \cap V(t_2)),\\
&V(t_1) \cap V(s_2) \subseteq V(s_1) \cap V(s_2).
\end{aligned}$$

So the following two triples are preterraces:

$$\begin{aligned}
&p^{\backslash,(1)}_{p_1,p_2} = \langle \mathbf{Prod}(s_1, t_1 \cup t_2), t_1, r_1\rangle,\\
&p^{\backslash,(2)}_{p_1,p_2} = \langle s_1 \cup s_2, t_1 \cup s_2, r_1\rangle.
\end{aligned}$$

Furthermore we have

$$t^{\backslash}_{p_1,p_2} = \{p \in \{p^{\backslash,(1)}_{p_1,p_2}, p^{\backslash,(2)}_{p_1,p_2}\} : \mathrm{supp}(p) \neq \emptyset\}$$

forms a terrace and

$$t^{\backslash}_{p_1,p_2}(a_1, \ldots, a_n) = r_1(a_1, \ldots, a_n) = t_1(a_1, \ldots, a_n)$$

for any $(a_1, \ldots, a_n) \in \mathrm{supp}(t_1)\backslash\mathrm{supp}(t_2) = \mathrm{supp}(p^{\backslash,(1)}_{p_1,p_2}) \sqcup \mathrm{supp}(p^{\backslash,(2)}_{p_1,p_2})$.

For the case $(a_1, \ldots, a_n) \in \mathrm{supp}(t_2)\backslash\mathrm{supp}(t_1)$, we define two preterraces $p^{\backslash,(1)}_{p_2,p_1}$ and $p^{\backslash,(2)}_{p_2,p_1}$ in a similar fashion to the above case

$$\begin{aligned}
&p^{\backslash,(1)}_{p_2,p_1} = \langle \mathbf{Prod}(s_2, t_1 \cup t_2), t_2, r_2\rangle,\\
&p^{\backslash,(2)}_{p_2,p_1} = \langle s_1 \cup s_2, t_2 \cup s_1, r_2\rangle.
\end{aligned}$$

Now the finite set

$$t = \{p \in \{p^{\cap}_{p_1,p_2}, p^{\backslash,(1)}_{p_1,p_2}, p^{\backslash,(2)}_{p_1,p_2}, p^{\backslash,(1)}_{p_2,p_1}, p^{\backslash,(2)}_{p_2,p_1}\} \ : \ \mathrm{supp}(p) \neq \emptyset\}$$

of preterraces forms a terrace and satisfy

$$t(a_1, \ldots, a_n) = t_1(a_1, \ldots, a_n) + t_2(a_1, \ldots, a_n)$$

for any $(a_1, \ldots, a_n) \in K^n$.

Using these notations, we define an additive operation on the set of the terraces. The following algorithm computes the addition of two terraces:

Algorithm (**TerraceAdd**).

Specification: $T \leftarrow$ **TerraceAdd**(T_1, T_2)
Input: T_1, T_2 terraces on $K[A_1, \ldots, A_n]$

Output: T a terrace on $K[A_1, \ldots, A_n]$

```
R := ∅
For each ⟨s, t, r⟩ ∈ T1 ∩ T2 DO
  R := R ∪ {⟨s, t, 2 · r⟩}
  T1 := T1 \ {⟨s, t, r⟩}
  T2 := T2 \ {⟨s, t, r⟩}
END
R := R ∪ T1 ∪ T2

T := ∅
WHILE R ≠ ∅ DO
  take p1 ∈ R
  Found := false
  FOR each p2 ∈ R \ {p1} DO
    IF (Found = false and
     (PreterraceIsZERO(p1 · p2) does not hold)) THEN
      Found := true
      R := R \ {p1, p2}
      S := {p^∩_{p1,p2}, p^{\,(1)}_{p1,p2}, p^{\,(2)}_{p1,p2}, p^{\,(1)}_{p2,p1}, p^{\,(2)}_{p2,p1}}
      FOR each p ∈ S DO
        IF PreterraceIsZERO(p) does not hold THEN
          R := R ∪ {p}
        ENDIF
      END
    ENDIF
  END
  IF Found = false THEN
    T := T ∪ {p1}
    R := R \ {p1}
  ENDIF
END
RETURN T
```

We define the terrace $t_1 + t_2$ as an output of **TerraceAdd** (t_1, t_2). It is easy to check that property 1 holds:

1. $(t_1 + t_2)(a_1, \ldots, a_n) = t_1(a_1, \ldots, a_n) + t_2(a_1, \ldots, a_n)$.

The definition of multiplication is rather simple. The following algorithm computes the multiplication of two terraces.

Algorithm (TerraceMul).

Specification: $T \leftarrow$ **TerraceMul**(T_1, T_2)
Input: T_1, T_2 terraces on $K[A_1, \ldots, A_n]$

Output: T a terrace on $K[A_1, \dots, A_n]$

$T := \emptyset$
FOR each $p_1 \in T_1$ and $p_2 \in T_2$ DO
  $p := p_1 \cdot p_2$
  IF **PreterraceIsZERO**(p) does not hold THEN
    $T := T \cup \{p\}$
  ENDIF
END
RETURN T

We define a terrace $t_1 \cdot t_2$ as an output of **TerraceMul** (t_1, t_2). It is easy to check that property 2 holds:

2. $(t_1 \cdot t_2)(a_1, \dots, a_n) = t_1(a_1, \dots, a_n) \cdot t_2(a_1, \dots, a_n)$.

For an arbitrary terrace t, the inverse t^{-1} of t is defined by $t^{-1} = \{p^{-1} : p \in t\}$. It is trivial that t^{-1} forms a terrace and that property 3 holds:

3. $t^{-1}(a_1, \dots, a_n) = \begin{cases} 1/t(a_1, \dots, a_n), & \text{if } t(a_1, \dots, a_n) \neq 0, \\ 0, & \text{if } t(a_1, \dots, a_n) = 0. \end{cases}$

Now we have defined algorithms to compute operations on the terraces satisfying properties 1, 2, 3.

We let $\textsc{Ter} = \textsc{Ter}(K[A_1, \dots, A_n])$ be the set of terraces on $K[A_1, \dots, A_n]$. We should note that, for a terrace $t \in \textsc{Ter}$, there are infinitely many terraces $t' \in \textsc{Ter}$ such that $t(a_1, \dots, a_n) = t'(a_1, \dots, a_n)$ for any $(a_1, \dots, a_n) \in K^n$.

Example 3.2. We consider the following two terraces t and t' on $\mathbb{C}[A, B]$:

$$t = \{\langle \{5A + B\}, \emptyset, 5A + B\rangle\},$$
$$t' = \{\langle \{B\}, \{A\}, B\rangle, \langle \{5A^2 + AB\}, \emptyset, 5A + B\rangle\}.$$

Then we have

$$t'(A, B) = \begin{cases} B, & \text{if } (A, B) \in V(\{A\}) \backslash V(\{B\}), \text{ i.e., } A = 0, B \neq 0, \\ 5A + B, & \text{if } (A, B) \in \mathbb{C}^2 \backslash V(\{5A^2 + AB\}), \\ & \qquad \text{i.e., } A \neq 0, 5A + B \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

So $t(a, b) = t'(a, b)$ for any $a, b \in \mathbb{C}$.

We define a binary relation $\sim$ on $\textsc{Ter}$ by

$$t \sim t' \iff t + \{\mathrm{pre}(-1)\} \cdot t' = \emptyset.$$

Then the relation $\sim$ is a computable equivalence relation on $\textsc{Ter}$.

Proposition 3.3. *For arbitrary two terraces t and t' on $K[A_1, \dots, A_n]$, $t \sim t'$ if and only if*

$$t(a_1, \dots, a_n) = t'(a_1, \dots, a_n)$$

for any $(a_1, \dots, a_n) \in K^n$.

Proof. Let $t_0 = t + \{\mathrm{pre}(-1)\} \cdot t'$. We want to show that $t_0 = \emptyset$ iff $(\forall \bar{a} \in K^n)\, t(\bar{a}) = t'(\bar{a})$.

First we assume that $t_0 = \emptyset$. We fix an arbitrary $\bar{a} \in K^n$. Then $t_0(a) = 0$. And we have

$$0 = t_0(\bar{a}) = t(\bar{a}) + (\{\mathrm{pre}(-1)\})(\bar{a}) \cdot t'(\bar{a}) = t(a) - t'(a).$$

So $t(a) = t'(a)$.

For the converse, we assume that $p \in t_0$. Then, by the definition of terraces, we have that $\mathrm{supp}(p) \neq \emptyset$. If we fix $\bar{a} \in \mathrm{supp}(p)$, we have $p(\bar{a}) \neq 0$ by the definition of preterraces. Thus $0 \neq p(\bar{a}) = t_0(\bar{a}) = t(\bar{a}) - t'(\bar{a})$. So $t(\bar{a}) \neq t'(\bar{a})$. $\square$

It should be noted that there is only one terrace, namely $\emptyset$, which represents 0. We denote the set of the equivalence class $\mathrm{TER}(K[A_1, \ldots, A_n])/\sim$ by $T_{(A_1,\ldots,A_n)}$. For a equivalence class $[t]_\sim \in T_{(A_1,\ldots,A_n)}$ and a sequence $(a_1, \ldots, a_n) \in K^n$, we define $[t]_\sim(a_1, \ldots, a_n) = t(a_1, \ldots, a_n) \in K$. The previous proposition witnesses the well-definedness of $[t]_\sim(a_1, \ldots, a_n) \in K$. Moreover, using the proposition, we can define addition, multiplication, and inverse on $T_{(A_1,\ldots,A_n)}$ by $[t]_\sim + [t']_\sim = [t+t']_\sim$, $[t]_\sim \cdot [t']_\sim = [t \cdot t']_\sim$, and $[t]_\sim^{-1} = [t^{-1}]_\sim$ for $t, t' \in \mathrm{TER}(K[A_1, \ldots, A_n])$.

We can easily check that $T_{(A_1,\ldots,A_n)}$ is a Von Neumann regular ring, actually it is isomorphic to the ring T defined at the beginning of this section as the closure of the image $\varphi[K[A_1, \ldots, A_n]]$.

For a given polynomial $f \in K[A_1, \ldots, A_n]$, we define the corresponding equivalence class on terraces $\mathrm{ter}_T(f) \in T_{(A_1,\ldots,A_n)}$ by

$$\mathrm{ter}_T(f) = \begin{cases} [\{\mathrm{pre}(f)\}]_\sim, & \text{if } f \in K[A_1, \ldots, A_n] \backslash \{0\}, \\ [\emptyset]_\sim, & \text{if } f = 0. \end{cases}$$

Note that $f(a_1, \ldots, a_n) = \mathrm{ter}_T(f)(a_1, \ldots, a_n)$ for any $(a_1, \ldots, a_n) \in K^n$. So we often identify f with $\mathrm{ter}_T(f)$ if there is no confusion.

4. ACGB

We give an alternative comprehensive Gröbner bases in this section. Let K be an algebraically closed field, TER be the set of the terraces on $K[A_1, \ldots, A_m]$ where $A_1, \ldots, A_m$ are variables, $T = \mathrm{TER}/\sim$, and $\mathrm{ter}_T : K[A_1, \ldots, A_m] \to T$ be the corresponding embedding. As we have seen in Section 3, $T = T_{(A_1,\ldots,A_m)}$ is a commutative Von Neumann regular ring.

Definition 4.1. We extend ter_T to the embedding

$$\mathrm{ter}_T : K[A_1, \ldots, A_m, X_1, \ldots, X_n] \to T[X_1, \ldots, X_n]$$

by

$$\mathrm{ter}_T(f_1\alpha_1 + \cdots + f_l\alpha_l) = \mathrm{ter}_T(f_1)\alpha_1 + \cdots + \mathrm{ter}_T(f_l)\alpha_l$$

where $f_1, \ldots, f_l \in K[A_1, \ldots, A_m]$ and $\alpha_1, \ldots, \alpha_l$ are terms of $X_1, \ldots, X_n$.

Definition 4.2. For each

$$f(X_1, \ldots, X_n) = c_1\alpha_1 + \cdots + c_l\alpha_l \in T[X_1, \ldots, X_n]$$

and elements $a_1, \ldots, a_m \in K$, we define

$$f_{(a_1,\ldots,a_m)}(X_1, \ldots, X_m) \in K[X_1, \ldots, X_m]$$

by

$$f_{(a_1,\ldots,a_m)}(X_1, \ldots, X_n) = c_1(a_1, \ldots, a_m)\alpha_1 + \cdots + c_l(a_1, \ldots, a_m)\alpha_l$$

where $c_i \in T$ and α_i are terms of $X_1, \ldots, X_n$.

We can calculate the stratified Gröbner basis for a given finite set of polynomials over a computable commutative Von Neumann regular ring. Now we prove the following theorem.

Theorem 4.3. *For an algebraically closed field K, let T be the canonical set of equivalence classes on the terraces on $K[A_1, \ldots, A_m]$, and let $\mathrm{ter}_T : K[A_1, \ldots, A_m, X_1, \ldots, X_n] \to T[X_1, \ldots, X_n]$ be the corresponding embedding. For a given set $F = \{f_1(A_1, \ldots, A_m, X_1, \ldots, X_n), \ldots, f_k(A_1, \ldots, A_m, X_1, \ldots, X_n)\} \subseteq K[A_1, \ldots, A_m, X_1, \ldots, X_n]$, we let $\mathrm{ter}_T(F) = \{\mathrm{ter}_T(f_i) : i = 1, \ldots, k\} \subseteq T[X_1, \ldots, X_n]$, and let $G = \{g_1(X_1, \ldots, X_n), \ldots, g_l(X_1, \ldots, X_n)\}$ be a Gröbner basis of $\mathrm{ter}_T(F)$ in $T[X_1, \ldots, X_n]$ such that each element g_i is Boolean closed. Then we have the following properties:*

(1) *For arbitrary $a_1, \ldots, a_m \in K$, $G_{(a_1,\ldots,a_m)} = \{g_{1_{(a_1,\ldots,a_m)}}(X_1, \ldots, X_n), \ldots, g_{l_{(a_1,\ldots,a_m)}}(X_1, \ldots, X_n)\}\backslash\{0\}$ is a Gröbner basis of the ideal generated by $F(a_1, \ldots, a_m) = \{f_1(a_1, \ldots, a_m, X_1, \ldots, X_n), \ldots, f_k(a_1, \ldots, a_m, X_1, \ldots, X_n)\}$ in $K[X_1, \ldots, X_n]$. Moreover, $G_{(a_1,\ldots,a_m)}$ becomes a reduced Gröbner basis, in case G is stratified.*

(2) *For any polynomial $h(X_1, \ldots, X_n) \in T[X_1, \ldots, X_n]$, we have*

$$(h\downarrow_G)_{(a_1,\ldots,a_m)}(X_1, \ldots, X_n) = h_{(a_1,\ldots,a_n)}(X_1, \ldots, X_n)\downarrow_{G_{(a_1,\ldots,a_m)}}.$$

Proof. We fix $a_1, \ldots, a_m \in K$ and denote $\bar{a}$ and $\bar{X}$ for "$a_1, \ldots, a_m$" and "$X_1, \ldots, X_n$" respectively.

It is easy to check that $G_{\bar{a}}$ and $F(\bar{a})$ generate the same ideal in $K[\bar{X}]$. In order to see that $G_{\bar{a}}$ is a Gröbner basis in $K[\bar{X}]$, it suffices to show that $SP(f, g) \xrightarrow{*}_{G_{\bar{a}}} 0$ for any pair f and g of polynomials in $G_{\bar{a}}$. This property follows from the following two claims.

Claim 1. *The homomorphism also preserves monomial reductions, that is $p_{\bar{a}}(\bar{X}) \xrightarrow{*}_{G_{\bar{a}}} q_{\bar{a}}(\bar{X})$ in case $p(\bar{X}) \to_G q(\bar{X})$.*

Proof of Claim 1. If $p(\bar{X}) \to_{g(\bar{X})} q(\bar{X})$, then p, g and q must have the following forms:

$$p(\bar{X}) = b\alpha\beta + p'(\bar{X}),$$
$$g(\bar{X}) = c\alpha + g'(\bar{X}),$$
$$q(\bar{X}) = p(\bar{X}) - bc^{-1}\beta g(\bar{X}).$$

An instantiation by $a_1, \ldots, a_m$ yields the following equations:

$$p_{\bar{a}}(\bar{X}) = b(\bar{a})\alpha\beta + p'_{\bar{a}}(\bar{X}),$$
$$g_{\bar{a}}(\bar{X}) = c(\bar{a})\alpha + g'_{\bar{a}}(\bar{X}),$$
$$q_{\bar{a}}(\bar{X}) = p_{\bar{a}}(\bar{X}) - b(\bar{a})c^{-1}(\bar{a})\beta g_{\bar{a}}(\bar{X}).$$

When $b(\bar{a})c(\bar{a}) \neq 0$, $c(\bar{a}) \neq 0$, so the leading term of g does not vanish. In this case, we have $p_{\bar{a}}(\bar{X}) \rightarrow_{g_{\bar{a}}(\bar{X})} q_{\bar{a}}(\bar{X})$. When $b(\bar{a})c(\bar{a}) = 0$, we have $b(\bar{a})c^{-1}(\bar{a}) = 0$. In this case, $p_{\bar{a}}(\bar{X})$ and $q_{\bar{a}}(\bar{X})$ are identical.

In either case, we have

$$p_{\bar{a}}(\bar{X}) \overset{*}{\rightarrow}_{g_{\bar{a}}(\bar{X})} q_{\bar{a}}(\bar{X}),$$

from which the assertion of the claim follows. □

Claim 2. *The homomorphism also preserves S-polynomial construction, that is*

$$SP(f_{\bar{a}}(\bar{X}), g_{\bar{a}}(\bar{X})) = SP(f, g)_{\bar{a}}(\bar{X})$$

for any pair f and g of G.

Proof of Claim 2. We first show that $SP(f, g)_{\bar{a}}(\bar{X}) = 0$ if $f_{\bar{a}}(X) = 0$ or $g_{\bar{a}}(X) = 0$. We first assume that $f_{\bar{a}}(X) = 0$. Since G is reduced, we know that f is Boolean closed, and so that $lc(f)^*(\bar{a}) = 0$. So $SP(f, g) = SP(lc(f)^* f, g) = lc(f)^* SP(f, g)$ implies that

$$SP(f, g)_{\bar{a}}(\bar{X}) = lc(f)^*(\bar{a}) SP(f, g)_{\bar{a}}(\bar{X}) = 0.$$

We also have $SP(f, g)_{\bar{a}}(\bar{X}) = 0$ if $g_{\bar{a}}(\bar{X}) = 0$ in the same way.

Next we assume that $f_{\bar{a}}(\bar{X}) \neq 0$ and $g_{\bar{a}}(\bar{X}) \neq 0$. We say $lm(f) = b\alpha\gamma$ and $lm(g) = c\beta\gamma$ where b and c are coefficients and α, β, and γ are terms with $\mathrm{GCD}(\alpha, \beta) = 1$. Now we note that f and g are Boolean closed since $f, g \in G$, and so $b(\bar{a}) \neq 0$ and $c(\bar{a}) \neq 0$ from the assumption. Thus $lm(f_{\bar{a}}) = b(\bar{a})\alpha\gamma$ and $lm(g_{\bar{a}}) = c(\bar{a})\beta\gamma$. Then we have $SP(f_{\bar{a}}, g_{\bar{a}}) = c(\bar{a})\beta f_{\bar{a}} - b(\bar{a})\alpha g_{\bar{a}} = (b\beta f - a\alpha g)_{\bar{a}} = SP(f, g)_{\bar{a}}$. □

The last assertion of 1 follows immediately by the definition of a stratified Gröbner basis.

In order to prove 2, we observe the following claim.

Claim 3. *$h_{\bar{a}}(\bar{X})$ is irreducible by $G_{\bar{a}}$ in $K[\bar{X}]$ for any polynomial $h(\bar{X})$ in $T[\bar{X}]$ which is irreducible by G and $\bar{a} \in K$.*

Proof of Claim 3. If $h_{\bar{a}}$ were reducible by $g_{\bar{a}}$ for some $\bar{a} \in K$ and $g \in G$, there were a monomial $c\alpha$ of h such that $lt(g) \mid \alpha$ and that $c_{\bar{a}} \cdot lc(g)_{\bar{a}} \neq 0$, and so $c \cdot lc(g) \neq 0$.

Then, we note that a polynomial h' is irreducible by g' if and only if $c' \cdot lc(g') = 0$ for any monomial $c'\alpha'$ of h' such that $lt(g')|\alpha'$.

Therefore we had that $h(\bar{X})$ were reducible by g. □

Now, by Claims 1 and 3, we have 2. □

By property 1, G can be considered as a kind of comprehensive Gröbner basis where $A_1, \ldots, A_m$ are parameters, and so we call G an *ACGB*. Note that in the standard comprehensive Gröbner bases, we can not define monomial reductions before instantiation. In our algorithm, we can define monomial reductions, furthermore they are preserved by any instantiation.

5. ACGB and CGB

In this section, we give an example of ACGB G such that there does not exist a comprehensive Gröbner basis G' that generates the same ideal as G for any instantiation.

Let G be a set $\{(V(\emptyset)\backslash V(\{A\}), 1)X, (V(\{A\})\backslash V(\{1\}), 1)\}$ of polynomials in a polynomial ring $T_{(A)}[X]$, where $T_{(A)}$ is a Von Neumann regular ring of the equivalence classes of the terraces on $K[A]$ with K an algebraically closed infinite field. Clearly G is a Gröbner basis in $T_{(A)}[X]$. Note that G generates an ideal $\langle X\rangle$ when A takes a non-zero value of K and $\langle 1\rangle$ when A takes a value 0.

For this G we show that there does not exists a finite set G' of polynomials in $K[A, X]$ such that G' becomes a Gröbner basis and generates the same ideal as G in $K[X]$ for any instantiation of A.

Proof. Let $G' = \{f_1(A, X), \ldots, f_l(A, X)\}$ and suppose that $\{f_1(a, X), \ldots, f_l(a, X)\}$ is a Gröbner basis and generates the ideal $\langle X\rangle$ when $a \neq 0$ and the ideal $\langle 1\rangle$ when $a = 0$. Since $\{f_1(0, X), \ldots, f_l(0, X)\}$ is a Gröbner basis, it must contain a non-zero constant $c \in K$. We can assume $f_1(0, X) = c$ w.l.o.g. Hence, $f_1(A, X)$ can be expressed as $f_1(A, X) = g(A, X)A + c$ for some polynomial $g(A, X)$ in $K[A, X]$. Express $g(A, X)$ further as $g(A, X) = g_1(A, X)X + g_2(A)$ with polynomials $g_1(A, X)$ in $K[A, X]$ and $g_2(A)$ in $K[A]$. So, we have $f_1(A, X) = g_1(A, X)AX + g_2(A)A + c$. Since $\{f_1(a, X), \ldots, f_l(a, X)\}$ is a Gröbner basis of the ideal $\langle X\rangle$ when $a \neq 0$, there must exist i such that $f_i(a, X) = dX$ for some non-zero constant $d \in K$. Certainly i is not equal to 1. Hence, we have $g_2(a)a + c = f_1(a, X) - f_i(a, X)g_1(a, X)a/d \in \langle X\rangle$ whenever $a \neq 0$. Since K is infinite, there must exist a non-zero element a of K such that $g_2(a)a + c \neq 0$, which produces a contradiction, since $\langle X\rangle$ contains a non-zero constant of K. $\square$

For any comprehensive Gröbner basis G', clearly there exists an ACGB G such that they generate the same ideal for any instantiation.

In this sense, we can say the class of ACGB is wider than the class of Weispfenning's original comprehensive Gröbner bases.

6. Applications and examples

We implemented the algorithm to compute ACGB in the case K is the field of the complex numbers $\mathbb{C}$. In this section, we give some computation examples of our implementation.

Example 6.1. Find the reduced Gröbner basis for the ideal generated by the following system of polynomials of the variables x, y with parameters a, b:

$$\begin{cases} ax^2y + 1, \\ bxy + abx + b. \end{cases}$$

In order to solve them simultaneously, compute a Gröbner basis of the ideal x in $T_{(a,b)}[x, y]$ where $T_{(a,b)}$ is the Von Neumann regular ring of equivalence classes on the terraces on $\mathbb{C}[a, b]$. Our program written in Risa/Asir Noro and Takeshima (1992) produces the following Gröbner basis in the graded reverse lexicographic order with $x > y$:

```
[[(V[a],1)]*1,
 [(V[0]-V[b*a],1)]*x+[(V[0]-V[b*a],(-1)/(-a^2))]*y+[(V[0]-V[b*a],(-2)/(-a))]*1,
 [(V[0]-V[b*a],1)]*y^2+[(V[0]-V[-b*a],3*a)]*y+[(V[0]-V[b*a],a^2)]*1,
 [(V[b*a]-V[a],1)]*y*x^2+[(V[b*a]-V[a],(1)/(a))]*1]
```

In this output, `(V[` t `] - V[` s `],` r `)` corresponds to the preterrace $\langle s, t, r\rangle$. So the above output means that the reduced Gröbner basis is

$$\begin{cases} \langle 1\rangle, & \text{if } a = 0,\\ \langle x + \frac{1}{a^2}y - \frac{2}{a}, y^2 + 3ay + a^2\rangle, & \text{if } ab \neq 0,\\ \langle x^2 y + \frac{1}{a}\rangle, & \text{if } ab = 0, a \neq 0. \end{cases}$$

Example 6.2. Let $h(a, b, x, y) \in \mathbb{C}[a, b, x, y]$ be such that

$$h(a, b, x, y) = (a^2 + b)x^3y^2 + 5a^2xy + (a - b^2).$$

Then, for each $a, b \in \mathbb{C}$, find the normal form under the reduced Gröbner basis for the ideal generated by the system of polynomials which appeared in Example 6.1:

$$\begin{cases} ax^2y + 1,\\ bxy + abx + b. \end{cases}$$

Our program calculates the normal form $nf(h)$ of h under the ACGB which we calculated at Example 6.1 as follows:

```
[(V[b*a]-V[b*a,5*a^2-a],(5*a^3-a^2-b)/(a))]*y*x+
[(V[0]-V[5*b*a^6-b*a^5+5*b^2*a^4-2*b^2*a^3-b^3*a],(5*a^3-a^2-b)/(a^2)),
 (V[-b*a^3-b^2*a]-V[-b*a],5*a)]*y+
[(V[b*a^3+b^2*a]-V[b*a^3+b^2*a,(-2*b^2-5*b)*a^2+(b^4+5*b^3-b)*a],5*a^2+a-b^2),
 (V[0]-V[-25*b*a^10+(25*b^3+5*b)*a^9-25*b^2*a^8+(-5*b^5+25*b^4-b^3+15*b^2)*a^7+
  (b^5-10*b^4-b^2)*a^6+(-5*b^6-b^4+10*b^3)*a^5+(2*b^6-10*b^5-2*b^3)*a^4+
  b^5*a^3+(b^7-b^4)*a^2+b^6*a],(-5*a^3+b^2*a+b)/(-a)),
 (V[5*b*a^6-b*a^5+5*b^2*a^4-2*b^2*a^3-b^3*a]-V[b*a^4-b^3*a^3+b^2*a^2-b^4*a,
  5*b*a^6-b*a^5+5*b^2*a^4-2*b^2*a^3-b^3*a],a-b^2),
  (V[b*a^4-b^3*a^3+b^2*a^2-b^4*a]-V[b*a^4-b^3*a^3+b^2*a^2-b^4*a,5*b*a^6-b*a^5+
  5*b^2*a^4-2*b^2*a^3-b^3*a],(-5*a^3+a^2+b)/(-a)),
 (V[b*a]-V[a],a-b^2),(V[a-b^2,-b*a^2-a,a^3+b*a]-V[-b,a],5*a^2)]*1
```

We can get much information using $nf(h)$. For example, we know that $h \in \langle ax^2y + 1, bxy + abx + b\rangle$ if and only if $a = 0$ or $(a - b^2 = 0 \wedge 5a^3 - a^2 - b = 0 \wedge a^2 + b \neq 0)$.

Example 6.3. Find the minimal polynomial of t in the ideal $\langle x^2 - a, y^3 - a, x + y - t\rangle$ with a parameter a.

It suffices to calculate the Gröbner basis of $\langle x^2 - a, y^3 - a, x + y - t\rangle$ with a term-order such that $x, y \gg t$ for each a, and find the polynomial consisting only of t. Our program produces eight polynomials for the given polynomials $\{x^2 - a, y^3 - a, x + y - t\}$ with a parameter a in the lexicographic order with $x > y > t$. The following three polynomials are the ones which contain only t as their variables.

```
[(V[-a],1)]*t^4,

[(V[-64*a^2-27*a]-V[a],1)]*t^5+[(V[-64*a^2-27*a]-V[-a],3/8)]*t^4+
```

```
[(V[-64*a^2-27*a]-V[-a],-10/3*a)]*t^3+[(V[-64*a^2-27*a]-V[a],-13/4*a)]*t^2+
[(V[-64*a^2-27*a]-V[a],7/3*a^2-3/2*a)]*t+[(V[-64*a^2-27*a]-V[-a],-91/24*a^2)]*1,

[(V[0]-V[-64*a^2-27*a],1)]*t^6+
[(V[-4096*a^3-1536*a^2+81*a]-V[64*a^2+27*a],-9/64),
(V[0]-V[-4096*a^3-1536*a^2+81*a],-3*a)]*t^4+
[(V[0]-V[-64*a^2-27*a],-2*a)]*t^3+[(V[0]-V[28672*a^4-17984*a^3-5202*a^2+3159*a],3*a^2),
(V[-896*a^3+198*a^2+243*a]-V[-64*a^2-27*a],2/3*a^2+3/2*a),
(V[28672*a^4-17984*a^3-5202*a^2+3159*a]-V[896*a^3-198*a^2-243*a],39/32*a)]*t^2+
[(V[5248*a^3+2790*a^2+243*a]-V[64*a^2+27*a],-7/8*a^2+9/16*a),
(V[896*a^3-198*a^2-243*a]-V[-64*a^2-27*a],-41/8*a^2-9/16*a),
(V[0]-V[-73472*a^4+8172*a^3+21708*a^2+2187*a],-6*a^2)]*t+
[(V[0]-V[-64*a^3+37*a^2+27*a],-a^3+a^2)]*1.
```

Looking at these polynomials, for example, we can see that the degree of the minimal polynomial is 6 if and only if $a \neq 0, -27/64$, and that it is 5 if and only if $a = -27/64$.

We should note that such conditions are derived also by `dispgb()` of DisPGB[1] Montes (2002) or by `gsys()` of CGB[2] as below:

DisPGB:

$$\begin{aligned}
Case &= [1,\ 1],\ [a \neq 0,\ 27 + 64\,a \neq 0],\ [t^6 - 3\,t^4\,a - 2\,a\,t^3 + 3\,t^2\,a^2 - 6\,a^2\,t + a^2 - a^3,\\
&\quad -91\,a^2 + 24\,t^5 - 78\,a\,t^2 + 9\,t^4 + (56\,a^2 - 36\,a)\,t - 80\,a\,t^3 + (27\,a + 64\,a^2)\,y,\\
&\quad 91\,a^2 - 24\,t^5 + 78\,a\,t^2 - 9\,t^4 + (9\,a - 120\,a^2)\,t + 80\,a\,t^3 + (27\,a + 64\,a^2)\,x]\\
Case &= [1,\ 0],\ [a \neq 0,\ 27 + 64\,a = 0],\\
&\quad [32\,768\,t^5 + 12\,288\,t^4 + 46\,080\,t^3 + 44\,928\,t^2 + 34\,344\,t - 22\,113,\\
&\quad 13\,824\,y\,t - 5184\,y + 4096\,t^4 - 3456\,t^2 + 6912\,t - 2187,\\
&\quad 6912\,y^2 - 5184\,y + 4096\,t^4 + 3456\,t^2 + 6912\,t + 729,\ x + y - t]\\
Case &= [0],\ [a = 0],\ [t^4,\ 3\,y\,t^2 - 2\,t^3,\ y^2 - 2\,y\,t + t^2,\ x + y - t]
\end{aligned}$$

CGB:

```
{{64*a + 27 <> 0 and a <> 0,
{x**2 - a,
  x + y - t,
  y**3 - a,
  y**2 - 2*y*t + t**2 - a,
  3*y*t**2 + a*y - 2*t**3 + (2*a)*t - a,
  (8*a)*y*t - (3*a)*y - t**4 - (2*a)*t**2 + (4*a)*t + 3*a**2,
  (64*a**2 + 27*a)*y + 24*t**5 + 9*t**4 - (80*a)*t**3 - (78*a)*t**2 + (56*a**2 - 36*a)*t
   - 91*a**2,
  t**6 - (3*a)*t**4 - (2*a)*t**3 + (3*a**2)*t**2 - (6*a**2)*t - (a**3 - a**2)}},
{a <> 0 and 64*a**2 + 27*a = 0,
{x**2 - a,
  x + y - t,
  y**3 - a,
  y**2 - 2*y*t + t**2 - a,
  3*y*t**2 + a*y - 2*t**3 + (2*a)*t - a,
  (8*a)*y*t - (3*a)*y - t**4 - (2*a)*t**2 + (4*a)*t + 3*a**2,
  (64*a**2 + 27*a)*y + 24*t**5 + 9*t**4 - (80*a)*t**3 - (78*a)*t**2 + (56*a**2 - 36*a)*t
   - 91*a**2}},
{a = 0,
{x**2 - a,
  x + y - t,
  y**3 - a,
```

[1] http://www-ma2.upc.es/~montes/

[2] http://www.fmi.uni-passau.de/~redlog/cgb/

```
y**2 - 2*y*t + t**2 - a,
3*y*t**2 + a*y - 2*t**3 + (2*a)*t - a,
(8*a)*y*t - (3*a)*y - t**4 - (2*a)*t**2 + (4*a)*t + 3*a**2}}}
```

7. Computations of functional equations

The following system of polynomial equations

$$\begin{cases} f_1(A_1, \ldots, A_m, \bar{X}) = 0 \\ \quad\quad\vdots \\ f_k(A_1, \ldots, A_m, \bar{X}) = 0 \end{cases} \tag{1}$$

in $K^{(K^m)}[\bar{X}]$ can be considered as a system of polynomial equations over functions, that is each A_i represents a function from K^m to K.

In this section, we also assume that K is an algebraically closed field. Our ACGBs give us direct information for such systems. First, we can decide whether the system has a solution.

We can easily extend Hilbert weak Nullstellensatz as follows.

Theorem 7.1. *The system of Eq. (1) has a solution if and only if* $\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X})\rangle \cap K^{(K^m)} = \{0\}$.

By this theorem, we know it has a solution if and only if the ACGB of $\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X})\rangle$ does not contain a constant.

Secondly, for each polynomial $h(A_1, \ldots, A_m, \bar{X})$ we can decide whether it vanishes at every solution of (1) by the following theorem.

Theorem 7.2. *Suppose that the system of Eq. (1) has a solution. Then, for each polynomial* $h(A_1, \ldots, A_m, \bar{X})$*, it vanishes at every solution of the system if and only if the ACGB of* $\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X}), h(A_1, \ldots, A_m, \bar{X})\, y + 1\rangle$ *is* $\{1\}$. *Where* y *is a new variable distinct from* $\bar{X}$.

Proof. Note that $h(A_1, \ldots, A_m, \bar{\alpha})$ vanishes at every solution $\bar{\alpha}$ of the system (1) if and only if $h(a_1, \ldots, a_m, \bar{b})$ vanishes at every solution $\bar{b}$ of the system of polynomial equation

$$\begin{cases} f_1(a_1, \ldots, a_m, \bar{X}) = 0 \\ \quad\quad\vdots \\ f_k(a_1, \ldots, a_m, \bar{X}) = 0 \end{cases} \tag{2}$$

in $K[\bar{X}]$ for each element $a_1, \ldots, a_m$ in K. We also have that $h(a_1, \ldots, a_m, \bar{b})$ vanishes at every solution $\bar{b}$ of (2) if and only if the polynomial ideal $\langle f_1(a_1, \ldots, a_m, \bar{X}), \ldots, f_k(a_1, \ldots, a_m, \bar{X}), h(a_1, \ldots, a_m, \bar{X})y + 1\rangle$ of $K[\bar{X}]$ includes 1. Hence, $h(A_1, \ldots, A_m, \bar{\alpha})$ vanishes at every solution $\bar{\alpha}$ of the system (1) if and only if the polynomial ideal $\langle f_1(a_1, \ldots, a_m, \bar{X}), \ldots, f_k(a_1, \ldots, a_m, \bar{X}), h(a_1, \ldots, a_m, \bar{X})y + 1\rangle$ of $K[\bar{X}]$ includes 1 for each element $a_1, \ldots, a_m$ of K, which is equivalent to $\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X}), h(A_1, \ldots, A_m, \bar{X})y + 1\rangle \ni \{1\}$. $\square$

This theorem also provide a decision procedure for ideal membership problems of the polynomial ring $K^{(K^m)}[\bar{X}]$.

Theorem 7.3. *Suppose the ideal* $\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X})\rangle \cap K^{K^m} = \{0\}$. *Then,*

$$h(A_1, \ldots, A_m, \bar{X}) \in \sqrt{\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X})\rangle}$$

if and only if the ACGB of

$$\langle f_1(A_1, \ldots, A_m, \bar{X}), \ldots, f_k(A_1, \ldots, A_m, \bar{X}), h(A_1, \ldots, A_m, \bar{X})\, y + 1\rangle$$

is $\{1\}$. *Where* y *is a new variable distinct from* $\bar{X}$.

Proof. It is a direct consequence of the above theorem since Hilbert strong Nullstellensatz holds in the polynomial ring $K^{(K^m)}[\bar{X}]$ (see Theorem I.4.3. in Saracino and Weispfenning, 1975). □

8. Conclusion and remarks

Our algorithm of ACGB does not have a canonical representation in a completely syntactic form. There are infinitely many forms of equivalent terraces, although there is only one form (i.e. an empty set) to represent 0 as is mentioned in Section 2. In this paper we employed rather naive methods to handle terraces. We did not use any sophisticated technique such as polynomial factorizations or computations of radical ideals or prime(primary) ideal decompositions. We need further computational experiments to find the most effective way.

We described our work under the assumption that K is algebraically closed. But this is not indispensable. What we actually need is the computability of terraces. If we can compute terraces, then we can define and calculate ACGB. For example, when K is a real closed field, we can handle terraces using standard quantifier elimination techniques.

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