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A note on (α, β)-derivations ☆

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ABSTRACT

We show that every multiplicative (α, β)-derivation of a ring $\mathcal{R}$ is additive if there exists an idempotent e' $(e' \neq 0, 1)$ in $\mathcal{R}$ satisfying the conditions (C1)–(C3): (C1) $\beta(e')\mathcal{R}x = 0$ implies $x = 0$; (C2) $\beta(e')x\alpha(e')\mathcal{R}(1 - \alpha(e')) = 0$ implies $\beta(e')x\alpha(e') = 0$; (C3) $x\mathcal{R} = 0$ implies $x = 0$. In particular, every multiplicative (α, β)-derivation of a prime ring with a nontrivial idempotent is additive. As applications, we could decompose a multiplicative (α, β)-derivation of the algebra $M_n(\mathbb{C})$ of all the $n \times n$ complex matrices into a sum of an (α, β)-inner derivation and an (α, β)-derivation on $M_n(\mathbb{C})$ given by an additive derivation f on $\mathbb{C}$.

1. Introduction

The problem when a multiplicative mapping is additive, which was first considered by Martindale in [1], is very well-known and interesting in the ring theory. Martindale and Daif answered this problem for a multiplicative isomorphism and a multiplicative derivation in [1] and [2], respectively. Recently, the similar problems are considered for Jordan mappings on some associative algebras, such as the triangular algebras, nest algebras and standard operator algebras, etc. [3–7]. Motivated by the Daif's ideas, in this note we consider the problem whether a multiplicative (α, β)-derivation of a ring is additive. Fortunately, we can give a full answer for this question under the existence of a single fixed idempotent satisfying some properties which are similar to Daif's conditions. In particular, we

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could show that every multiplicative (α, β)-derivation of a prime ring with a nontrivial idempotent is additive.

Let $\mathcal{R}$ be an associative ring, α and β be ring automorphisms of $\mathcal{R}$. By a multiplicative (α, β)-derivation from $\mathcal{R}$ into itself, we call a mapping $d : \mathcal{R} \to \mathcal{R}$ such that

$$d(xy) = d(x)\alpha(y) + \beta(x)d(y), \quad \text{for all } x, y \text{ in } \mathcal{R}. \tag{1}$$

In addition, if d is additive, we call d an (α, β)-derivation of $\mathcal{R}$. If there exists $x_0 \in \mathcal{R}$ such that $d(x) = \beta(x)x_0 - x_0\alpha(x)$ holds for each x in $\mathcal{R}$, then d is called an (α, β)-inner derivation. Obviously, if α and β are the identity mapping id on $\mathcal{R}$, then a multiplicative (id, id)-derivation is an ordinary multiplicative derivation defined in [2].

Similarly, we can define the notion of a multiplicative (α, β)-derivation on an associative algebra $\mathcal{A}$ over $\mathbb{C}$, in which we only assume that α and β are algebraic automorphisms of $\mathcal{A}$. It is natural to consider the linearity problem of a multiplicative (α, β)-derivation of $\mathcal{A}$. In this note, we will describe the problem for the multiplicative (α, β)-derivations of the algebra $M_n(\mathbb{C})$ of all the $n \times n$ complex matrices. By [2] or Corollary 2 in this note, every multiplicative (id, id)-derivation on $M_n(\mathbb{C})$ is additive; and in [8], Šemrl obtained the existence of additive derivations without linearity on $M_n(\mathbb{C})$. In section 3, we show that each linear (α, β)-derivation on $M_n(\mathbb{C})$ is inner, and prove that each multiplicative (α, β)-derivation on $M_n(\mathbb{C})$ can be expressed a sum of an (α, β)-inner derivation and an additive (α, β)-derivation induced by an additive derivation of $\mathbb{C}$.

2. Additivity of multiplicative (α, β)-derivations on rings

In this section, we have the following main result.

Theorem 1. *Let $\mathcal{R}$ be a ring (not necessarily containing an identity), α and β be ring automorphisms of $\mathcal{R}$. Suppose that there exists an idempotent e ($e \neq 0, e \neq 1$) such that the following conditions hold:*

(C1) $\tilde{e}\mathcal{R}x = 0$ *implies* $x = 0$ (*and hence* $\mathcal{R}x = 0$ *implies* $x = 0$);
(C2) $\tilde{e}xe\mathcal{R}(1 - e) = 0$ *implies* $\tilde{e}xe = 0$ (*and hence* $\tilde{e}xe\mathcal{R} = 0$ *implies* $\tilde{e}xe = 0$);
(C3) $x\mathcal{R} = 0$ *implies* $x = 0$;

where $\tilde{e} = \beta\alpha^{-1}(e)$. Then every multiplicative (α, β)-derivation of $\mathcal{R}$ is additive.

Remark. Let $\mathcal{R}$ be a ring. For convenience, we replace $y - xy$ with $(1 - x)y$ for x, y in $\mathcal{R}$. Hence for an automorphism α of $\mathcal{R}$, the equality $\alpha((1 - x)y) = (1 - \alpha(x))\alpha(y)$ is well-defined. Let d be a multiplicative (α, β)-derivation on $\mathcal{R}$. If $\mathcal{R}$ has an identity I, then $d(I) = 0$. If $\mathcal{R}$ has no identity, we let $\mathcal{R}_1 = \{(x, n) : x \in \mathcal{R}, n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the integer ring. Then, under the following addition and multiplication:

$$(x, n) + (y, m) = (x + y, n + m), \quad (x, n)(y, m) = (xy + ny + mx, nm),$$

$\mathcal{R}_1$ is a ring with unit $\mathbf{1} = (0, 1)$, and contains $\mathcal{R}$ as a ideal if we identify x in $\mathcal{R}$ with $(x, 0)$ in $\mathcal{R}_1$. For an automorphism α of $\mathcal{R}$, we define the mapping $\tilde{\alpha}$ of $\mathcal{R}_1$ by $\tilde{\alpha}(x, n) = (\alpha(x), n)$. Then $\tilde{\alpha}$ is an automorphism of $\mathcal{R}_1$ such that $\tilde{\alpha}|_{\mathcal{R}} = \alpha$. Obviously, the mapping $\tilde{d}$ of $\mathcal{R}_1$ into itself, defined by $\tilde{d}(x, n) = d(x)$ for all $(x, n) \in \mathcal{R}_1$, is a multiplicative $(\tilde{\alpha}, \tilde{\beta})$-derivation on $\mathcal{R}_1$ if and only if $d(xy + mx + ny) = d(xy) + md(x) + nd(y)$ for all $x, y \in \mathcal{R}$ and $m, n \in \mathbb{Z}$. In particular, if d is additive, then $\tilde{d}$ is an (additive) $(\tilde{\alpha}, \tilde{\beta})$-derivation.

Recall that an associative ring $\mathcal{R}$ is prime if, for each a, b in $\mathcal{R}$, $a\mathcal{R}b = 0$ implies $a = 0$ or $b = 0$. It is well known that the matrix ring $M_n(\mathbb{C})$, and generally each factor von Neumann algebra, is prime. By Theorem 1, we have the following corollary.

Corollary 2. *Every multiplicative (α, β)-derivation of a prime ring with a nontrivial idempotent is additive. In particular, each multiplicative (α, β)-derivation of a factor von Neumann algebra, and hence, of $M_n(\mathbb{C})$, is additive.*

Let d be a multiplicative (α,β)-derivation of $\mathcal{R}$, e and $\tilde{e}$ be as in Theorem 1. If let $e'=\alpha^{-1}(e)$, then $\alpha(e')=e, \beta(e')=\tilde{e}$. As in [2], the two-sided Peirce decomposition of $\mathcal{R}$ relative to the idempotent e' takes the form $\mathcal{R}=\mathcal{R}'_{11}\oplus\mathcal{R}'_{12}\oplus\mathcal{R}'_{21}\oplus\mathcal{R}'_{22}$, where $\mathcal{R}'_{11}=e'\mathcal{R}e'$, $\mathcal{R}'_{12}=e'\mathcal{R}(1-e')$, $\mathcal{R}'_{21}=(1-e')\mathcal{R}e'$ and $\mathcal{R}'_{22}=(1-e')\mathcal{R}(1-e')$. Relative to the idempotents $\tilde{e}$ and e, we have the generalized two-sided Peirce decomposition of $\mathcal{R}$, $\mathcal{R}=\mathcal{R}_{11}\oplus\mathcal{R}_{12}\oplus\mathcal{R}_{21}\oplus\mathcal{R}_{22}$, where $\mathcal{R}_{11}=\tilde{e}\mathcal{R}e$, $\mathcal{R}_{12}=\tilde{e}\mathcal{R}(1-e)$, $\mathcal{R}_{21}=(1-\tilde{e})\mathcal{R}e$, $\mathcal{R}_{22}=(1-\tilde{e})\mathcal{R}(1-e)$.

From the definition of d, we have $d(0)=0$ and $d(e')=d(e')e+\tilde{e}d(e')$. Hence $(1-\tilde{e})d(e')(1-e)=0$ and $\tilde{e}d(e')e=2\tilde{e}d(e')e$, which implies $\tilde{e}d(e')e=0$. So we can decompose $d(e')$ into $a_{12}+a_{21}$, where $a_{12}=\tilde{e}d(e')(1-e)$, $a_{21}=(1-\tilde{e})d(e')e$.

Let f be a mapping of $\mathcal{R}$ into itself, defined by $f(x)=\beta(x)(a_{12}-a_{21})-(a_{12}-a_{21})\alpha(x)$. Since α and β are automorphisms, we have that f is additive and satisfies that $f(x_1x_2)=f(x_1)\alpha(x_2)+\beta(x_1)f(x_2)$ for each x_1,x_2 in $\mathcal{R}$, so f is an (α,β)-inner derivation of $\mathcal{R}$. It follows from the definitions of a_{12} and a_{21} that $f(e')=a_{12}+a_{21}=d(e')$.

Define $D=d-f$. Then D is a multiplicative (α,β)-derivation of $\mathcal{R}$, and D is additive if and only if so is d. Hence in order to complete the proof of Theorem 1, we only show that D is additive. We remark that $D(e')=0$ and $D(0)=0$.

Lemma 1. $D(\mathcal{R}'_{ij})\subseteq\mathcal{R}_{ij}$.

Proof. For x'_{11} in $\mathcal{R}'_{11}$, since $D(e')=0$, we have $D(x'_{11})=D(e'(x'_{11}e'))=\beta(e')D(x'_{11}e')=\tilde{e}D(x'_{11})\alpha(e')=\tilde{e}D(x'_{11})e\in\mathcal{R}_{11}$. For x'_{12} in $\mathcal{R}'_{12}$, we have $D(x'_{12})=D(e'x'_{12})=\tilde{e}D(x'_{12})$ and $0=D(x'_{12}e')=D(x'_{12})\alpha(e')=D(x'_{12})e$, so that $(1-\tilde{e})D(x'_{12})=D(x'_{12})e=0$, which implies that $D(x'_{12})$ is in $\mathcal{R}_{12}$.

For x'_{21} in $\mathcal{R}'_{21}$, we have $D(x'_{21})=D(x'_{21}e')=D(x'_{21})e$ and $0=D(e'x'_{21})=\tilde{e}D(x'_{21})$. Hence $D(x'_{21})\in\mathcal{R}_{21}$. For x'_{22} in $\mathcal{R}'_{22}$, we have $0=D(e'x'_{22})=\tilde{e}D(x'_{22})$ and $0=D(x'_{22}e')=D(x'_{22})e$. Hence $D(x'_{22})\in\mathcal{R}_{22}$. □

Lemma 2. *For each x'_{ii} in $\mathcal{R}'_{ii}$ and x'_{jk} in $\mathcal{R}'_{jk}$ with $1\leqslant i,j,k\leqslant 2$ and $j\neq k$, we have* $D\left(x'_{ii}+x'_{jk}\right)=D(x'_{ii})+D\left(x'_{jk}\right)$.

Proof. Obviously, we only need to show $D(x'_{ii})+D\left(x'_{jk}\right)-D\left(x'_{ii}+x'_{jk}\right)=0$. By the hypothesis, we consider four cases.

Case 1: $i=j=1$ and $k=2$. Using the condition (C3), we only show that $(D(x'_{11})+D(x'_{12})-D(x'_{11}+x'_{12}))\mathcal{R}=0$.

For $t_1\in e\mathcal{R}$, using Lemma 1, we have $D(x'_{12})t_1=0$. Let $s_1=\alpha^{-1}(t_1)$. Then $s_1=\alpha^{-1}(et_1)=e's_1$, and thus, $x'_{12}s_1=0$. Since $x'_{12}e'=0$, it follows that $\beta(x'_{12})D(s_1)=\beta(x'_{12})D(e's_1)=\beta(x'_{12})\beta(e')D(s_1)=\beta(x'_{12}e')D(s_1)=0$. Hence

$$\begin{aligned}\left(D\left(x'_{11}\right)+D\left(x'_{12}\right)\right)t_1 &= D\left(x'_{11}\right)t_1+D\left(x'_{12}\right)t_1=D\left(x'_{11}\right)t_1=D\left(x'_{11}\right)\alpha(s_1)\\ &= D\left(x'_{11}s_1\right)-\beta\left(x'_{11}\right)D(s_1)=D\left(\left(x'_{11}+x'_{12}\right)s_1\right)-\beta\left(x'_{11}\right)D(s_1)\\ &= D\left(x'_{11}+x'_{12}\right)\alpha(s_1)+\beta\left(x'_{11}+x'_{12}\right)D(s_1)-\beta\left(x'_{11}\right)D(s_1)\\ &= D\left(x'_{11}+x'_{12}\right)\alpha(s_1)+\beta\left(x'_{12}\right)D(s_1)\\ &= D\left(x'_{11}+x'_{12}\right)t_1.\end{aligned}$$

Consequently, $(D(x'_{11})+D(x'_{12})-D(x'_{11}+x'_{12}))t_1=0$ for each $t_1\in e\mathcal{R}$.

On the other hand, for $t_2\in(1-e)\mathcal{R}$, it follows from Lemma 1 that $D(x'_{11})t_2=0$. Let $s_2=\alpha^{-1}(t_2)$. Then $s_2=\alpha^{-1}((1-e)t_2)=\alpha^{-1}(t_2)-\alpha^{-1}(e)\alpha^{-1}(t_2)=(1-e')s_2$, and hence $x'_{11}s_2=0$. Since $0=D(0)=D(e's_2)=\beta(e')D(s_2)$, we have $\beta(x'_{11})D(s_2)=\beta(x'_{11}e')D(s_2)=\beta(x'_{11})\beta(e')D(s_2)=0$. Hence

$$\begin{aligned}\left(D\left(x'_{11}\right)+D\left(x'_{12}\right)\right)t_2 &= D\left(x'_{12}\right)t_2=D\left(x'_{12}\right)\alpha(s_2)=D\left(x'_{12}s_2\right)-\beta\left(x'_{12}\right)D(s_2)\\ &= D\left(\left(x'_{11}+x'_{12}\right)s_2\right)-\beta\left(x'_{12}\right)D(s_2)\\ &= D\left(x'_{11}+x'_{12}\right)\alpha(s_2)+\beta\left(x'_{11}+x'_{12}\right)D(s_2)-\beta\left(x'_{12}\right)D(s_2)\\ &= D\left(x'_{11}+x'_{12}\right)\alpha(s_2)+\beta\left(x'_{11}\right)D(s_2)\\ &= D\left(x'_{11}+x'_{12}\right)t_2.\end{aligned}$$

Consequently, $(D(x'_{11})+D(x'_{12})-D(x'_{11}+x'_{12}))\,t_2=0$. Since t_1 and t_2 are arbitrary, we obtain that $(D(x'_{11})+D(x'_{12})-D(x'_{11}+x'_{12}))\,t=0$ for each $t\in\mathcal{R}$.

Case 2: $i=j=2, k=1$. By a similar way to case 1, we show $(D(x'_{22})+D(x'_{21})-D(x'_{22}+x'_{21}))\,\mathcal{R}=0$.

For $t_1\in e\mathcal{R}$, it follows from Lemma 1 that $D(x'_{22})\,t_1=0$. Let $s_1=\alpha^{-1}(t_1)$. Then $s_1=\alpha^{-1}(et_1)=e's_1$, and thus, $x'_{22}s_1=0$. Also since $\beta(x'_{22})\,D(s_1)=\beta(x'_{22})\,D(e's_1)=\beta(x'_{22}e')\,D(s_1)=0$, we have

$$\begin{aligned}\left(D\left(x'_{22}\right)+D\left(x'_{21}\right)\right)t_1 &= D\left(x'_{21}\right)t_1=D\left(x'_{21}\right)\alpha(s_1)=D\left(x'_{21}s_1\right)-\beta\left(x'_{21}\right)D(s_1)\\ &= D\left(\left(x'_{22}+x'_{21}\right)s_1\right)-\beta\left(x'_{21}\right)D(s_1)\\ &= D\left(x'_{22}+x'_{21}\right)\alpha(s_1)+\beta\left(x'_{22}\right)D(s_1)\\ &= D\left(x'_{22}+x'_{21}\right)t_1.\end{aligned}$$

For $t_2\in(1-e)\mathcal{R}$, using Lemma 1, we have $D(x'_{21})\,t_2=0$. Let $s_2=\alpha^{-1}(t_2)$. Then $s_2=\alpha^{-1}((1-e)t_2)=(1-e')s_2$, and thus $e's_2=0$, which implies that $\beta(x'_{21})\,D(s_2)=\beta(x'_{21}e')\,D(s_2)=\beta(x'_{21})\,D(e's_2)=0$. Hence $(D(x'_{22})+D(x'_{21}))\,t_2=D(x'_{22})\,t_2=D(x'_{22})\,\alpha(s_2)=D(x'_{22}s_2)-\beta(x'_{22})\,D(s_2)=D(x'_{22}+x'_{21})\,\alpha(s_2)+\beta(x'_{21})\,D(s_2)=D(x'_{22}+x'_{21})\,t_2$.

Since t_1 and t_2 are arbitrary, we have $(D(x'_{22})+D(x'_{21})-D(x'_{22}+x'_{21}))\,\mathcal{R}=0$.

Case 3: $i=k=1, j=2$. For the case, we use the condition (C1).

For $t_1\in\mathcal{R}\tilde{e}$, it follows from Lemma 1 that $t_1D(x'_{21})=0$. Let $s_1=\beta^{-1}(t_1)$. Then $s_1=s_1e'$. Hence $s_1x'_{21}=0$ and $D(s_1)\alpha(x'_{21})=D(s_1e')\,\alpha(x'_{21})=D(s_1)\alpha(e'x'_{21})=0$. So

$$\begin{aligned}t_1\left(D\left(x'_{11}\right)+D\left(x'_{21}\right)\right) &= t_1D\left(x'_{11}\right)=\beta(s_1)D\left(x'_{11}\right)=D\left(s_1x'_{11}\right)-D(s_1)\alpha\left(x'_{11}\right)\\ &= D\left(s_1\left(x'_{11}+x'_{21}\right)\right)-D(s_1)\alpha\left(x'_{11}\right)\\ &= D(s_1)\alpha\left(x'_{11}+x'_{21}\right)+\beta(s_1)D\left(x'_{11}+x'_{21}\right)-D(s_1)\alpha\left(x'_{11}\right)\\ &= D(s_1)\alpha\left(x'_{21}\right)+\beta(s_1)D\left(x'_{11}+x'_{21}\right)=\beta(s_1)D\left(x'_{11}+x'_{21}\right)\\ &= t_1D\left(x'_{11}+x'_{21}\right).\end{aligned}$$

Hence $t_1\,(D(x'_{11})+D(x'_{21})-D(x'_{11}+x'_{21}))=0$.

For $t_2\in\mathcal{R}(1-\tilde{e})$, using Lemma 1, we have $t_2D(x'_{11})=0$. Let $s_2=\beta^{-1}(t_2)$. Then $s_2=s_2(1-e')$, and thus, $s_2e'=0$ and $s_2x'_{11}=0$, which implies $D(s_2)\alpha(e')D(s_2e')=0$. It follows that $D(s_2)\alpha(x'_{11})=D(s_2)\alpha(e'x'_{11})=0$. Hence

$$\begin{aligned}t_2\left(D\left(x'_{11}\right)+D\left(x'_{21}\right)\right) &= t_2D\left(x'_{21}\right)=\beta(s_2)D\left(x'_{21}\right)=D\left(s_2x'_{21}\right)-D(s_2)\alpha\left(x'_{21}\right)\\ &= D\left(s_2\left(x'_{11}+x'_{21}\right)\right)-D(s_2)\alpha\left(x'_{21}\right)\\ &= D(s_2)\alpha\left(x'_{11}+x'_{21}\right)+\beta(s_2)D\left(x'_{11}+x'_{21}\right)-D(s_2)\alpha\left(x'_{21}\right)\\ &= t_2D\left(x'_{11}+x'_{21}\right).\end{aligned}$$

Consequently, $t_2\left(D\left(x'_{11}\right)+D\left(x'_{21}\right)-D\left(x'_{11}+x'_{21}\right)\right)=0$. Since t_1 and t_2 are arbitrary, we have $\mathcal{R}\left(D\left(x'_{11}\right)+D\left(x'_{21}\right)-D\left(x'_{11}+x'_{21}\right)\right)=0$. It follows from (C1) that $D\left(x'_{11}+x'_{21}\right)=D\left(x'_{11}\right)+D\left(x'_{21}\right)$.

Case 4: $i=k=2, j=1$. The proof is similar to that of the Case 3, we omit it. □

Lemma 3. *D is additive on* $\mathcal{R}'_{12}$.

Proof. Let x'_{12} and y'_{12} be in $\mathcal{R}'_{12}$. Using Lemma 1, we have $D\left(x'_{12}\right), D\left(y'_{12}\right), D\left(x'_{12}+y'_{12}\right)\in\mathcal{R}_{12}$. Hence $\left(D\left(x'_{12}\right)+D\left(y'_{12}\right)-D\left(x'_{12}+y'_{12}\right)\right)t_1=0$ for each $t_1\in e\mathcal{R}$.

For $t_2\in(1-e)\mathcal{R}$, let $s_2=\alpha^{-1}(t_2)$. Then $s_2=(1-e')s_2$, which implies that $e'\left(s_2+y'_{12}s_2\right)=y'_{12}s_2$. It follows that $D\left(y'_{12}s_2\right)=D\left(e'\left(s_2+y'_{12}s_2\right)\right)=\beta(e')D\left(s_2+y'_{12}s_2\right)$. Since $e'\in\mathcal{R}'_{11}$ and $x'_{12}\in\mathcal{R}'_{12}$, it follows from Lemma 2 that $D\left(e'+x'_{12}\right)=D(e')+D\left(x'_{12}\right)=D\left(x'_{12}\right)$. Also since $\left(x'_{12}+y'_{12}\right)s_2=\left(e'+x'_{12}\right)\left(s_2+y'_{12}s_2\right)$, we have

$$\begin{aligned}
D\left(\left(x'_{12}+y'_{12}\right)s_2\right) &= D\left(\left(e'+x'_{12}\right)\left(s_2+y'_{12}s_2\right)\right)\\
&= D\left(x'_{12}\right)\alpha\left(s_2+y'_{12}s_2\right)+\beta\left(e'+x'_{12}\right)D\left(s_2+y'_{12}s_2\right)\\
&= D\left(x'_{12}\right)\alpha\left(s_2+y'_{12}s_2\right)+\beta\left(x'_{12}\right)D\left(s_2+y'_{12}s_2\right)+\beta(e')D\left(s_2+y'_{12}s_2\right)\\
&= D\left(x'_{12}\left(s_2+y'_{12}s_2\right)\right)+D\left(y'_{12}s_2\right)\\
&= D\left(x'_{12}s_2\right)+D\left(y'_{12}s_2\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(D\left(x'_{12}\right)+D\left(y'_{12}\right)-D\left(x'_{12}+y'_{12}\right)\right)t_2\\
&= D\left(x'_{12}\right)\alpha(s_2)+D\left(y'_{12}\right)\alpha(s_2)-D\left(x'_{12}+y'_{12}\right)\alpha(s_2)\\
&= D\left(x'_{12}s_2\right)+D\left(y'_{12}s_2\right)-D\left(\left(x'_{12}+y'_{12}\right)s_2\right)\\
&=0.
\end{aligned}$$

Since t_1 and t_2 are arbitrary, we have $\left(D\left(x'_{12}\right)+D\left(y'_{12}\right)-D\left(x'_{12}+y'_{12}\right)\right)\mathcal{R}=0$. It follows from the condition (C3) that $D\left(x'_{12}+y'_{12}\right)=D\left(x'_{12}\right)+D\left(y'_{12}\right)$. Hence D is additive on $\mathcal{R}'_{12}$. □

Lemma 4. *D is additive on* $\mathcal{R}'_{11}$.

Proof. Fix $x'_{11}, y'_{11}\in\mathcal{R}'_{11}$. It follows from Lemma 1 that $D\left(x'_{11}\right), D\left(y'_{11}\right)$ and $D\left(x'_{11}+y'_{11}\right)$ are in $\mathcal{R}_{11}$. Hence $\left(D\left(x'_{11}\right)+D\left(y'_{11}\right)-D\left(x'_{11}+y'_{11}\right)\right)t_{22}=0$ for each $t_{22}\in\mathcal{R}_{22}$.

For $t_{12}\in\mathcal{R}_{12}$, let $s_{12}=\alpha^{-1}(t_{12})$. Then $s_{12}=\alpha^{-1}(t_{12}(1-e))=s_{12}(1-e')\in\mathcal{R}(1-e')$. Hence $x'_{11}s_{12}$ and $y'_{11}s_{12}$ are in $\mathcal{R}'_{12}$. It follows from Lemma 3 that $D\left(x'_{11}s_{12}+y'_{11}s_{12}\right)=D\left(x'_{11}s_{12}\right)+D\left(y'_{11}s_{12}\right)$. So

$$\begin{aligned}
&\left(D\left(x'_{11}\right)+D\left(y'_{11}\right)-D\left(x'_{11}+y'_{11}\right)\right)t_{12}\\
&= D\left(x'_{11}\right)\alpha(s_{12})+D\left(y'_{11}\right)\alpha(s_{12})-D\left(x'_{11}+y'_{11}\right)\alpha(s_{12})\\
&= D\left(x'_{11}s_{12}\right)+D\left(y'_{11}s_{12}\right)-D\left(\left(x'_{11}+y'_{11}\right)s_{12}\right)\\
&= 0.
\end{aligned}$$

Since t_{12} and t_{22} are arbitrary, we have $\left(D\left(x'_{11}\right)+D\left(y'_{11}\right)-D\left(x'_{11}+y'_{11}\right)\right)\mathcal{R}(1-e)=0$. By Lemma 1, we note that $D\left(x'_{11}\right)+D\left(y'_{11}\right)-D\left(x'_{11}+y'_{11}\right)\in\mathcal{R}_{11}$. Hence it follows from the condition (C2) that $D\left(x'_{11}+y'_{11}\right)=D\left(x'_{11}\right)+D\left(y'_{11}\right)$. Consequently, D is additive on $\mathcal{R}'_{11}$. □

Lemma 5. *D is additive on* $e'\mathcal{R}=\mathcal{R}'_{11}\oplus\mathcal{R}'_{12}$.

Proof. By Lemmas 3 and 4, D is additive on $\mathcal{R}'_{11}$ and $\mathcal{R}'_{12}$, respectively. Using Lemma 2, for each $x'_{11} \in \mathcal{R}'_{11}$ and $x'_{12} \in \mathcal{R}'_{12}$, we have $D\left(x'_{11} + x'_{12}\right) = D\left(x'_{11}\right) + D\left(x'_{12}\right)$. Hence D is additive on $e'\mathcal{R} = \mathcal{R}'_{11} \oplus \mathcal{R}'_{12}$. $\square$

The Proof of Theorem 1. Let x and y be in $\mathcal{R}$. For each $t \in \tilde{e}\mathcal{R}$, let $s = \beta^{-1}(t)$. Then $s = \beta^{-1}(\tilde{e}t) = e's$, and hence $sx,\ sy \in e'\mathcal{R}$. By Lemma 5, we have that $D(sx + sy) = D(sx) + D(sy)$. Consequently, $t(D(x) + D(y)) = \beta(s)D(x) + \beta(s)D(y) = D(sx) + D(sy) - D(s)\alpha(x) - D(s)\alpha(y) = D(s(x + y)) - D(s)\alpha(x + y) = \beta(s)D(x + y) = tD(x + y)$.

Since t is arbitrary, we have $\tilde{e}\mathcal{R}(D(x + y) - D(x) - D(y)) = 0$. It follows from the condition (C1) that $D(x + y) = D(x) + D(y)$. Hence D is additive.

3. Linearity of multiplicative (α, β)-derivations on $M_n(\mathbb{C})$

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$, α and β be algebraic automorphisms of $\mathcal{A}$. Recall that a mapping D from $\mathcal{A}$ into itself is called a multiplicative (α, β)-derivation of $\mathcal{A}$, if the derivation condition (1) holds, i.e., $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ for all x, y in $\mathcal{A}$. Moreover, D is called inner if there exists x_0 in $\mathcal{A}$ such that $D(x) = \beta(x)x_0 - x_0\alpha(x)$ for each x in $\mathcal{A}$. Obviously, if $\mathcal{A}$ has an identity I, then $D(I) = 0$. In this section, we consider the linearity problems of multiplicative (α, β)-derivations on $M_n(\mathbb{C})$. It follows from Corollary 2 that every multiplicative (α, β)-derivation on $M_n(\mathbb{C})$ is additive.

It is well known that each algebraic automorphism α on $M_n(\mathbb{C})$ is inner, i.e., there is an invertible matrix T_0 in $M_n(\mathbb{C})$ such that $\alpha(A) = T_0AT_0^{-1}$ for each A in $M_n(\mathbb{C})$. Like the ordinary derivation, we can show that every linear multiplicative (α, β)-derivation is inner, which may be a known fact.

Theorem 3. *Let D be a multiplicative (α, β)-derivation on $M_n(\mathbb{C})$. If D is linear, then D is inner.*

Proof. Let E_{ij}, $i, j = 1, 2 \ldots, n$, be the standard matrix unit of $M_n(\mathbb{C})$. Let

$$T_0 = \sum_{j=1}^{n} \beta(E_{j1})D(E_{1j}).$$

Then, for each E_{kl}, using that $D(I) = 0$, we have

$$\begin{aligned}
\beta(E_{kl})T_0 - T_0\alpha(E_{kl}) &= \sum_{j=1}^{n} \beta(E_{kl})\beta(E_{j1})D(E_{1j}) - \sum_{j=1}^{n} \beta(E_{j1})D(E_{1j})\alpha(E_{kl}) \\
&= \beta(E_{k1})D(E_{1l}) - \sum_{j=1}^{n}(D(E_{j1}E_{1j}) - D(E_{j1})\alpha(E_{1j}))\alpha(E_{kl}) \\
&= \beta(E_{k1})D(E_{1l}) - D(I)\alpha(E_{kl}) + D(E_{k1})\alpha(E_{1l}) \\
&= D(E_{k1}E_{1l}) \\
&= D(E_{kl}),
\end{aligned}$$

where I is the identity matrix. Since α, β and D are linear, we have $D(A) = \beta(A)T_0 - T_0\alpha(A)$ for each A in $M_n(\mathbb{C})$. Hence D is inner. $\square$

The following lemma is similar to Lemma 1 in [8].

Lemma 6. *Let D be a multiplicative (α, β)-derivation on $M_n(\mathbb{C})$. Then there exist an additive derivation $f: \mathbb{C} \to \mathbb{C}$ and an invertible matrix V_0 in $M_n(\mathbb{C})$ such that $D(tI) = f(t)V_0$ holds for all t in $\mathbb{C}$.*

Proof. For arbitrary $A \in M_n(\mathbb{C})$ and $t \in \mathbb{C}$, we have

$$D(tA) = D((tI)A) = D(tI)\alpha(A) + tD(A).$$

On the other hand,

$$D(tA) = D(A(tI)) = tD(A) + \beta(A)D(tI).$$

Hence $D(tI)\alpha(A) = \beta(A)D(tI)$. Since α and β are inner, there exist invertible matrices T_0 and S_0 such that $\alpha(A) = T_0AT_0{}^{-1}$ and $\beta(A) = S_0AS_0{}^{-1}$ for all A in $M_n(\mathbb{C})$. Thus, $D(tI)T_0AT_0{}^{-1} = S_0AS_0{}^{-1}D(tI)$, and hence, $\left(S_0^{-1}D(tI)T_0\right)A = A\left(S_0^{-1}D(tI)T_0\right)$ holds for all A in $M_n(\mathbb{C})$. Consequently, $S_0^{-1}D(tI)T_0$ is in the center of $M_n(\mathbb{C})$, so there exists $f(t) \in \mathbb{C}$ such that $S_0^{-1}D(tI)T_0 = f(t)I$, hence

$$D(tI) = f(t)V_0, \tag{2}$$

where $V_0 = S_0T_0^{-1}$. Since D is additive, one can see easily that the mapping $f: \mathbb{C} \to \mathbb{C}$ defined by the Eq. (2) is an additive derivation. □

Remark. The proof of Lemma 6 implies that the multiplicative (α, β)-derivation D is linear if and only if so is f, i.e, f is a trivial derivation.

Theorem 4. *A mapping D on $M_n(\mathbb{C})$ is a multiplicative (α, β)-derivation if and only if there exist an additive derivation $f: \mathbb{C} \to \mathbb{C}$, a matrix A_0 and invertible matrices S_0 and T_0 such that*

$$D((a_{ij})) = S_0(f(a_{ij}))T_0^{-1} + S_0(a_{ij})S_0^{-1}A_0 - A_0T_0(a_{ij})T_0^{-1},$$

where $\alpha(A) = T_0AT_0^{-1}$ and $\beta(A) = S_0AS_0^{-1}$.

Proof. Let D be a multiplicative (α, β)-derivation on $M_n(\mathbb{C})$, f be the additive derivation on $\mathbb{C}$ defined by $D(tI) = f(t)S_0T_0^{-1}$, as in the proof of Lemma 6. Let $F(A) = S_0(f(a_{ij}))T_0^{-1}$ for each $A = (a_{ij}) \in M_n(\mathbb{C})$. Then F is additive on $M_n(\mathbb{C})$. For all $A = (a_{ij})$ and $B = (b_{ij})$ in $M_n(\mathbb{C})$, we have

$$\begin{aligned}
F(A)\alpha(B) + \beta(A)F(B) &= S_0(f(a_{ij}))T_0^{-1}T_0BT_0^{-1} + S_0AS_0^{-1}S_0(f(b_{ij}))T_0^{-1} \\
&= S_0(f(a_{ij}))(b_{ij})T_0^{-1} + S_0(a_{ij})(f(b_{ij}))T_0^{-1} \\
&= S_0\left(\sum_{k=1}^{n}(f(a_{ik})b_{kj} + a_{ik}f(b_{kj}))\right)T_0^{-1} \\
&= S_0\left(\sum_{k=1}^{n}f(a_{ik}b_{kj})\right)T_0^{-1} \\
&= S_0\left(f\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right)\right)T_0^{-1} \\
&= F(AB).
\end{aligned}$$

Consequently, F is a multiplicative (α, β)-derivation on $M_n(\mathbb{C})$.

Define $\widetilde{D} = D - F$. Then $\widetilde{D}$ is a multiplicative (α, β)-derivation on $M_n(\mathbb{C})$. Obviously, $\widetilde{D}(tI) = D(tI) - F(tI) = f(t)S_0T_0^{-1} - f(t)S_0T_0^{-1} = 0$. By the Remark of Lemma 6, $\widetilde{D}$ is linear. It follows from Theorem 3 that there exists A_0 in $M_n(\mathbb{C})$ such that $\widetilde{D}(A) = \beta(A)A_0 - A_0\alpha(A)$ for each A in $M_n(\mathbb{C})$. Hence

$$D((a_{ij})) = S_0(f(a_{ij}))T_0^{-1} + S_0(a_{ij})S_0^{-1}A_0 - A_0T_0(a_{ij})T_0^{-1}. \quad \square \tag{3}$$

Remark. For fixed α and β, putting $(a_{ij}) = tI$ in (3), we can see that the additive derivation f is uniquely determined. Hence all such matrices A_0 are different from λI.

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